MSC2010: 93C15, 49N30, 49N35

© *M. Larbani, V. I. Zhukovskii*

BERGE EQUILIBRIUM IN NORMAL FORM STATIC GAMES: A LITERATURE REVIEW

We present a literature review of Berge equilibrium (BE) in normal form static games. The review shows that research on BE has gained momentum in the last few years as this equilibrium is now grounded in game theory, philosophy and social interaction. It captures mutual support, cooperation and coordination, and models altruism and the moral Golden Rule in normal form games. Mathematical investigation of Berge equilibrium is advanced but not complete; more research is needed in the areas related to its existence and computation. Application of BE in real-world socio-economic interactions where players are mutually supportive is an almost unexplored area of research.

Keywords: Berge equilibrium, mutual support, cooperation, Golden Rule, altruism, determination of BE, existence of BE, computation of BE.

DOI: 10.20537/2226-3594-2017-49-04

Dedicated to the memory of Konstantin Semenovich Vaisman[1]

Introduction

C. Berge [7][2] published a short book on game theory that contained a wealth of concepts that did not receive enough attention from game theory scholars for decades. M. Shubik's [43] negative review of Berge's book ("... no attention is paid to the application to the economy ... the book is of little interest for economists") "discouraged" economists from exploring its concepts for a long time. Berge's book was translated into Russian in 1961. In the 1980s V. I. Zhukovskiy broke away from this state of affairs; he started to investigate one of the concepts of equilibrium in normal form games introduced by Berge in his book: the concept of equilibrium of a coalition P with respect to another coalition K, or P/K-equilibrium. Such an equilibrium is reached when the coalition K does its best to maximize the payoffs of players in coalition P. Using this concept, Zhukovskiy [52] introduced the Berge equilibrium (BE) that is a $\{i\}/I - \{i\}$-equilibrium for all players $i \in I$, where $I = \{1, 2, \ldots, n\}$ is the set of players, in static normal form games. That is, a Berge equilibrium is reached when for each player $i \in I$, all the other players in the coalition $I - \{i\}$ do their best to maximize his/her payoff. For a decade BE did not receive attention in the West as most of related publications appeared in the Russian language and within Russia and were mainly authored by Zhukovskiy, his students and colleagues only. Here it is important to pay tribute to Konstantin Semenovich Vaisman, a student of V. I. Zhukovskiy, who conducted a pioneering first in-depth study of BE and its properties and improved its definition, in his PhD thesis [45]. Unfortunately, he died at the early age of 36 after a struggle with cancer. The main contributions of Vaisman are as follows. He (i) constructed a counterexample showing that BE may not satisfy the well-known individual rationality condition, thereby, improving it, (ii) pointed out that BE is immune against deviation of coalitions of the form, (iii) initiated the investigation of BE in games involving uncertainty in the payoff functions, (iv) was the first to investigate the Nash bargaining solution involving uncertainty in payoff functions in cooperative games [46], and (v) introduced in non-cooperative games the concept

[1] Konstantin Semenovich Vaisman was a young Russian mathematician who died on March 10, 1998, after a struggle with cancer. He was born on August 29, 1962 in Moscow Region; his father was an electrical engineer graduating from the famous Moscow Energy Institute; his mother dedicated her life to raising her two children. He went through Orekhovo-Zuevo Pedagogical Institute (1984) and Moscow State University (1993). In 1995, he received his Ph.D. from Saint-Petersburg State University. The theme of his thesis was Berge Equilibrium. In the last years of his short life he worked as an Associate Professor at Orekhovo-Zuevo Pedagogical Institute. He has published 26 works in the field of game theory. Most of them are related to Berge Equilibrium.

[2] The book was published in 1957 when Claude Berge, a French mathematician, was visiting professor at the Institute of Advanced Study at Princeton (Courtois et al. [2015]). Berge is an outstanding mathematician; his research and highly significant contributions cover a large spectrum of areas including operations research, optimization, graph theory and game theory.

of hybrid equilibrium where some players select their strategies from Nash equilibrium, while others select their strategies based on the concept of threats and counter-threats equilibrium [54]. Later this concept was developed in [74]. We will come back BE-related Vaisman's contributions (i)–(iv) mentioned above.

In the last decade, BE received a great deal of attention as this equilibrium presents a possible alternative besides Nash equilibrium in the new challenging and interdependent world that is the result of globalization and the information and communication technology revolution. Indeed, initially, Zhukovskiy introduced this equilibrium as an alternative to Nash equilibrium when Nash equilibrium does not exist in a game. Further investigations have revealed that BE is a rich concept as it has many interpretations and reflects many socio-economic behaviors in human interaction. Indeed, it can express the moral Golden Rule, the principle of mutual support or "positive" reciprocation, coordination, cooperation in non-cooperative settings and altruism. Later, we will see these interpretations in more detail.

All the above-mentioned socio-economic behaviors in human interaction are not captured by the concept of Nash equilibrium in normal form games. Numerous works have been published on these behaviors in human interaction; however, no formal conceptual framework has been put forward for their formal analysis in normal form games. There is growing evidence that BE is an appropriate concept to fill this huge gap. Therefore, BE has a great application potential in socio-economic interactions.

At this stage of evolution of research on BE, it is time to make a small pause and analyze what has been done and what needs to be done next regarding this equilibrium. The objective of this work is to conduct a literature review of the published works on BE to evaluate the theoretical and applied achievements made so far and to show some directions of further research. We do not pretend that the review is exhaustive as many publications appear in Russia (local journals, conferences, etc.) but not at an international level. However, the main results of Russian colleagues are reviewed here as most of them are published by Zhukovskiy's team or co-authored by him. Moreover, this review focuses on static normal form games, whereas the most important works on BE in differential games are only mentioned with few comments.

The work is organized as follows. In Section 1, to help the reader and make the work self-contained, we provide the definition of BE and discuss it. Section 2 is devoted to the explanation of different interpretations of BE. Section 3 reviews publications on the challenging problems of existence, determination and computation of this equilibrium. Section 4 reviews publications on applications of BE. Section 5 concludes and shows some research directions.

§ 1. Definition of Berge Equilibrium

Consider the following normal form game

$$G = \langle I, \{X_i\}_{i\in I}, \{f_i(x)\}_{i\in I}\rangle,$$

where $I = \{1, \dots, n\}$ is the set of players, X_i is the set of strategies of player $i \in I$, $f_i(\cdot) : X \to \mathbb{R}$ ($\mathbb{R}$ is the real line) is the payoff function of player $i \in I$; $X = \prod_{i\in I} X_i$ is the set of strategy profiles, $x = (x_1, \dots, x_n) \in X$ is a strategy profile and $x_i \in X_i \subset \mathbb{R}^{p_i}$ is the strategy selected by player $i \in I$; for any non-empty subset K of I and strategy profile $x = (x_1, \dots, x_n)$, we use the notation $x = (x_K, x_{I\setminus K})$, where $x_K \in X_K = \prod_{i\in K} X_i$ and $x_{I\setminus K} = \prod_{i\in I\setminus K} X_i$. Particularly, when $K = \{i\}$, that is, a singleton, the counter coalition $I \setminus K = I \setminus i$ is denoted by $I \setminus i$; the strategy profile x is denoted by $(x_i, x_{I\setminus i})$. The payoff functions $f_i(\cdot)$, $i \in I$, are such that the more the better.

D e f i n i t i o n 1.1. A strategy profile $\widetilde{x} = (\widetilde{x}_1, \dots, \widetilde{x}_n) \in X$ is said to be a *Berge Equilibrium* (BE) of the game G if for all $i \in I$, $y_{I\setminus i} \in X_{I\setminus i}$

$$f_i(\widetilde{x}_i, y_{I\setminus i}) \leqslant f_i(\widetilde{x}). \tag{1.1}$$

Condition (1.1) means that in BE each player's payoff function is maximized by the coalition of all the other players $I \setminus i$ (in some publications BE is referred to as *simple Berge equilibrium* or *Berge–Zhukovskiy* equilibrium).

Condition (1.1) can be equivalently formulated as follows:

$$\max_{y_{I\setminus i}\in X_{I\setminus i}} f_i(\widetilde{x}_i, y_{I\setminus i}) = f_i(\widetilde{x}), \quad i \in I.$$

To help the reader understand BE, let us compare it to the well-known concept of Nash equilibrium that is the most used equilibrium in normal form games [33,34].

D e f i n i t i o n 1.2. A strategy profile $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_n) \in X$ is said to be a *Nash Equilibrium* (NE) of the game G if for all $i \in I$, $y_i \in X_i$

$$f_i(y_i, \widetilde{x}_{I\setminus i}) \leqslant f_i(\widetilde{x}). \tag{1.2}$$

Condition (1.2) means that in NE it is not beneficial to any player to unilaterally deviate from NE. Indeed, if player i deviates from $\widetilde{x}_i$ to another strategy $y_i \in X_i$, while the other players remain in their equilibrium strategy, his payoff will remain the same or decrease. That is why NE is said to be a self-enforcing equilibrium. Once the players are in NE, no player has an incentive to unilaterally deviate from it. Let us give two examples for illustration.

E x a m p l e 1.1. Consider the following Prisoner's Dilemma game:

$$A = \begin{array}{c} \\ RP \\ CP \end{array} \begin{array}{c} \begin{array}{cc} RP & CP \end{array} \\ \begin{pmatrix} (20,20) & (5,25) \\ (25,5) & (10,10) \end{pmatrix} \end{array}. \tag{1.3}$$

This is a Prisoner's Dilemma game where two firms selling the same product are competing for market share, they can stick to a regular price or cut the price. The two players are the row player and the column player, each of them has two strategies, regular price (RP) and cut down price (CP). The strategies of the row player are displayed on the left side of the payoff matrix A, while the strategies of the column player are displayed at the top of the matrix A. The first number in each entry of the matrix A represents the payoff of the row player, while the second number represents the payoff of the column player. For instance, if the players select the strategy profile (RP, CP), i.e. the corresponding entry in the matrix A is $(5, 25)$, then the row player receives 5 units and the column player receives 25 units.

It is easy to see that the strategy profile (RP, RP) is a BE with the payoffs $(20, 20)$. Indeed, if the row player unilaterally deviates from RP to CP, the column player's payoff drops from 20 to 5, while if the column player unilaterally deviates from RP to CP, the row player's payoff drops from 20 to 5. Thus, at (RP, RP) both players maximize each other's payoff. The situation (CP, CP) is an NE of the game (1.3). Indeed, if either of the players unilaterally deviates from CP, his own payoff drops from 10 to 5. Thus, once the two players are in (CP, CP), none of them will have an incentive to unilaterally deviate from CP. Finally, note that if the players (the two firms) stay in the price cutting strategy profile NE (CP, CP), one of the players will have to leave the market: the one whose unit cost equals the current price first. Therefore, BE seems more suitable for both firms if they want to survive.

E x a m p l e 1.2. Imagine that in a market there are three sellers: a man (husband), his wife and their son. At their disposal they have resources X_h, X_w and X_s, respectively, and for gaining some profit they allocate part of their resources $x_i \in X_i$ $(i = h, w, s)$ to each family member. Which of these values will ensure that every one of them gets the greatest (possible) profit (revenues–cost) $P_i(x_h^e, x_w^e, x_s^e)$ $(i = h, w, s)$? To date, the NE concept "reigns" in such decision problems or game situations. According to Definition 1.2, a strategy profile $x^e = (x_h^e, x_w^e, x_s^e)$ is a Nash equilibrium if the following three equalities are satisfied:

$$\begin{aligned}
\max_{x_h} P_h(x_h, x_w^e, x_s^e) &= P_h(x_h^e, x_w^e, x_s^e), \\
\max_{x_w} P_w(x_h^e, x_w, x_s^e) &= P_w(x_h^e, x_w^e, x_s^e), \\
\max_{x_s} P_s(x_h^e, x_w^e, x_s) &= P_s(x_h^e, x_w^e, x_s^e).
\end{aligned}$$

Here, clearly "selfish nature" appears because everyone tends to increase (maximize) his/her profit only, ignoring the interests of other family members. The concept of BE put forward in Berge's book is the opposite of Nash equilibrium. According to Definition 1.1, BE at $x^B = (x_h^B, x_w^B, x_s^B)$ is characterized by the following three equalities:

$$\max_{x_w, x_s} P_h(x_h^B, x_w, x_s) = P_h(x_h^B, x_w^B, x_s^B),$$
$$\max_{x_h, x_s} P_w(x_h, x_w^B, x_s) = P_w(x_h^B, x_w^B, x_s^B),$$
$$\max_{x_h, x_w} P_h(x_h, x_w, x_s^B) = P_s(x_h^B, x_w^B, x_s^B).$$

These very equalities realize the Golden Rule that states "do to others as you would like them to do to you". Here, each family member wants more profit, therefore, he/she has to act in such a way as to maximize the profit of the other members so that the other members act in the same way and maximize his profit. Thus, the husband, by selecting $x_h = x_h^B$, does his best to maximize the profit of the wife and son, P_w and P_s, respectively, as it appears in the last two equalities

$$\max_{x_h, x_s} P_w(x_h, x_w^B, x_s) = P_w(x_h^B, x_w^B, x_s^B) \quad \text{and} \quad \max_{x_h, x_w} P_s(x_h, x_w, x_s^B) = P_s(x_h^B, x_w^B, x_s^B).$$

The wife and the son, by selecting $x_w = x_w^B$ and $x_s = x_s^B$, respectively, reciprocate by maximizing the husband's profit, P_h, as it appears in the first equality

$$\max_{x_w, x_s} P_h(x_h^B, x_w, x_s) = P_h(x_h^B, x_w^B, x_s^B).$$

The wife acts in exactly the same way. By selecting $x_w = x_w^B$, she maximizes the profit of the husband and the son, P_h and P_s, respectively, as evidenced by the first and last equalities

$$\max_{x_w, x_s} P_h(x_h^B, x_w, x_s) = P_h(x_h^B, x_w^B, x_s^B) \quad \text{and} \quad \max_{x_h, x_w} P_s(x_h, x_w, x_s^B) = P_s(x_h^B, x_w^B, x_s^B).$$

Reciprocating, both the husband and the son maximize the wife's profit by selecting $x_h = x_h^B$ and $x_s = x_s^B$, respectively, as it clearly appears in the second equality

$$\max_{x_h, x_s} P_w(x_h, x_w^B, x_s) = P_w(x_h^B, x_w^B, x_s^B).$$

A similar reasoning shows that the son maximizes the profit of both the husband and the wife, while the husband and the wife maximize the son's profit when all of them select as their strategy BE $x^B = (x_h^B, x_w^B, x_s^B)$. Thus, in BE each of the members maximizes the profit of the other two members and enjoys the same behavior from the other two members, which means that BE fully realizes or characterizes the Golden Rule. We later come back to the Golden Rule.

Comparisons between BE and NE appear in almost all publications related to BE (see, for example, [10, 12, 22, 54]).

Later, K. S. Vaisman [44] discovered that BE may not satisfy the individual rationality condition, that is, in BE some players may get a payoff that is less than what they can guarantee (maximin value or security level) for themselves, whatever the other players in $I \setminus i$ do. The security level for a player i in I is defined as follows:

$$\alpha_i = \max_{x_i \in X_i} \min_{x_{I \setminus i} \in X_{I \setminus i}} f_i(x_i, x_{I \setminus i}).$$

The following example by K. S. Vaisman's [45] illustrates the individual rationality problem in BE.

E x a m p l e 1.3. Assume that in the game G there are two players, $I = \{1, 2\}$, the strategy sets are $X_1 = (-\infty, +\infty)$ and $X_2 = [-1, 1]$, and the payoff functions are

$$f_1(x_1, x_2) = -2x_1^2 + 2x_1x_2 + x_2^2, \quad f_2(x_1, x_2) = -(x_1 - 1)^2 + 5.$$

Let us verify that $x^b = (1,1)$ is a BE of this game. We have

$$f_1(x_1^B, x_2) = f_1(1, x_2) = -4 + 2x_2 + x_2^2, \quad x_2 \in X_2 = [-1,1].$$

As $\dfrac{\partial^2 f_1(x_1^B, x_2)}{\partial x_2^2} = 2 > 0$, this function is strictly convex, therefore, it reaches its maximum at the boundary point $x_2 = x_2^B = 1$. We also have

$$f_2(x_1, x_2^B) = f_2(x_1, 1) = -(x_1 - 1)^2 + 5, \quad x_1 \in X_1 = (-\infty, +\infty).$$

As $-(x_1 - 1)^2 \leqslant 0$, this function reaches its maximum at $x_1 = x_1^B = 1$. Consequently, $f_1(x_1, x_2^B)$ reaches its maximum at $x_1 = x_1^B = 1$. Thus, we have the relations

$$\max_{x_2} f_1(x_1^B, x_2) = f_1(x_1^B, x_2^B), \quad \max_{x_1} f_2(x_1, x_2^B) = f_2(x_1^B, x_2^B),$$

which means $x^B = (1,1)$ is a BE. Let us now check whether $x^B = (1,1)$ satisfies the individual rationality condition. We have

$$f_1(x_1^B, x_2^B) = -1, \quad \alpha_1 = \max_{x_1 \in X_1} \min_{x_2 \in X_2} f_1(x_1, x_2) = 0.$$

This implies that $f_1(x_1^B, x_2^B) < \alpha_1$. Therefore, the individual rationality condition is not satisfied for the BE $x^B = (1,1)$.

Consequently, K. S. Vaisman [44] proposed the following new definition of BE to eliminate this drawback.

D e f i n i t i o n 1.3. A strategy profile $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_n) \in X$ is said to be a *Berge–Vaisman Equilibrium* (BVE) of the game G if for all $i \in I$

$$f_i(\widetilde{x}_i, y_{I\setminus i}) \leqslant f_i(\widetilde{x}), \quad y_{I\setminus i} \in X_{I\setminus i}, \quad \alpha_i \leqslant f_i(\widetilde{x}).$$

It is important to note that in some games all or some of the maximin values α_i, $i \in I$, may not exist (especially in linear-quadratic games). In this case, the corresponding individual rationality conditions in BVE can be dropped. Moreover, let

$$\beta_i = \max_{x_{I\setminus i} \in X_{I\setminus i}} \min_{x_i \in X_i} f_i(x_i, x_{I\setminus i}), \quad i \in I.$$

The following result is a sufficient practical condition for the individual rationality of BVE.

P r o p o s i t i o n 1.1 (see [27]). *Suppose the following inequalities are satisfied*

$$\alpha_i \leqslant \beta_i, \quad i \in I.$$

Then the individual rationality condition of BVE is satisfied.

Note that this result appeared later in [10].

§ 2. Different Interpretations of BE

BE is a rich concept of solution for normal form games. It can be interpreted in many ways. From the published literature, at least three interpretations have been put forward: the moral Golden Rule, capturing cooperation and mutual support in non-cooperative settings and altruism. Each of them can be used to handle games related to different contexts of socio-economic interaction. These interpretations show also that the scope of application of BE could be large and cover socio-economic interactions where NE may not be appropriate. In the following three sections we explain the three interpretations.

§ 2.1. The Moral Golden Rule Interpretation

Guseinov et al. [22] provide a philosophical and moral basis and justification of BE. In fact, BE can be seen as a straightforward expression of one of the most ancient rules of social interaction, the moral Golden Rule. This rule states that "do unto others as you would have them do unto you" or "you should treat people the way you would like other people to treat you" or "and just as you want men to do to you, you also do to them likewise" (Gospel of Luke 6:31). Written evidence of the existence of this rule could be traced back to the period 705–681 B.C.; it is also mentioned and given as a guidance for conduct and interaction in all major religions such as Christianity, Islam, Judaism, Buddhism and Confucianism (see [22]). In contemporary language, this rule can be said to express the principle of Positive Reciprocation in human behavior. The moral Golden Rule interpretation of BE was introduced recently in [62] and developed extensively by Zhukovskiy and his colleagues and students. Its mathematical foundations can be found in [57] and [62, 68].

Let us explain how BE is a mathematical expression of the moral Golden Rule through Definition 1.1. We start by illustrating the rule through Example 1.1. As mentioned above, the strategy profile (RP, RP) is a BE. It is easy to see that in this strategy profile both players follow the moral Golden Rule. Indeed, the row player does what he would like the column player to do to him, maximize his payoff. Indeed, by playing RP, the row player allows the column player to get the maximum payoff of 20 units. Similarly, by playing RP, the column player does what he would like the row player to do to him, maximize his payoff. Indeed, by selecting the strategy RP, the column player allows the row player to get the maximum payoff of 20 units. Clearly, if one of the players does not follow the moral Golden Rule, by deviating to the strategy CP, the other player gets the worst payoff of the game, 5 units.

Now we turn to the general case. Consider BE in Definition 1.1. Without loss of generality, select any two players $i, j \in I$. We show that in BE both i and j follow the moral Golden Rule with respect to each other, and thereafter all players follow the moral Golden Rule with respect to each other. Consider player i. From (1.1), it is clear that all the other players in $I \setminus i$ do their best to player i by selecting the bundle of strategies $\widetilde{x}_{I\setminus i}$, as in BE $\widetilde{x}$, he/she gets the maximum payoff $f_i(\widetilde{x})$. Any deviation of the players in $I \setminus i$ from their BE bundle of strategies $\widetilde{x}_{I\setminus i}$ would make player i's payoff worse or at most stay at the same level as $f_i(\widetilde{x})$. In other words, the remaining players in $I \setminus i$ behave *as player i would like them to behave towards him.* This means that 50 % of the moral Golden Rule is satisfied in the game G. For the rule to be completely satisfied, the player i should also behave according to the moral Golden Rule towards all the other remaining players in $I \setminus i$, i.e. do to them what they would like him to do to them (maximize their payoffs). Take player j, who is also in the set of the remaining players $I \setminus i$. Now, considering (1.1) with respect to player j, we get

$$f_j(\widetilde{x}_j, y_{I\setminus j}) \leqslant f_j(\widetilde{x}) \quad \text{for all} \quad y_{I\setminus j} \in X_{I\setminus j}.$$

Here also it is clear that the remaining players in $I \setminus j$ are doing what the player j would like them to do for him, i.e. maximizing his payoff by adopting their BE bundle of strategies $\widetilde{x}_{I\setminus j}$. Any deviation of players in $I \setminus j$ from $\widetilde{x}_{I\setminus j}$ would worsen player j's payoff or at best keep it at the same level as $f_j(\widetilde{x})$. Therefore, all the players in $I \setminus j$ are following the moral Golden Rule in their behavior towards player j. As player i is also in the set $I \setminus j$, he is also following the moral Golden Rule towards player j. Therefore, player i is implementing the moral Golden rule towards all the other players in $I \setminus i$. Thus, all players observe the moral Golden Rule with respect to each other.

From economic, social and global points of view, adopting the moral Golden Rule behavior would solve many long-lasting conflicts and problems at local and global levels. And solutions reached through the moral Golden Rule would be more stable than those reached through NE, as in BE all parties act in such a way as to maximize the satisfaction of the other parties. The moral Golden Rule includes the well-known win-win situation.

Van Dam [49] has discussed the compatibility of the Golden Rule and rationality. He concluded that the former and the latter are consistent in two player games, while this may not occur in games with more than two players. He formalized the Golden Rule in a symmetric normal form game, which is a special case of Definition 1.1. The drawbacks of the Golden Rule pointed out by Van

Dam are expressed in terms of BE in Example 1.3 (inconsistency with rationality) and Section 3 (existence).

§ 2.2. Capturing Cooperation and Mutual Support in Non-Cooperation Setting

Efforts to provide a rational justification for BE in game theory are made in [10, 12, 32, 38]. BE provides a compelling model of cooperation in social dilemmas, including the Prisoner's Dilemma and n-Player Prisoner's Dilemma games [10]. Unlike zero-sum games, normal form games require cooperation or coordination to reach acceptable solutions. For instance, when a game has several NEs, it is not always clear for all the players which NE will be played; some pre-play consultations must be conducted, otherwise, players may select strategies from different NEs or even from non-NE strategy profiles, and the game ends up in a strategy profile that is not an NE at all. Similarly, when a game has no NE and it is a one-shot game, it is not clear which strategy profile the players will select, and some form of cooperation must take place to avoid worse scenarios. Thus, normal form games are generally not 100 % non-cooperative. In fact, when players see that it is in their interest to cooperate, they do. This means that players may rationally resort to cooperation in non-cooperative settings. BE is an equilibrium concept that captures a form of cooperation within the framework of normal form games. The first hint at this fact appeared in [36]. It was further investigated and justified in the above-mentioned series of works.

The main principle put forward is mutual support among players. Courtois et al. [12] contend that in zero-sum games players play NE, that is, they do not cooperate; however, in non-zero-sum games players do not always play Nash equilibrium, and cooperation may take place depending on the context they are in and the society in which they live. They also adopt different behavior rules depending on which type of game they are playing. For instance, meta-analysis of experimental results shows that on average, about 50 % of subjects cooperate in the Prisoner's Dilemma game [9, 29, 40]. Similar anomalies related to Nash's predictions have been documented in studies of the Chicken game. The behavioral hypothesis put forward is that the choice in many interactive situations requires that each player make the welfare of the others a key feature of his or her reasoning. The cooperation among players takes the form of reciprocation or mutual support. The above-mentioned hypothesis is also supported by the Nobel Prize winner Sen [42] as quoted by Musy et al. [32]: "individuals who maximize their personal interests may adopt a mutual support behavior since they consider the goals of others in recognizing the nature of the mutual interdependence of the results achieved by each of them. Sen [41] adds that, by considering these interactions and adopting a different behavior, everyone eventually finds themselves in a better situation regarding their own objective; this modification of the behavior can then be justified."

Let us illustrate the mutual support principle using the Prisoner's Dilemma game of Example 1.1. In this game, the strategy profile (RP, RP) is a BE, while the strategy profile (CP, CP) is an NE. The outcome of the game depends on what type of behavior the players adopt. If the players adopt individualistic behavior, they end up in (CP, CP), with payoffs $(10, 10)$. However, if they adopt mutual support behavior, they will end up in (RP, RP), with payoffs $(20, 20)$, which are double of the payoffs yielded by the strategy profile (CP, CP). Thus, in (RP, RP) the players support each other for a better reward. This behavior is in line with Sen's statement above. The players reach a cooperative outcome in no-cooperative settings through mutual support.

§ 2.3. Interpretation Based on Altruism

The interpretation of BE based on altruism appeared in [10]. The authors state that players are invariably motivated to maximize their expected utilities in any situations in which they find themselves, but that these expected utilities are not necessarily individualistic — they may be altruistic, cooperative, competitive, or equality-seeking, depending on the psychological characteristics of the decision maker and the circumstances of the social interaction. Then they show that Berge equilibria arise in circumstances in which utility maximizing players are motivated by the altruistic social value orientation. Indeed, in BE each player selects a strategy that is individually rational and maximizes the payoff of the others, ignoring the maximization of his own payoff, which is a form of altruism. Thus, BE can be selected by players in a game when all of them are motivated by altruism. However, if only part of players is altruistic and the other part is individualistic, altruistic players

may lose. This is in line with the very nature of altruism as being altruistic is accepting or being ready to give without compensation, especially in the short term. Colman et al. [10] argued that BE may also arise because of coordination in situations of common interest games. The altruistic interpretation of BE is also developed in the framework of Bertrand Duopoly model in [6].

Let us illustrate the altruistic interpretation of BE by the Prisoner's Dilemma game (1.3). If both row and column players are motivated by altruism in their strategy selection processes, both would select the BE strategy RP. Indeed, if the row player selects the non BE strategy CP, the game would end up in one of the two outcomes: (CP, RP) with payoffs $(25, 5)$ if the column player selects the strategy RP or (CP, CP) with payoffs $(10, 10)$ if the column player selects CP as well. In the first outcome, the column player gets 5 units, the worst outcome in the game, and in the second outcome he/she gets only 10 units. In both cases the column player does not get the maximum payoff he can get in BE, 20 units. Consequently, being motivated by altruism, the row player must play the BE strategy RP. A similar reasoning shows that being motivated by altruism, the column player must play the BE strategy RP. Thus, when both players are motivated by altruism in the strategy selection process, they naturally and rationally converge to BE.

As mentioned above, if part of the players is not motivated by altruism, BE may not be selected. For instance, in the Prisoner's Dilemma game (1.3), if the row player, by altruism, adopts the BE strategy RP, while the other player, being individualistic, adopts the strategy CP, the payoffs are $(5, 25)$, that is the row player (altruist) receives the worst payoff in the game, 5 units, while the column player (the individualist) receives the highest payoff of the game, 25 units.

§ 2.4. Properties of BE

Besides the previous three different interpretations, BE has many properties that make it interesting and suitable in applications. The first is that it may exist in games where NE does not exist, hence, players can use it as a solution in such games. The second is that in many cases it yields a payoff that is better than in NE for all players (see Section 5 and the Prisoner's Dilemma game (1.3)). The third is that under reasonable conditions, if the set of BEs is non-empty, there exists a BE that is Pareto optimal. The properties of BEs are discussed in [27, 44, 45, 48]. However, finding and/or computing BE is more difficult than finding and/or computing NE as we will see in Sections 3 and 4.

§ 3. Existence and Determination of BE

In this section, we review some studies on the problem of existence and determination of BE. Establishing sufficient conditions for the existence of BE has received a great deal of attention in the literature, while works on constructing and developing numerical methods and procedures for its determination and computation are scarce. After the landmark thesis of Vaisman [45], Larbani and his student Fariza Krim carried out another landmark study of BE in [26] and [25]. They conducted an in-depth and comprehensive study of BE, including its properties, existence results using fixed point theorems, and methods of effective computation of BE in static non-linear and linear quadratic n-person games, as well as non-linear and linear-quadratic differential games. Only a small part of these results is published in [26]; the remaining part will be published soon in journals. We mention some of them below.

§ 3.1. The Problem of BE Existence

The problem of existence of BE is a challenging one as common sufficient conditions for the existence of NE, such as convexity and compactness of strategy sets, continuity of payoff functions and concavity of these functions with respect to strategies of players are not sufficient for the existence of BE. To illustrate the difficulty of BE existence problem, many related publications that appeared in international standard journals involve significant mistakes, as we will discuss below. BE existence has been investigated in most types of normal form games. We will present the existing results for each type of game. Basically, researchers use the same tools as those used for establishing the existence of NE, but with some significant changes or adjustments.

1. BE in Two Person Games. The first work on BE in two-person games appeared in [10, 19, 27]. The existence of BE in bimatrix games has been studied in simple cases involving two

strategies for each player, generally, for illustration and justification of the rationale behind BE. Without being exhaustive, we mention [10, 12, 32]. In the general case of two-person games, it has been proven that BE is just a NE of the game obtained from the initial game by permutation (or exchange) of the payoff functions of players [19]. Therefore, BE existence conditions can be established via NE's. However, in n-person games, it is not the case as we discuss below.

2. BE in n-Person Games. As mentioned above, establishing general sufficient existence conditions for BE is a difficult task. The first results on the existence of BE were established for differential games in [17,18] (stochastic case) and [39]. Later Dinovsky [15] established some existence results for BE in static games. As stated above, Vaisman [45] discovered that BE may not satisfy the individual rationality condition; therefore, he suggested adding this condition to Definition 1.1, which led to a new BE definition, Definition 1.3. Vaisman [44,45] established the existence of BVE in two-person static games and three-person differential games with linear-quadratic payoff functions. Vaisman's results on BE appear also in [51]. His doctoral thesis [45] is the first landmark in the study of BE. It is an in-depth study of BVE of three player static and differential linear-quadratic games. Existence conditions are formulated in terms of properties of the matrices and vectors involved in the payoffs.

In [57], the problem of determination of BE is transformed into the problem of finding a saddle point as follows. Consider the following functions:

$$\varphi_i(x,y) = f_i(x_i, y_{I\setminus i}) - f_i(x) \quad \text{for all} \quad (x,y) \in X \times X,\ i \in I,$$
$$\varphi(x,y) = \max_{i\in I} \varphi_i(x,y). \tag{3.1}$$

Next, the following two-person zero-sum game is associated with the initial game G:

$$G_2 = \langle I = \{1,2\}, \{X, Y = X\}, \varphi(x,y)\rangle.$$

A strategy profile $(x^0, y^0) \in X \times X$ is said to be a *saddle point* of the function $\varphi(x,y)$ (or Nash equilibrium of the game G_2), if it satisfies the following relationship:

$$\varphi(x^0, y) \leqslant \varphi(x^0, y^0) \leqslant \varphi(x, y^0) \text{ for all } (x,y) \in X \times X.$$

P r o p o s i t i o n 3.1 (see [57]). *If $(x^0, y^0) \in X \times X$ is a saddle point of the function $\varphi(x,y)$, then x^0 is a BE of the initial game G.*

P r o o f. Indeed, if $(x^0, y^0) \in X \times X$ is a saddle point of the function $\varphi(x,y)$, then

$$\varphi(x^0, y) \leqslant \varphi(x^0, y^0) \leqslant \varphi(y^0, y^0) = 0 \quad \text{for all} \quad y \in X.$$

Hence,

$$\varphi(x^0, y) \leqslant 0 \quad \text{for all} \quad y \in X,$$

which means

$$\varphi_i(x^0, y) \leqslant \varphi(x^0, y^0) \leqslant \varphi(y^0, y^0) = 0 \quad \text{for all } y \in X \text{ and } i \in I.$$

This implies

$$f_i(x_i^0, y_{I\setminus i}) \leqslant f_i(x^0) \quad \text{for all} \quad y_{I\setminus i} \in X_{I\setminus i},\ i \in I.$$

That is, x^0 is a BE of the game G. □

Thus, the problem of determining BE of the game G is transformed into the problem of determining a saddle point of the function $\varphi(x,y)$. In [57], it has been established that the set of BEs of the game G may be *internally unstable* in the sense that a given BE may be dominated by another BE. Let X^B be the set of all BEs of G. Let x^{B_1} and x^{B_2} be in X^B. We say that x^{B_1} dominates x^{B_2} if

$$f_i(x^{B_2}) < f_i(x^{B_1}), \quad i = 1, \ldots, n.$$

The following example illustrates the internal instability of X^B [57].

E x a m p l e 3.1. Consider the game G where $I = \{1, 2\}$, $X_1 = X_2 = [-1, 1]$ and

$$f_1(x_1, x_2) = -x_2^2 + 2x_1x_2, \quad f_2(x_1, x_2) = -x_1^2 + 2x_1x_2.$$

One can easily verify that $X^B = \{x^{B_\beta} = (\beta, \beta), \beta \in [-1, 1]\}$. Thus, the set of all BEs is infinite. Let us now consider two particular BEs, namely, $(0, 0)$ and $(1, 1)$ obtained for $\beta = 0$ and $\beta = 1$, respectively. Then the payoffs are $f_1(0, 0) = 0$, $f_2(0, 0) = 0$ for the BE $(0, 0)$ and $f_1(1, 1) = 1$, $f_2(1, 1) = 1$ for the BE $(1, 1)$. Clearly, $f_i(0, 0) < f_i(1, 1), i = 1, 2$, that is, the BE $(1, 1)$ dominates the BE $(0, 0)$. Thus, when there are multiple BEs, it is advised to select a non-dominated BE from the set of all BEs. Such BE can be determined using an extension of the saddle point approach presented above and the notion of Pareto optimality as follows from [57].

P r o p o s i t i o n 3.2. *The set X^B of all BEs of the game G is compact if:*
(1) *the payoff functions $f_i(x)$, $i = 1, \dots, n$, are continuous over X;*
(2) *the strategy sets X_i, $i = 1, \dots, n$, are compact.*

Let us now recall the definition of Pareto optimality. A BE x^B of the game G is said to be *Pareto optimal* (*non-dominated*) with respect to X^B if for all $x \in X^B$ the system of inequalities

$$f_i(x^B) \leqslant f_i(x), \quad i = 1, \dots, n,$$

with at least one strict inequality is impossible. It is well-known that if the maximum $\max\limits_{x \in X^B} \sum\limits_{i \in I} f_i(x)$ is reached at some BE x^B, then x^B is Pareto optimal with respect to X^B. Using this result and Proposition 3.2, a sufficient condition for the existence of a Pareto optimal BE (with respect to X^B) can be established. First, along with the functions (3.1), consider the following functions:

$$\varphi_{n+1}(x, y) = \sum_{i \in I} f_i(x) - \sum_{i \in I} f_i(y) \text{ for all } (x, y) \in X \times X,$$
$$\phi(x, y) = \max_{i = 1, \dots, n+1} \varphi_i(x, y) \text{ for all } (x, y) \in X \times X.$$

P r o p o s i t i o n 3.3. *If $(x^0, y^0) \in X \times X$ is a saddle point of the function, then x^0 is a Pareto optimal BE of the game G.*

The drawback of the saddle point approach is that the functions $\varphi(x, y)$ and $\phi(x, y)$ are generally not differentiable, which makes it difficult to use the well-known numerical methods of saddle point determination that use derivatives.

For *games with linear-quadratic payoff functions,* explicit forms of BE were obtained in Vaisman [45] and Belskikh et al. [6]. As an illustration, we consider the game G where $I = \{1, 2\}, X_i = \mathbb{R}^{n_i}$ and

$$f_1(x_1, x_2) = x_1'A_1x_1 + 2x_1'B_1x_2 + x_2'C_1x_2 + 2a_1'x_1 + 2c_1'x_2,$$
$$f_2(x_1, x_2) = x_1'A_2x_1 + 2x_1'B_2x_2 + x_2'C_2x_2 + 2a_2'x_1 + 2c_2'x_2,$$

where the apostrophe means the transposition operation, A_i is an $n_1 \times n_1$ square matrix, C_i is an $n_2 \times n_2$ square matrix, B_i is an $n_1 \times n_2$ matrix, a_i is an n_1-vector and c_i is an n_2-vector for $i = 1, 2$. The notation $A > 0$ ($A < 0$) means that the quadratic form $x'Ax$ is positive (negative) definite. Let us introduce the following determinant related conditions:

$$\det[C_1 - B_1'A_2^{-1}B_2] \neq 0, \tag{3.2}$$
$$\det[A_2 - B_2'C_1^{-1}B_1'] \neq 0, \tag{3.3}$$
$$\det[C_2 - B_2'A_1^{-1}B_1] \neq 0, \tag{3.4}$$
$$\det[A_1 - B_1C_2^{-1}B_2'] \neq 0. \tag{3.5}$$

Then we have the following table for the existence of BE and NE [6].

Table 1. Explicit Form of BE in Two-Person Linear-Quadratic Games

Only one of the Equilibria exists					BE	NE	
$A_1 > 0$	$A_2 > 0$	$C_1 < 0$		(3.2)	Yes	No	$\forall C_2, B_i, a_i, c_i$
	$A_2 < 0$	$C_1 < 0$	$C_2 > 0$	(3.3)	Yes	No	$\forall A_1, B_i, a_i, c_i$
$A_1 < 0$	$A_2 > 0$		$C_2 < 0$	(3.4)	No	Yes	$\forall C_1, B_i, a_i, c_i$
$A_1 < 0$		$C_1 > 0$	$C_2 < 0$	(3.5)	No	Yes	$\forall A_2, B_i, a_i, c_i$
Neither BE nor NE Exist							
$A_1 > 0$	$A_2 > 0$				No	No	$\forall B_i, C_i, a_i, c_i$
$A_1 > 0$		$C_1 > 0$			No	No	$\forall A_2, C_2, B_i, a_i, c_i$
	$A_2 > 0$		$C_2 > 0$		No	No	$\forall A_1, C_1, B_i, a_i, c_i$
		$C_1 > 0$	$C_2 > 0$		No	No	$\forall A_1, A_2, B_i, a_i, c_i$
Both BE and NE Exist							
$A_1 < 0$	$A_2 < 0$	$C_1 < 0$	$C_2 < 0$	(3.2) and (3.4)	Yes	Yes	$\forall B_i, a_i, c_i$
$A_1 < 0$	$A_2 < 0$	$C_1 < 0$	$C_2 < 0$	(3.3) and (3.5)	Yes	Yes	$\forall B_i, a_i, c_i$

Moreover, when BE $x^B = (x_1^B, x_2^B)$ exists, its explicit form can be of two types.

a) If (3.2) is true, then

$$x_1^B = -A_2^{-1}B_2[C_1 - B_1'A_2^{-1}B_2]^{-1}(B_1'A_2^{-1}a_2 - c_1) - A_2^{-1}a_2,$$
$$x_2^B = [C_1 - B_1'A_2^{-1}B_2]^{-1}(B_1'A_2^{-1}a_2 - c_1).$$

b) If (3.3) is true, then

$$x_1^B = [A_2 - B_2C_1^{-1}B_1']^{-1}(B_2C_1^{-1}c_1 - a_2),$$
$$x_2^B = -C_1^{-1}B_1'[A_2 - B_2C_1^{-1}B_1']^{-1}(B_2C_1^{-1}c_2 - a_1) - C_1^{-1}c_1.$$

When NE $x^e = (x_1^e, x_2^e)$ exists, its explicit form can be of two types.

c) If (3.4) is true, then

$$x_1^e = -A_1^{-1}B_1[C_2 - B_2'A_1^{-1}B_1]^{-1}(B_2'A_1^{-1}a_1 - c_2) - A_1^{-1}a_1,$$
$$x_2^e = [C_2 - B_2'A_1^{-1}B_1]^{-1}(B_2'A_1^{-1}a_1 - c_2).$$

d) If (3.5) is true, then

$$x_1^e = [A_1 - B_1C_2^{-1}B_2']^{-1}(B_1C_2^{-1}c_2 - a_1),$$
$$x_2^e = -C_2^{-1}B_2'[A_1 - B_1C_2^{-1}B_2']^{-1}(B_1C_2^{-1}c_2 - a_1) - C_2^{-1}c_2.$$

In [25] the following more general n-person games with linear-quadratic payoff functions are studied:

$$G_{LQ} = \langle I, \{X_i\}_{i\in I}, \{f_i(x) = x'A_ix + b_ix\}_{i\in I}\rangle,$$

where $x = (x_1, \ldots, x_n)$, $x_i \in X_i \subset \mathbb{R}^{n_i}$, the matrices A_i and the vectors b_i, $i = 1, \ldots, n$, are of corresponding dimensions. Explicit forms of BE are given using a matrix partitioning approach. Both the constrained and unconstrained strategy sets cases are investigated.

General sufficient conditions for BE existence in n-person static games were established in [26] using the Fan minimax inequality and the Kakutani [23] fixed point theorem. Let us briefly present the two approaches. Consider the following real-valued function:

$$H(x, \widehat{y}) = \sum_{i\in I}\big[f_i(x_i, \widehat{y}_{I\setminus i}) - f_i(x)\big], \tag{3.6}$$

where $x \in X$ and $\widehat{y} \in \prod\limits_{i\in I} X_{I\setminus i}$. We have the following sufficient condition for the existence of BE.

L e m m a 3.1. *If a strategy profile $\overline{x} \in X$ satisfies the inequality*

$$H(\overline{x}, \widehat{y}) \leqslant 0 \quad \textit{for all} \quad \widehat{y} \in \widehat{X},$$

then it is a BE of the game G.

P r o o f. Assume $\overline{x} \in X$ satisfies the inequality of Lemma 3.1. Consider a player $i \in I$. Then for each $j \in I$ such that $j \neq i$ let $\widehat{y}_{I\setminus i} = \overline{x}_{I\setminus i}$, and leave $\widehat{y}_{I\setminus i}$ free in $X_{I\setminus i}$ in the inequality of Lemma 3.1 and (3.6), then we get

$$H(\overline{x}, \widehat{y}) = f_i(\overline{x}_i, \widehat{y}_{I\setminus i}) - f_i(\overline{x}) + \sum_{j \in I\setminus i} \left[f_j(\overline{x}_j, \overline{x}_{I\setminus i}) - f_j(\overline{x}) \right] \leqslant 0 \quad \text{for all} \quad \widehat{y}_{I\setminus i} \in X_{I\setminus i}.$$

In the last inequality, all the terms equal zero except for the i-th term, then

$$f_i(\overline{x}_i, \widehat{y}_{I\setminus i}) - f_i(\overline{x}) \leqslant 0 \quad \text{for all} \quad \widehat{y}_{I\setminus i} \in X_{I\setminus i}.$$

Since i is arbitrarily selected, $\overline{x}$ is a BE according to (1.1). □

Then we have the following existence theorem.

T h e o r e m 3.1. *Assume the following conditions are satisfied:*
(1) *the strategy sets X_i, $i \in I$, are nonempty, compact and convex;*
(2) *the function $y_{I\setminus i} \to f_i(x_i, y_{I\setminus i})$ is concave for all $x_i \in X_i$ and $i \in I$;*
(3) *for all $x \in X$ there exists $z \in X$ such that*

$$f_i(x_i, t_{I\setminus i}) \leqslant f_i(x_i, z_{I\setminus i}) \quad \textit{for all} \quad t_{I\setminus i} \in X_{I\setminus i},\ i \in I.$$

Then the game G has a BE.

The proof of Theorem 3.1 is based on Lemma 3.1 and the Fan minimax inequality.

The approach based on the function (3.6) is more practical that the approach using the functions in (3.1) as finding a saddle point is generally more difficult than solving an inequality.

Moreover, the functions in (3.1) involve the "max" operator, which makes it difficult to use numerical methods to find BE, as functions involving this operator are generally not differentiable. In Section 3.2, we present a method for computing BE based on the function (3.6), Lemma 3.1 and Theorem 3.1.

Another approach to establishing the existence of BE is to use fixed point theorems. This approach was first developed in [27] as follows. First the following correspondence is constructed. Let $x \in X$ define the correspondences

$$F_i(x) = \left\{ z \in X \;\middle|\; f_i(x_i, z_{I\setminus i}) = \max_{y_{I\setminus i} \in X_{I\setminus i}} f_i(x_i, y_{I\setminus i}) \right\} \text{ for all } x \in X,\ i \in I,$$

$$F(x) = \bigcap_{i \in I} F_i(x) \text{ for all } x \in X.$$

L e m m a 3.2. *If a strategy profile $\overline{x} \in X$ is a fixed point of the correspondence $x \to F(x)$, that is, $\overline{x} \in F(\overline{x})$, then $\overline{x}$ is a BE of the game G.*

The existence of BE is established under the conditions of Theorem 3.1.

Another approach for BE existence was developed in Krim [25] based on Brouwer's fixed point theorem. This approach is more practical than the correspondence fixed point approach as there are effective numerical methods for finding fixed points of ordinary functions.

In a series of papers Abalo and Kostreva [1–5] published BE existence results for games with an infinite number of players, abstract strategy spaces and weak compactness and continuity conditions using the following condition.

The set

$$\operatorname{Arg}\max(x_i) = \left\{ z_{I\setminus i} \in X_{I\setminus i} \mid f_i(x_i, z_{I\setminus i}) = \max_{y_{I\setminus i}\in X_{I\setminus i}} f_i(x_i, y_{I\setminus i}) \right\} \tag{3.7}$$

is a *singleton* for each $i \in I$ and $x_i \in X_i$.

In [28] and [36], it has been established that condition (3.7) is not sufficient to prove the existence of BE; therefore, all the existence theorems stated in the Abalo and Kostreva series of papers mentioned above are invalid, and a correction of these theorems is proposed. It is important to know that a condition similar to (3.7) works well for the existence of NE, but (3.7) does not work for BE. The same conclusion can be drawn on existence theorems of Radjef [39] as the condition (3.7) was first used in this work.

To compare the difficulty of establishing the existence of BE with that of NE, a relationship has been found between BE and NE. This relationship was first discovered in [27]; later it appeared in [10] as Lemma 3.2, Theorem 3.1 and Corollary 1. Let us explain briefly this relationship. Consider the game G. First, define the special set of permutations of the set of players $I = \{1, \ldots, n\}$,

$$\Sigma = \{\sigma(\cdot) \mid \sigma(\cdot) \text{ is a permutation of the set } I \text{ such that } \sigma(i) \neq i, \text{ for all } i \in I\}$$

It is the set of all permutations $\sigma(\cdot)$ of the set of players I such that for each player i, $\sigma(i) \neq i$ for all $i \in I$. Σ is called the set of deranged permutations of $I = \{1, \ldots, n\}$. Now consider the set of games

$$G_{\sigma(\cdot)} = \langle I, \{X_i\}_{i\in I}, \{f_{\sigma(\cdot)}(x)\}_{i\in I}\rangle \text{ for all } \sigma(\cdot) \in \Sigma. \tag{3.8}$$

For each $\sigma(\cdot) \in \Sigma$, the game $G_{\sigma(\cdot)}$ is obtained from the game G by permutation of the payoff functions of players according to $\sigma(\cdot)$. As $\sigma(i) \neq i$, each player is assigned a new payoff function different from his initial payoff function, but he/she keeps his strategy set.

Then the following relation between a BE of the game G and NE of the games $G_{\sigma(\cdot)}, \sigma(\cdot) \in \Sigma$ in (3.8) has been established. Each BE of the game G is at the same time a NE of *all the games* in (3.8), the number of which is equal to the cardinality of Σ, $\text{Card}\{\Sigma\} = n! \sum_{s=0}^{n} \frac{(-1)^s}{s!}$. For instance, for a game G with $n = 4$ players, $\text{Card}\{\Sigma\} = 9$, there are 9 games of type (3.8) and for game G with $n = 5$ players, $\text{Card}\{\Sigma\} = 44$, there are 44 games of type (3.8). Formally, we have the following implication between BE and NE

$$x \in X \text{ is a BE of } G \Rightarrow x \text{ is a NE of all } G_{\sigma(\cdot)}, \sigma \in \Sigma.$$

In other words, denoting by $\text{BE}(G)$ the sets of BEs of the initial game G and by $\text{NE}(G_{\sigma(\cdot)})$ the set of NEs of the game $G_{\sigma(\cdot)}, \sigma(\cdot) \in \Sigma$, the previous implication can be written as

$$BE(G) \subset \prod_{\sigma(\cdot)\in\Sigma} NE(G_{\sigma(\cdot)}).$$

This implication gives an idea about the relative difficulty of finding BE. It also implies that if one of the games has no NE, the initial game G has no BE. Later, Pottier and Nessah [38] investigated further this relationship between BE and NE.

Another interesting relationship between BE and NE has been established in [27] as follows. Consider the n two-person games

$$G_{(i)} = \langle I = \{i, I \setminus i\}, \{X_i, X_{I\setminus i}\}, \{f_i(x), \sum_{s\in I\setminus i} f_s(x)\}\rangle \ (i = 1, \ldots, n).$$

That is, each $G_{(i)}$ is a game between player i with his/her initial set of strategies X_i and payoff function $f_i(x)$ against the coalition of the rest of players $I \setminus i$ with the strategy set $X_{I\setminus i}$ and the

payoff function $\sum_{s\in I\setminus i} f_s(x)$ consisting of the sum of the payoff functions of all players in $I\setminus i$. Then it is proved that

$$\overline{x}\in X \text{ is a BE } \Rightarrow \overline{x} \text{ is NE for all games } G_{(i)}, i\in I.$$

In other words, denoting by $\mathrm{NE}(G_{(i)})$ the set of NEs of the game $G_{(i)}$, $i\in I$, the previous equivalence can be written as

$$BE(G)=\bigcap_{i\in I} NE(G_{(i)}).$$

Later, this relationship and the result appeared in [10] as Theorem 3.

Using the g-maximum inequality in [35], which is a generalization of the Fan minimax inequality, some sufficient conditions of BVE existence like Theorem 1 are established in [37]. Recently, Deghdak [13] have established sufficient existence results of BE when payoff functions of players are pseudo-continuous using the correspondence $x\to F(x)$ of Lemma 2, without referring to [26] or [25]. Moreover, it is claimed that Deghdak [13] generalizes the existence results in [37] and [36], while they generalize the results in [26] only, as in the previous works the Abalo and Kostreva approach developed in the above-mentioned series of papers is discussed and existence results are based on condition (3.6) not on the correspondence $x\to F(x)$ of Lemma 2. In [32], the following result is presented as a theorem. Consider a player $i\in I$ and his strategy $x_i\in X_i$, one can define the best support correspondence of players in for player $I\setminus i$ by

$$BS_i(x_i)=\left\{y_{I\setminus i}\in X_{I\setminus i}\mid f_i(x_i,y_{I\setminus i})=\max_{z_{I\setminus i}\in Y_{I\setminus i}} f_i(x_i,z_{I\setminus i})\right\},$$

which is like the correspondence $x\to F(x)$ of Lemma 2. The graph $GR(BS_(\cdot))$ of this correspondence is a subset of the strategy profile set $X_i\times X_{I\setminus}=X$ of the game G. We get the relation

$$x\in GR(BS_i(\cdot))\Leftrightarrow x_{I\setminus}\in BS_i(x_i).$$

The set X^B of all BEs of the game G is characterized by the equality

$$X^B=\bigcap_{i\in I} GR(BS_i(\cdot)). \tag{3.9}$$

In fact, this result is just an equivalent formulation of Definition 1.1 in terms of correspondences. Indeed, Condition (1.1) of Definition 1.1 can be reformulated as follows:

$$\overline{x}\in X \text{ is a BE } \Leftrightarrow \overline{x}_{I\setminus i}\in BS_i(\overline{x}_i) \text{ for all } i\in I.$$

Therefore, it seems that stating that this result is a theorem is an overstatement; a proposition or a lemma may be more appropriate, as it is similar to Lemma 3.2. Similarly, in the problem of NE existence, the best response correspondence is used as a preliminary result in the proof of the existence of Nash equilibrium by Kakutani [23] correspondence fixed point theorem [34]. In contrast, a result related to the difficult problem of finding sufficient conditions for the non-emptiness of the intersection in (3.9) in terms of properties of the strategy sets X_i and payoff functions $f_i(x)$, $i\in I$, for the existence and computation of BE could be stated as a theorem.

The existence of *mixed strategy* BE in an infinite game has been established in [57]. A game in mixed strategies is associated to the initial game G in pure strategies as follows. Assume that the strategy sets X_i, $i\in I$, are compact and the payoff functions $f_i(\cdot)$, $i\in I$, are continuous, in the game G. To each player $i\in I$ a set of mixed strategies ν_i is associated. A mixed strategy ν_i of the player i is a countable-additive, non-negative and normed on $[0,1]$ function with domain the set of Borel σ-algebra of subsets of the compact strategy set X_i. In other words, ν_i is a probability measure on his/her pure strategy set X_i The strategy profile set $\{\nu(\cdot)\}$ consists of probability measures on the

set $X = \prod_{i\in I} X_i$, of the form $\nu(\cdot) = (\nu_1(\cdot), \dots, \nu_n(\cdot))$, where $\nu_i(\cdot) \in \{\nu_i\}$, $i \in I$. Thanks to Fubini theorem, for any continuous real-valued function $g(x)$, with domain X, we have

$$g[\nu] = \int_X g(x)\nu(dx) = \int_{X_1} \dots \int_{X_n} g(x)\nu_n(dx_n)\dots\nu_1(dx_1) \text{ for all } \nu \in \{\nu(\cdot)\},$$

where the order of the integrals can be changed. Particularly, the expected value of player $i \in I$ is given by

$$f_i[\nu] = \int_X f_i(x)\nu(dx) = \int_{X_1} \dots \int_{X_n} f_i(x)\nu_n(dx_n)\dots\nu_1(dx_1) \text{ for all } \nu \in \{\nu(\cdot)\}.$$

To the initial pure strategy game G, the following *mixed strategy game* is associated

$$\widetilde{G} = \langle I, \{\nu\}, \{f_i[\nu]\}_{i\in I}\rangle.$$

A mixed strategy profile $\nu^*(\cdot) \in \{\nu\}$ is said to be a BE of the game $\widetilde{G}$ if

$$\max_{\nu_{I\setminus i}(\cdot)} f_i[\nu_i^*, \nu_{I\setminus i}] = f_i[\nu^*] \text{ for all } i \in I,$$

where $\{\nu_{I\setminus i}\} = \{(\nu_1, \dots, \nu_{i-1}, \nu_{i+1}, \dots, \nu_n)\}$. Now consider the functions defined in (3.1), the following zero-sum two-person game is introduced

$$\widetilde{G}_2 = \langle I = \{1, 2\}, \{\nu\}, \{u\}, \varphi[\nu, u]\rangle,$$

where $\varphi[\nu, u] = \int_{X\times X} \varphi(x, y)\,\nu(dx)u(dy)$. A strategy profile $(\nu^0, u^0) \in \{\nu\} \times \{u\}$ is said to be a saddle point of the function $\varphi[\nu, u]$ (or NE of the game $\widetilde{G}_2$) if it satisfies the relationship

$$\varphi[\nu^0, u] \leqslant \varphi[\nu^0, u^0] \leqslant \varphi[\nu, u^0] \text{ for all } (\nu, u) \in \{\nu\} \times \{u\}.$$

P r o p o s i t i o n 3.4. *If $(\nu^0, u^0) \in \{\nu\} \times \{u\}$ is a saddle point of the function $\varphi[\nu, u]$ (or NE of the game $\widetilde{G}_2$), then ν^0 is a mixed strategy BE of the game $\widetilde{G}$.*

Thus, the existence of BE is a consequence of the Glicksberg [20] mixed strategies NE existence theorem of a zero-sum two-person game.

Let $\widetilde{X}^B$ be the set of mixed strategy BE of the game $\widetilde{G}$. Following the case of pure strategy BE (Propositions 3.1 and 3.2), a theorem of existence of a Pareto optimal (with respect to $\widetilde{X}^B$ and $\{f_i[\nu]\}_{i\in I}\}$) mixed strategy BE is proved in [57].

The investigation of the problem of existence of BE in *games involving uncertainty* in payoff functions has been initiated in [47]. Almost all non-trivial decision problems in all human activities involve uncertainty. The quality of our decisions depends substantially on how we deal with unknowns. In games, generally, uncertainty appears in the following different forms:

(1) uncertainty can appear as actions of persons or entities having their goals, but are not players in the game, e.g., a government;

(2) uncertainty can reflect fuzzy knowledge of the players have of their own objectives or strategies;

(3) uncertainty can appear when processes or quantities are not sufficiently studied or identified;

(4) uncertainty may arise in the process of collecting, processing and transmitting information.

Therefore, a great deal of research efforts is devoted to the construction of decision models that incorporate uncertainty.

BE in games involving *unknown parameters* in the payoff functions (in the form $f_i(x, y)$, $i = 1, \dots, n$, where $y \in Y$ is the unknown parameter) has been investigated for the first time in [45, 47, 48, 55, 62–65, 69, 71].

At the substantive level, the presence of uncertainty requires basically using BE and at the same time considering any possible realization of the uncertainty to formalize concepts of solution of the

following extension of the initial deterministic game G to normal form games under uncertainty of the form

$$G_U = \langle I, \{X_i\}_{i\in I}, Y, \{f_i(x,y)\}_{i\in I}\rangle,$$

where $y = (y_1, \ldots, y_p) \in Y$ is an unknown vector-parameter with range in $Y \subset \mathbb{R}^p$ and the remaining parts are defined as in the initial game G. The parameter y affects all the players' payoffs and only its range is known to them; no information is available about its behavior. The game G_U was introduced in [47] and further studied for NE in [59,60].

Two ways for dealing with normal form games involving uncertainty of type G_U are introduced: the first one is based on the concept of saddle point analysis (*balanced* BE) [59], while the second one is based on maximin principle analysis (*guaranteed* BE) [60]. Here the basic idea is simple.

The following two problems are derived from the game G_U: the game

$$\Gamma_1 = \langle I, \{X_i\}_{i\in I}, \{f_i(x, y^S)\}_{i\in I}\rangle,$$

where y^S is fixed and the multi-criteria problem

$$\Gamma_2 = \langle Y, \{f_i(x^B, y)\}_{i\in I}\rangle,$$

where x^B is fixed. Next, find a BE x^B in the game Γ_1 and a Slater minimum (Pareto weak) y^S with respect to the parameter value y of the problem Γ_2. A balanced BE of the game is defined as the pair $(x^B, \{f_i(x^B, y^S)\}_{i\in I})$, where the players select their strategy from the BE x^B and their guaranteed payoffs is $f(x^B, y^S) = (f_1(x^B, y^S), \ldots, f_n(x^B, y^S))$, and the system of inequalities

$$f_i(x^B, y) < f_i(x^B, y^S), \quad i = 1, \ldots, n,$$

is impossible for all $y \in Y$. Formally, we have the following definition.

D e f i n i t i o n 3.1. A pair $(\overline{x}^B, \overline{f}^S) \in X \times \mathbb{R}^n$ (is said to be a *Slater guaranteed balanced BE* of the game G_U if there is an uncertainty value $\overline{y}^S \in Y$ such that:

(1) $\overline{x}^B$ is a BE of the deterministic game

$$\langle I, \{X_i\}_{i\in I}, \{f_i(x, \overline{y}^S)\}_{i\in I}\rangle,$$

that is,

$$\text{for all } i \in I \max_{x_{I\setminus i}\in X_{I\setminus i}} f_i(\overline{x}_i^B, x_{I\setminus i}, \overline{y}^S) = f_i(\overline{x}^B, \overline{y}^S);$$

(2) the uncertainty value $\overline{y}^S$ is a Slater minimal (Pareto weak optimal) solution of the minimization multiple criteria problem

$$\langle Y, \{f_i(\overline{x}^B, y)\}_{i\in I}\rangle,$$

that is, there is no uncertainty value $y \in Y$ such that the system of inequalities

$$f_i(\overline{x}^B, y) < f_i(\overline{x}^B, \overline{y}^S) \quad \forall\, i \in I$$

is satisfied;

(3) denote by $\{\overline{x}^B, y^S\}$ the set of pairs of BE and corresponding uncertainty value that satisfy conditions 1 and 2. Then there is no pair $(x, y) \in \{\overline{x}^B, \overline{y}^S\}$ such that the system of inequalities

$$\overline{f}_i^S = f_i(\overline{x}^B, \overline{y}^S) < f_i(x, y) \quad \forall\, i \in I$$

is true.

Then $\overline{x}^B$ is called the *Slater guaranteeing strategy profile* and the value $\overline{f}^S$ is called the *guaranteed payoff vector*.

We have the following Slater guaranteed balanced BE existence theorem in two-person games when payoff functions are separated with respect to x and y in the game G_U, that is,

$$G_{US} = \langle I = 1, 2, \{X_i\}_{i\in I}, \{g_i(x) + h_i(y)\}_{i\in I}\rangle.$$

We have the following BE existence theorem.

T h e o r e m 3.2. *Assume the following conditions are satisfied in the game G_{US}:*
(1) *X_i and Y are compact, and X_i is convex for all $i \in I$;*
(2) *real-valued functions g_i, h_i are continuous on X and Y, respectively, for all $i \in I$;*
(3) *the function $g_i(x)$ is strictly concave with respect to x_j $(i, j = 1, 2, j \neq i)$, the other variable being constant $(i \in \{1, 2\})$.*
Then there exists a Slater guaranteed balanced BE in the game G_{US}.

By analogy with the maximin, in the value $\max\limits_{x\in X}\min\limits_{y\in Y} f(x, y)$, the operator min is used in $\min\limits_{y\in Y} f_i(x, y) = f_i[x]$ and the max operator is devoted to the construction of BE in the following deterministic game:

$$\Gamma_3 = \langle I, \{X_i\}_{i\in I}, \{f_i[x]\}_{i\in I}\rangle.$$

Here one needs to recall some important result of operations research.
1. If $f_i(x, y)$ is continuous over $X \times Y$, then $f_i[x]$ is continuous over X.
2. If, in addition to the continuity of $f_i(x, y)$ and the compactness of X, the set Y is convex and $f_i(x, y)$ is strictly convex with respect to y for each $x \in X$, then the vector function $x \to y^{(i)}(x)$ defined by

$$\min_{y\in Y} f_i(x, y) = f_i(x, y^{(i)}(x)) = f_i[x] \text{ for all } x \in X,$$

is well-defined and continuous with respect to x.

Consider an extension of the game G_U to games in mixed strategies and with uncertainty of the following form:

$$\widetilde{G}_U = \langle I, \{\nu\}, \{\mu\}, \{f_i[\nu, \mu]\}_{i\in I}\rangle,$$

where $\{\nu\}, \{\mu\}, \{f_i[\nu, \mu]\}_{i\in I}$ are defined as in the game $\overline{G}$ and $\{\mu\}$ represents the set of probability distributions on the set Y. Definition 3.1 has been extended to the case of mixed strategies for the game $\overline{G}_{ij}$. We have the following existence theorem of Slater guaranteed mixed strategy BE of the game $\widetilde{G}_{ij}$ [58].

T h e o r e m 3.3. *Assume that the sets X_i and Y are compact and the function $f_i(x, y)$ is continuous on $X \times Y$, for all $i \in I$. Then the game $\widetilde{G}_U$ has a Slater guaranteed mixed strategy BE.*

Let us present an analog of maximin. For this purpose, consider the game

$$G_S = \langle I, \{X_i\}_{i\in I}, Y^X, \{f_i(x, y)\}_{i\in I}\rangle,$$

where $I, \{X_i\}_{i\in I}, \{f_i(x, y)\}_{i\in I}$ are defined as in the game $\widetilde{G}_U$ and Y^X is the set of m-vector functions $x \to y^{(i)}(x)$ with domain X and range Y, which are called uncertainties (informational strategies) in the game G_S, and $f_i(x, y) = f_i(x, y(x))$ is the payoff function of player $i \in I$. One shot of the game G_S takes place as follows. The players simultaneously select their strategies $x_i \in X_i$, $i \in I$. Thus, a strategy profile $x = (x_1, \ldots, x_n) \in X = X_1 \times \ldots \times X_n$ is obtained. Informational discrimination of the players and additional informational uncertainty are proposed, as in a hierarchical game.

The first move from the players: they select their strategies $x_i \in X_i$, $i \in I$, and inform the decision maker (DM), the player responsible for selecting or constructing the uncertainty function. In the second move, the DM selects or constructs the n uncertainties in the form of m-vector functions, $y^{(i)}(x)$, $i \in I$, and informs all the n players. And it is assumed that uncertainty is constructed in

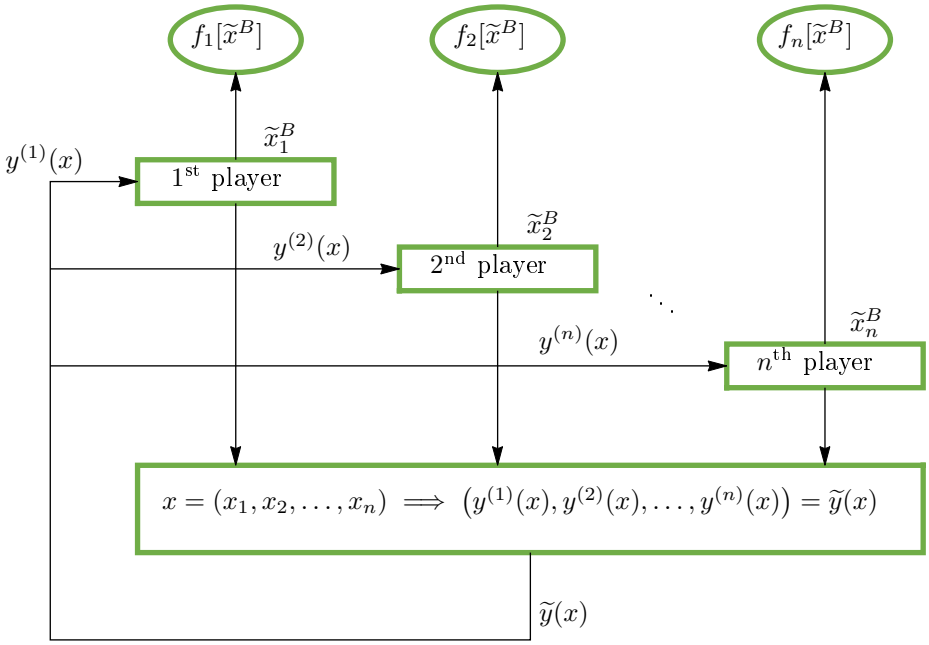

Figure 1. The way the game G_S takes place

such a way as to reduce maximally the payoff of each player individually. Using this information, the players select a BE $x^B \in X$.

Following this way of playing the game G_S, the players select such a "good" BE $\overline{x}^B$ from the set of all BEs x^B using the Slater (Pareto weak) maximization. By the way, as we have seen in Example 3.1, the set of all BEs X^B is not internally stable: one may find two BE $x^{(1)}, x^{(2)}$ such that one dominates the other, for instance,

$$f_i(x^{(1)}, y(x^{(1)}) > f_i(x^{(2)}, y(x^{(2)}) \quad \text{for all} \quad i \in I.$$

To eliminate this drawback, the Slater maximization is used to select non-dominated BE $\overline{x}^B$. Such a hierarchical decision making procedure is explained in the Figure 1.

The strictly guaranteed Berge equilibrium is constructed in three steps.

Step 1. To each player $i \in I$ associate a unique continuous vector-function $x \to y^{(i)}(x)$ over X such that

$$\min_{y \in Y} f_i(x, y) = f_i(x, y^{(i)}(x)) = f_i[x], \quad x \in X.$$

Step 2. To the game G_U associate the following deterministic (without uncertainty) normal form game Γ_3, called *Guarantee Game*. Next, find BE $x^B \in X$ of the game Γ_3; recall that BE is characterized by the following relation:

$$\max_{x_{I\setminus i} \in X_{I\setminus i}} f_i[x_i^B, x_{I\setminus i}] = f_i[x^B], \quad i \in I.$$

Step 3. From the set of all BEs x^B of the game Γ_3 select the maximal (in the vector sense) BE $\overline{x}^B$, that is, find the Slater maximum (Pareto weak) strategy profile $\overline{x}^B$ in the n-criteria optimization problem

$$\langle X^B, \{f_i[x]\}_{i \in I} \rangle.$$

In the case of Slater maximum, it is sufficient to determine $\overline{x}^B$ as follows:

$$\max_{x \in X^B} \sum_{i \in I} \alpha_i f_i[x] = \sum_{i \in I} \alpha_i f_i[\overline{x}^B],$$

where α_i, $i \in I$, are such that $\alpha_i \geqslant 0$, $i \in I$, and $\sum_{i \in I} \alpha_i > 0$. We have the following theorem.

T h e o r e m 3.4. *Assume that the following conditions are satisfied in the game G_{ij}:*

(1) *the strategy sets $\{X_i\}_{i\in I}$ are compact and the unknown parameter set Y is compact and convex;*

(2) *payoff functions $\{f_i(x,y)\}_{i\in I}$ are continuous over $X\times Y$ and strictly convex with respect to the parameter y over Y, for each $x\in X$.*

Then the game $\widetilde{G}_{ij}$ has a strong guaranteed mixed strategy BE.

Similar results can be found in [55], where the existence of BE is established for such games in mixed strategies.

At an abstract level, BE has been analyzed using players' preferences and lattice methods. Mashchenko [31] has investigated BE in games where players' payoff functions are not available. He used preferences of players to characterize BE. Keskin and Saglam [24] analyzed the existence of BE by lattice theoretical methods using the correspondence $x\to F(x)$ of Lemma 2 and introduced Berge modular games, then proved that BE set is a complete lattice.

As this literature review focuses on BE in static games, we only mention the most important results on BE analysis in *differential games* without detailed comments. BE analysis *in positional differential games* is one area that is extensively explored by Zhukovskiy and his team in Russia. Gaidov [17,18] provided some results on the existence of BE in stochastic differential two-person and n-person games. Boribekova and Jarkynbayev [8] investigated BE in differential-difference games involving uncertainty. A BE existence result based on condition (3.7) above appeared in [39]. The first work on BVE in differential games appeared in [44, 66] investigated BVE in linear-quadratic differential games. Then an in-depth study of BVE in linear-quadratic differential three person games followed by Vaisman in his doctoral thesis [45]. Vaisman published his works in [44,72]. After Vaisman's early death at the age of 36, Zhukovskiy published the book [53] dedicated to Vaisman. The approach used by Zhukovskiy's team is based on the application of Lyapunov's function in differential games. BE is investigated as the Golden Rule in n-person differential positional games in [70]. As an exception, a full chapter in the thesis [25], is devoted to the existence of BE n-person non-linear and linear-quadratic *open loop differential games;* this work has been conducted in Algeria.

For the last two years, V. I. Zhukovskiy and his students have been actively investigating BE in feedback differential games (FDG) in the framework of N. N. Krasovky's FDG mathematical formalization. The specificity of BE with respect to FDG required considering the following three factors. First, in their well-known counterexamples, A. I. Subbotin and A. F. Kononenko had to somehow change and modernize the above-mentioned formalization (see the fundamental results in [66]). Second, active use of the idea of "guide system" proposed by Krasovsky. Third, using the Y. B. Germayer guaranteeing convolution $\max\limits_{i\in I}\varphi_i(\cdot)$ introduced in (3.1). The results of [68] are based on these three factors. The existence and uniqueness of a BE is established for FDG with separated dynamics. Further results on such games are obtained in [56,67,68]. Finally, using a dynamic programming approach, BE in multi-step differential feedback games related to Cournot and Bertrand oligopoly mathematical models has been investigated in the series of works [21,50,58,73].

§ 3.2. Determination and Computation of BE

Establishing sufficient conditions for the existence of a concept of equilibrium is an important step towards its implementation in real-life situations, but this is not enough. If the determination and/or computation of this equilibrium concept is not possible with existing methods and computer processing power, such a concept cannot be useful for solving real-world problems. In this section, we give an account of the publications related to BE determination and computation. Our review reveals that there are interesting results, however, this research area is not well explored. Most of the results are obtained in linear-quadratic static or differential games and in finite games.

As an exception, Larbani and his student Krim [25] developed numerical methods of BE determination and computation based on [26]. Many effective methods for computing BE in n-person nonlinear and linear-quadratic infinite static n-person games are presented. A method of determination of BE based on the minimax Fan inequality is presented in [26] based on the function (3.6),

Lemma 3.1 and Theorem 3.1 as follows. Compute the value

$$\delta = \max_{\widehat{y}\in\widehat{Y}} \min_{x\in X} H(x, \widehat{y}).$$

Let $\overline{x}$ be a solution related to the variable x to this problem. Then if $\delta = 0$, $\overline{x}$ is a BE of the game G. If the conditions of Theorem 3.1 are satisfied, then $\delta = 0$. We have the general result. If the strategy sets are compact and the payoff functions are continuous in the game G, the following equivalence takes place

$$G \text{ has a } BE \Leftrightarrow \delta = 0.$$

Here the well-known numerical methods of solving the maximin problem can be used to compute the value δ. This result is used in [28,36,37]. A generalization of this method using the g-maximum inequality [35] is presented in [37].

For games with linear-quadratic payoffs an explicit form of BE is computed (see also the Table 1 above and the following page) and in linear-quadratic differential games involving uncertainty in payoffs [69]. Explicit forms of BE are obtained for Cournot oligopoly and Bertrand duopoly models (see application Section 4 below) in [50,57,61].

ε-BE is introduced in [30]. This concept is characterized by using a generative relation and a procedure for its computation based on evolutionary multiobjective optimization algorithms with illustrative examples. Corley and Kwain [11] present an algorithm based on the notion of disappointment matrix for computing all BVE in finite games with an illustration. Another algorithm for finding BE in finite games is presented in [32], but without illustration. In [16], the concept of meta-strategy is described that allows players to have different rationality types. This concept is the basis of an evolutionary approach for BE detection that is illustrated by numerical examples. The structure of BE is given in [48].

Computation of BE in differential games is being investigated by Zhukovskiy and his team. Explicit forms of BE in linear-quadratic games are obtained in [45,51,53,54].

Here also, it is important to emphasize that computing BE is more difficult than computing NE, and well-known methods of computing NE cannot be used directly to compute BE. Some significant changes and/or adjustments must be made to adapt them to BE determination.

§ 4. Applications of BE

As BE is an equilibrium that reflects the moral Golden Rule, mutual support, cooperation and altruism in normal form games, it has a great application potential in socio-economic interactions. So far few publications on the application of BE have appeared. Only two BE applications are known. Both are in economics, namely, on Cournot oligopoly and Bertrand duopoly models. The former model appeared in [50,57,58,61], while the latter model appeared in [73] and [21]. Explicit forms of BE are computed and a comparison is made with NE. Note that in [58] and [73], Cournot and Bertrand models are investigated in the framework of multi-step differential games. Let us give some results as an illustration.

1. Cournot oligopoly model. (See [50,61]). Consider a market that is dominated by few big firms producing the same product. It is assumed that these firms compete for the market share and the price is determined by the law of demand and supply. Precisely, assume there are n firms and $I = \{1, \ldots, n\}$ is the set of these firms. Let q_i be the supplied quantity by the firm $i \in I$. In oligopoly, the quantity q_i is subjected to the following constraints:

$$\alpha \leqslant q_i \leqslant \beta,$$

where the inequality $q_i \leqslant \beta$ means that the production capacity of the firms is limited, while $\alpha \leqslant q_i$ means that each firm must supply the market with a guaranteed minimum quantity α to be admitted to the market (which is the case for example in electricity market). The quantity α is generally imposed by the government; a firm that cannot supply this quantity is not allowed in

the market. Next, we assume that the production cost of the quantity q_i is a linear function of this quantity for the firm $i \in I$, that is,

$$cq_i + d,$$

where c and d are constants and the same for all the firms. The price p is determined by the supply and demand law depending on the supplied total quantity $\overline{q} = q_1 + \ldots + q_n$. It is assumed that the price p is a linear function of the total supply $\overline{q}$ as follows:

$$p(\overline{q}) = a - b\overline{q},$$

where a is the initial price, a positive constant and b is the elasticity of the price, a positive constant as well. Thus, the revenue of firm $i \in I$ from selling the quantity q_i is

$$p(\overline{q})q_i = (a - b\overline{q})q_i = \left(a - b\sum_{j\in I} q_j\right) q_i$$

and its profit (revenue–cost) is $p(\overline{q})q_i - (cd_i + d)$ or

$$\pi_i(q_1, \ldots, q_n) = \left(a - b\sum_{j\in I} q_j\right) q_i - (cd_i + d).$$

Thus, we obtain the following normal form game for the Cournot oligopoly game

$$G_{COL} = \langle I = \{1, \ldots, n\}, \{Q_i = [\alpha, \beta]\}_{i\in I}, \{\pi_i(q_1, \ldots, q_n)\}_{i\in I}\rangle.$$

We have the following explicit form of BE [58, 61].

P r o p o s i t i o n 4.1. *If $a > c$, then the game G_{COL} has a BE equilibrium $q^B = (q_1^B, \ldots, q_n^B)$ where $q_i^B = \alpha$, and $\pi_i(q^B) = (a - c)\alpha - bn\alpha^2 - d$ for all $i \in I$.*

A detailed comparison with NE shows that the payoffs (profits) of the firms can be better in BE than in NE depending on the values of the constants n, a, b, c, α and β. For instance, when $a > c$ and

$$\frac{a - c}{n(n + 1)b} < \alpha < \frac{a - c}{(n + 1)b} < \beta,$$

the payoffs of all the firms at BE are larger than their payoffs at NE.

Further, explicit forms of BE are obtained in Cournot duopoly model (Firm 1 and Firm 2) involving uncertainty in the payoffs function ([61] and [22]) via approaches used to deal with uncertainty in games of the form G_U. The uncertainty appears as a third firm that imports the same product and sells it in the same market. The problem is that Firms 1 and 2 do not know the quantity y, the entering Firm 3 will put into market, they just know its range $y \in (0, \infty)$. Based on the form of payoff functions of the Cournot model above, the profits of both functions are affected by the import Firm 3 (its supply quantity, the unknown parameter y) as follows:

$$P_i(q_1, q_2, y) = [a - b(q_1 + q_2 + y)]\, q_i - (cq_i + d), \quad i = 1, 2.$$

The parameter y appears in the total supplied quantity $q_1 + q_2 + y = \overline{q}$. At the same time both Firms 1 and 2 want to minimize the effect of the import Firm 3. This is considered in the payoff functions of both firms as follows:

$$\pi_i(q_1, q_2, y) = P_i(q_1, q_2, y) - y^2, \quad i = 1, 2.$$

Each of these payoff functions involve two criteria, P_i is to be maximized and the effect of the uncertainty y is to be minimized. The following game is obtained:

$$G_{COLIM} = \langle I = \{1, 2\}, \{X_i = (0, \infty)\}_{i\in I}, Y = (0, \infty), \{\pi_i(q_1, q_2, y)\}_{i\in I}\rangle.$$

This game is a game under uncertainty of type G_U presented in Section 3. Using similar methods to those used to solve G_U, explicit BE is constructed.

2. Bertrand Duopoly Model. (See [6, 21, 73]). In a different and more natural approach, instead of the quantity supplied-based firm's strategy in the Cournot model, Bertrand proposed a model of competition in a market involving two firms producing the same product, duopoly, in which the firm's strategy is price-based. The game takes place as follows. Each firm $i \in I = \{1, 2\}$ announces its price $p_i \geqslant 0$, a situation $p = (p_1, p_2)$ is obtained. This price situation creates a demand for the product of each firm. We assume that the demand is linear for both firms:

$$Q_1(p) = q - l_1 p_1 + l_2 p_2, \quad Q_2(p) = q - l_1 p_2 + l_2 p_1,$$

where q is the initial demand, and l_1 and l_2 are elasticity coefficients with respect to prices. Denoting by a positive number c the unit cost of the product, the profits of the firms are

$$f_1(p) = [q - l_1 p_1 + l_2 p_2](p_1 - c), \quad f_2(p) = [q - l_1 p_2 + l_2 p_1](p_2 - c).$$

As each $i \in I$ firm rationally selects a unit price $p_i > c$, the prices will vary in an interval of the form $(c, \beta]$, with β being the maximum price that is the result of market equilibrium (when demand equals supply). Thus, we obtain the following two-person normal form game for Bertrand duopoly:

$$G_{BDU} = \langle I = \{1, 2\}, \{P_i = (c, \beta]\}_{i \in I}, \{f_i(p)\}_{i \in I} \rangle.$$

We have the following explicit form of BE [6, 21, 61].

P r o p o s i t i o n 4.2. *If $l_2 > l_1$ the game G_{BDU} has a BE $p^B = (p_1^B, p_2^B)$ of the form*

$$p_1^B = \beta \text{ and } p_2^B = \beta$$

and the two firms' profits are

$$f_i(p) = (l_2 - l_1)(\beta - c)^2 + [q + c(l_2 - l_1)](\beta - c), \ i = 1, 2.$$

Here also, a detailed comparison with NE shows that in some cases BE provides bigger profits than NE for both firms depending on the values l_1, l_2, c, q, β. For instance, when the relation

$$0 < l_1 < l_2 < 2L_1 \quad \text{and} \quad \beta > \frac{q + cl_1}{2l_1 - l_2}$$

is satisfied, the payoffs of both firms at BE are larger than their payoffs at NE.

Further, similarly to Cournot duopoly, a Bertrand duopoly model involving uncertainty related to import like G_{COLIM} above has been also investigated and explicit forms of BE are constructed by Gorbatov and Zhukovskiy [21].

§ 5. Conclusion and Further Research

In this literature review, we have reviewed most of the published works on BE in static normal form games. We can say that research on BE is gaining momentum as more scholars are attracted to this equilibrium. Moreover, new interesting properties and application potential of BE are discovered, especially, in the last five years (45 % of the publications directly related to BE). Indeed, we have seen that BE is a rich concept of equilibrium, as it has many interpretations:

(i) it expresses the moral Golden Rule;

(ii) it captures mutual support and cooperation among players and

(iii) models altruistic behavior in games.

Thus, it is more suitable than NE in game situations or contexts where players behave according to one of the types (i)–(iii). These behaviors are an integral part of socio-economic human interaction besides the competitive or NE behavior. The literature review reveals that BE is accepted and established as a well-grounded concept from game theory, social and philosophical points of view.

From a *mathematical perspective,* BE analysis is well advanced, but not complete. More research is needed in the following areas:

(a) finding simpler or more practical sufficient existence conditions of BE and/or BVE in n-person non-linear games,

(b) investigating BE in finite games with mixed strategies (open area);

(c) investigating n-person games involving uncertainty of different types;

(d) developing more effective and efficient methods of determination and computation of BE in all types of games and

(e) investigating BE in extensive form games (open area).

In what follows we provide some research problems related to BE existence and computation.

Problem 1. For the existence of BE, we have the following challenge: Consider the multi-valued mapping (best support correspondence) of Lemma 3.2,

$$x \ \to F(x) = \bigcap_{i \in I} f_i(x), \text{ for all } x \in X.$$

Unlike the best response correspondence of NE, there is no guarantee that for BE $F(x)$ is nonempty for all $x \in X$, when the strategy sets are compact and the payoff functions are continuous. Condition 3 of Theorem 3.1 is equivalent to the nonemptiness of $F(x)$. Moreover, it is an extremal condition that may not be easy to verify. Under what additional conditions is the nonemptiness of $F(x)$ guaranteed or can be dropped in BE existence investigation? Note that in almost all the publications on BE existence, it is just assumed that $F(x)$ is non-empty for all $x \in X$, e.g., [13,24].

Problem 2. Let X and Z be two subsets of $\mathbb{R}^m$. Consider n real-valued functions $f_i(x,z) : X \times Z \to \mathbb{R}$, $i \in I = \{1, \ldots, n\}$ and the Y. B. Germayer convolution

$$\varphi(x,z) = \max_{i \in I} f_i(x,z), \quad (x,z) \in X \times Z.$$

The problem is to develop effective methods to find a saddle point of $\varphi(x,z)$, that is, find $(x^0, z^0) \in X \times Z$ that satisfies the following inequalities:

$$\varphi(x^0, z) \leqslant \varphi(x^0, y^0) \leqslant \varphi(x, y^0) \quad \text{for all} \quad (x,z) \in X \times Z.$$

Such methods have never been developed so far. They can be used in the computation of NE (see [58]) and BE (see [57]). The complexity of the operator $\max_{i=1\ldots n}$ spoils the properties of convexity and differentiability of the convolution $\varphi(x,z)$, which make it difficult to use most of existing optimization methods. We think that it is worth trying ti introduce new approaches, mathematical tools of concepts to find saddle points of $\varphi(x,z)$ as it is done to find its extrema in Saint Petersburg University under the supervision of D. V. Dem'yanov (see [14]).

Problem 3. Find practical sufficient conditions for the existence of BE that is at the same time Pareto optimal (BEP). Further, develop numerical methods for computation of BEP.

Problem 4. Find practical sufficient conditions for the existence of BE that is at the same time NE (BNE). Further, develop numerical methods for computation of BNE. BNE is very interesting as in this equilibrium self-interest and altruism are aligned, which makes it an individually and collectively acceptable solution to conflicts.

BE justification through experimental research is also an unexplored area. Determining experimentally in which situations players play BE is a major topic in this area.

BE application in real-world is the area where more efforts and research are needed as there are just few related publications; it is a new and wide open area. The three interpretations (i)–(iii) mentioned above show that BE has potentially a very large application spectrum. Social problem resolution, politics, geopolitics and global problems (e.g. Global Warming) are instances where BE can be applied successfully. Finally, we hope that this literature review will trigger interest of many scholars and researchers in investigating unexplored theoretical, practical and application aspects of BE.

Acknowledgements

We would like to thank M. V. Boldyrev, a student of the Faculty of Computational Mathematics and Cybernetics, Moscow State University, for helping in the process of preparation of this survey paper for publication.

REFERENCES

1. Abalo K.Y., Kostreva M.M. Equi-well-posed games, *Journal of Optimization Theory and Applications*, 1996, vol. 89, issue 1, pp. 89–99. DOI: 10.1007/BF02192642
2. Abalo K.Y., Kostreva M.M. Fixed points, Nash games and their organization, *Topological Methods in Nonlinear Analysis*, 1996, vol. 8, no. 1, pp. 205–215.
3. Abalo K.Y., Kostreva M.M. Intersection theorems and their applications to Berge equilibria, *Applied Mathematics and Computation*, 2006, vol. 182, issue 2, pp. 1840–1848. DOI: 10.1016/j.amc.2006.06.021
4. Abalo K.Y., Kostreva M.M. Berge equilibrium: some recent results from fixed-point theorems, *Applied Mathematics and Computation*, 2005, vol. 169, issue 1, pp. 624–638. DOI: 10.1016/j.amc.2004.09.080
5. Abalo K.Y., Kostreva M.M. Some existence theorems of Nash and Berge equilibria, *Applied Mathematics Letters*, vol. 17, issue 5, pp. 569–573. DOI: 10.1016/S0893-9659(04)90127-9
6. Bel'skikh Yu.A., Zhukovskii V.I., Samsonov S.P. Altruistic (Berge) equilibrium in the model of Bertrand duopoly, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2016, vol. 26, issue 1, pp. 27–45 (in Russian). DOI: 10.20537/vm160103
7. Berge C. *Theorie generale des jeux a n personnes*, Paris: Gauthier-Villars, 1957, 114 p.
8. Boribekova K.A., Jarkynbayev S. Equilibrium of Berge in one differential-difference game, *Mnogokriterial'nye dinamicheskie sadachi pri neopredelennosti: tez. dokl. Vserossiskoi konferentsii* (Multicriteria dynamical problems under uncertainty: proc. of All-Russian conf.), Orekhovo-Zuevo, Russia, 1991, pp. 83–86 (in Russian).
9. Colman A.M. *Game theory and its applications in the social and biological sciences*, Oxford: Butterworth-Heinemann, 1995, 375 p.
10. Colman A.M., Körner T.W., Musy O., Tazdaït T. Mutual support in games: some properties of Berge equilibria, *Journal of Mathematical Psychology*, 2011, vol. 55, issue 2, pp. 166–175. DOI: 10.1016/j.jmp.2011.02.001
11. Corley H.W., Kwain P. An algorithm for computing all Berge equilibria, *Game Theory*, 2015, vol. 2015, Article ID 862842, 2 p. DOI: 10.1155/2015/862842
12. Courtois P., Nessah R., Tazdaït T. How to play games? Nash versus Berge behaviour rules, *Economics and Philosophy*, 2015, vol. 31, issue 1, pp. 123–139. DOI: 10.1017/S026626711400042X
13. Deghdak M. On the existence of Berge equilibrium with pseudocontinuous payoffs, *ROMAI J.*, 2014, vol. 10, no. 1, pp. 25–37.
14. Demyanov V.F., Malozemov V.I. *Vvedenie v minimaks* (Introduction to minimax), Moscow: Nauka, 1972 (in Russian).
15. Dinovsky V.D. Existence d'un point d'equilibre de Berge? *Mnogokriterial'nye dinamicheskie sadachi pri neopredelennosti: tez. dokl. Vserossiskoi konferentsii* (Multicriteria dynamical problems under uncertainty: proc. of All-Russian conf.), Orekhovo-Zuevo, Russia, 1991, pp. 75–77 (in Russian).
16. Dumitrescu D., Lung R., Gasko N., Nagy R. Equilibria detection in non-cooperative game theory: an evolutionary approach, *Proceedings of the 4th World Congress of the Game Theory Society*, Istanbul, Turkey, July 22–26, 2012.
17. Gaidov S.D. Optimal strategies in two-player stochastic differential games, *C. R. Acad. Bulg. Sci.*, 1986, vol. 39, pp. 33–36.
18. Gaidov S.D. Berge-equilibrium in stochastic differential games, *Mathematica Balkanica. New Series*, 1987, vol. 1, pp. 25–32.
19. Gintchev I.A. Method to obtain Berge equilibrium in bimatrix games, *Mnogokriterial'nye dinamicheskie sadachi pri neopredelennosti: tez. dokl. Vserossiskoi konferentsii* (Multicriteria dynamical problems under uncertainty: proc. of All-Russian conf.), Orekhovo-Zuevo, Russia, 1991, pp. 78–82 (in Russian).
20. Glicksberg I.L. A further generalization of the Kakutani fixed theorem, with application to Nash equilibrium points, *Proc. Amer. Math. Soc.*, 1952, vol. 3, issue 1, pp. 170–174. DOI: 10.1090/S0002-9939-1952-0046638-5
21. Gorbatov A.S., Zhukovskiy V.I. Berge equilibrium in Bertrand oligopoly with import, *Tavricheskii Vestnik Informatiki i Matematiki*, 2015, no. 2 (27), pp. 55–64 (in Russian).
22. Guseinov A.A., Zhukovskiy V.I., Kudryavtsev K.N. *Matematicheskie osnovy zolotogo pravila nravstvennosti* (Mathematical foundations of moral Golden rule), Moscow: Editorial URSS, 2016, 280 p.
23. Kakutani S.A. A generalization of Brouwer's fixed point theorem, *Duke Math. J.*, 1941, vol. 8, no. 3, pp. 457–459. DOI: 10.1215/S0012-7094-41-00838-4

24. Keskin K., Çagri Sağlam H. On the existence of Berge equilibrium: an order theoretic approach, *Int. Game Theory Rev.*, 2015, vol. 17, issue 3, 1550007, 9 p. DOI: 10.1142/S0219198915500073
25. Krim F. *An in depth study of Berge equilibrium*, Master's Dissertation, Faculty of Exact Sciences, University of Tizi-Ouzou, Algeria, 2001.
26. Larbani M. Sur l'existence de l'equilibre de Berge pour un jeu a n-personnes, *Optimisation et decision: Actes des Deuxiemes journees francophones de recherche operationnelle, Francoro'98*, Eds.: F. Ben Abdelaziz, M. Haouari, K. Mellouli. Centre de Publication Universitaire de Tunisie, 2000, pp. 291–300.
27. Larbani M., Krim F. Berge Equilibrium, *An in depth study of Berge equilibrium*, Master's Dissertation, Faculty of Exact Sciences, University of Tizi-Ouzou, Algeria, 2001, pp. 14–44.
28. Larbani M., Nessah R. A note on the existence of Berge and Berge–Nash equilibria, *Mathematical Social Sciences*, 2008, vol. 55, issue 2, pp. 258–271. DOI: 10.1016/j.mathsocsci.2007.07.004
29. Ledyard J.O. Public goods: a survey of experimental researc, *The Handbook of Experimental Economics*, Eds.: J.H. Kagel, A.E. Roth. Princeton: Princeton University Press, 1995, pp. 111–194.
30. Lung R.I., Suciu M., Gaskó N., Dumitrescu D. Characterization and detection of ε-Berge–Zhukovskii equilibria, *PLOS ONE*, 2015, vol. 10, no. 7, pp. e0131983. DOI: 10.1371/journal.pone.0131983
31. Mashchenko S.O. Individually-optimal equilibria of non-cooperative games in preference relations, *Cybernetics and Systems Analysis*, 2009, vol. 45, issue 1, pp. 153–159. DOI: 10.1007/s10559-009-9086-4
32. Musy O., Pottier A., Tazdait T. A new theorem to find Berge equilibria, *Int. Game Theory Rev.*, 2012, vol. 14, issue 1, 1250005, 10 p. DOI: 10.1142/S0219198912500053
33. Nash J. Non-cooperative games, *Annals of Mathematics*, 1951, vol. 54, no. 2, pp. 286–295. DOI: 10.2307/1969529
34. Nash J.F. Equilibrium points in n-person games, *Proc. Natl. Acad. Sci. USA.*, 1950, vol. 36, no. 1, pp. 48–49. DOI: 10.1073/pnas.36.1.48
35. Nessah R., Larbani M. g-maximum equality, *Proceedings of the 3rd International Conference on Nonlinear Analysis and Convex Analysis*, Tokyo, 2003. Eds.: W. Takahashi, T. Tanaka. Yokohama: Yokohama Publishers, 2004, pp. 391–400.
36. Nessah R., Larbani M., Tazdait T. A note on Berge equilibrium, *Appl. Math. Lett.*, 2007, vol. 20, issue 8, pp. 926–932. DOI: 10.1016/j.aml.2006.09.005
37. Nessah R., Larbani M., Tazdait T. On Berge equilibrium, *CIRED Working Papers*, 2007, HAL Id: halshs-00271452. `https://halshs.archives-ouvertes.fr/halshs-00271452`
38. Pottier A., Nessah R. Berge–Vaisman and Nash equilibria: transformation of games, *Int. Game Theory Rev.*, 2014, vol. 16, issue 4, 1450009, 8 p. DOI: 10.1142/S0219198914500091
39. Radjef M. Sur l'existence d'un equilibre de Berge pour un jeu différentiel à n-personnes, *Cahiers Mathématiques de l'Université d'Oran*, 1988, vol. 1, pp. 89–93.
40. Sally D. Conversation and cooperation in social dilemmas: a meta-analysis of experiments from 1958 to 1992, *Rationality and Society*, 1995, vol. 7, issue 1, pp. 58–92. DOI: 10.1177/1043463195007001004
41. Sen A. Goals, commitment and identity, *The Journal of Law, Economics, and Organization*, 1985, vol. 1, issue 2, pp. 341–355. DOI: 10.1093/oxfordjournals.jleo.a036895
42. Sen A. *On ethics and economics*, Oxford: Basil Blackwell, 1987, 146 p.
43. Shubik M. Review: the general theory of n-person games by Claude Berge, *Econometrica*, 1961, vol. 29, no. 4, p. 821. DOI: 10.2307/1911828
44. Vaisman K.S. *The Berge equilibrium*, in book: Zhukovskii V.I., Chikrii A.A. *Lineino-kvadratichnye differensial'nye igry* (Linear-quadratic differential games), Kiev: Naukova Dumka, 1994, pp. 119–142 (in Russian).
45. Vaisman K.S. *The Berge equilibrium*, Cand. Sci. (Phys.-Math.) Dissertation, St. Petersburg, 1995, 110 p. (In Russian).
46. Vaisman K.S. *Nash bargaining solution under uncertainty*, in book: Zhukovskiy V.I. *Kooperativnye igry pri neopredelennosti* (Cooperative games under uncertainty), Moscow: Editorial URSS, 1999, pp. 231–249 (in Russian).
47. Vaisman K.S., Zhukovskiy V.I. The Berge equilibrium under uncertainty, *Mnogokriterial'nye dinamicheskie sadachi pri neopredelennosti: tez. dokl. Vserossiskoi konferentsii* (Multicriteria dynamical problems under uncertainty: proc. of All-Russian conf.), Orekhovo-Zuevo, Russia, 1994, pp. 97–98 (in Russian).
48. Vaisman K.S., Zhukovskiy V.I. Properties of Berge equilibrium, *Matematicheskie problemy ekologii: tez. dokl. Vserossiiskoi konferentsii* (Mathematical problems of ecology: proc. of All-Russian conf.), Institute of Natural Resources, Ecology and Cryology of the Siberian Branch of the Russian Academy of Sciences, Chita, 1994, pp. 27–28 (in Russian).
49. van Dam E. *Rationality and the golden rule*, Working Paper, Tilburg: Tilburg University, 2014, 18 p.
50. Vysokos M.I., Zhukovskii V.I. Golden rule in Cournot duopoly model, *Tavricheskii Vestnik Informatiki i Matematiki*, 2015, no. 2 (27), pp. 46–54 (in Russian).
51. Zhitomirskiy G.I., Vaisman K.S. Berge equilibrium, *Slozhnye upravlyaemye sistemy* (Complex dynamical systems), Pskov: Pskov Pedagogical Institute, 1994, pp. 52–57 (in Russian).

52. Zhukovskiy V.I. Some problems of non-antagonistic differential games, *Matematicheskie metody v issledovanii operatsii* (Mathematical methods in operations research), Sofia: Bulgarian Academy of Sciences, 1985, pp. 103–195 (in Russian).
53. Zhukovskiy V.I. *Vvedenie v differentsial'nye igry pri neopredelennosti. Ravnovesie po Berzhu–Vaismanu* (Introduction to differential games under uncertainty. Berge–Vaisman equilibrium), Moscow: Editorial URSS, 2010, 176 p.
54. Zhukovskii V.I., Chikrii A.A. *Lineino-kvadratichnye differentsial'nye igry* (Linear quadratic differential games), Kiev: Naukova dumka, 1994, 320 p.
55. Zhukovskii V.I., Chikrii A.A., Soldatova N.G. The Berge equilibrium in the conflicts under uncertainty, *XII Vserossiiskoe soveshchanie po problemam upravleniya (VSPU–2014): Trudy* (Proc. XII All-Russian conf. on control problems (RCCP-2014)), Moscow: Inst. of Control Problems, 2014, pp. 8290–8302 (in Russian).
56. Zhukovskii V.I., Gorbatov A.S. Connection between the golden rule and Berge equilibrium, *Control Theory and Theory of Generalized Solutions of Hamilton–Jacobi Equations CGS'2015: Abstracts of II International Seminar dedicated to the 70th Anniversary of Academician A.I. Subbotin*, Yekaterinburg, 2015, pp. 58–60 (in Russian).
57. Zhukovskii V.I., Kudryavtsev K.N. Mathematical foundations of the Golden Rule. I. Static variant, *Mat. Teor. Igr Pril.*, 2015, vol. 7, no. 3, pp. 16–47 (in Russian).
58. Zhukovskiy V.I., Kudryavtsev K.N. Pareto-optimal Nash equilibrium: sufficient conditions and existence in mixed strategies, *Automation and Remote Control*, 2016, vol. 77, issue 8, pp. 1500–1510. DOI: 10.1134/S0005117916080154
59. Zhukovskiy V.I., Kudryavtsev K.N. Equilibrating conflicts under uncertainty. I. Analogue of a saddle-point, *Mat. Teor. Igr Pril.*, 2013, vol. 5, no. 1, pp. 27–44 (in Russian).
60. Zhukovskii V.I., Kudryavtsev K.N. Equilibrating conflicts under uncertainty. II. Analogue of a maximin, *Mat. Teor. Igr Pril.*, 2013, vol. 5, no. 2, pp. 3–45 (in Russian).
61. Zhukovskii V.I., Kudryavtsev K.N., Gorbatov A.S. The Berge equilibrium in Cournot's model of oligopoly, *Vestn. Udmurt. Univ. Mat. Mekh. Komp'yut. Nauki*, 2015, vol. 25, issue 2, pp. 147–156 (in Russian). DOI: 10.20537/vm150201
62. Zhukovskiy V.I., Sachkov S.N. An unusual way of solving conflicts, *Sovremennye tekhnologii upravleniya–2014* (Contemporary technology of management–2014), Moscow, 2014, pp. 337–354 (in Russian).
63. Zhukovskii V.I., Sachkov S.N. Bilanciamento conflitti friendly, *Italian Science Review*, 2014, issue 9 (18), pp. 169–179 (in Italian). `http://www.ias-journal.org/archive/2014/september/Zhukovsky.pdf`
64. Zhukovskiy V.I., Sachkov S.N., Smirnova L.V. Existence of Berge equilibrium in mixed strategies, *Uchenye Zapiski Tavricheskogo Natsional'nogo Universiteta im. V.I. Vernadskogo*, 2014, vol. 27 (66), no. 1, pp. 261–279.
65. Zhukovsky V.I., Sachkov S.N., Smirnova L.V. Berge equilibrium, *Analiz, modelirovanie, upravlenie, razvitie ekonomicheskikh sistem: materialy nauchnykh trudov VIII Mezhdunarodnoi shkoly-simposiuma AMUR-2014* (Analysis, modeling, management, development of economic systems: proceedings of VIII international school-symposium AMUR-2014), Crimean Federal University, Simferopol, 2014, pp. 124–133.
66. Zhukovskii V.I., Salukvadze M.E. *The vector-valued maximin*, N.Y.: Academic Press, 1994, 407 p.
67. Zhukovskii V.I., Smirnova L.V. The existence of Berge equilibrium in differential game with "separated" dynamics, *Tavricheskii Vestnik Informatiki i Matematiki*, 2015, no. 2 (27), pp. 77–81 (in Russian).
68. Zhukovskii V.I., Smirnova L.V., Gorbatov A.S. Mathematical foundations of the Golden Rule. II. Dynamic variant, *Mat. Teor. Igr Pril.*, 2016. vol. 8, no. 1, pp. 27–62 (in Russian).
69. Zhukovskiy V.I., Topchishvili A.L. The explicit form of Berge strongly guaranteed equilibrium in linear-quadratic non-cooperative games, *International Journal of Operations and Quantitative Management,* 2015, vol. 21, no. 4, pp. 265–273.
70. Zhukovskiy V.I., Topchishvili A.L. Mathematical model of golden rule in the form of differential positional game of many persons, *International Journal of Operations and Quantitative Management*, 2016, vol. 22, no. 3, pp. 203–229.
71. Zhukovskiy V., Topchishvili A., Sachkov S. Application of probability measures to the existence problem of Berge–Vaisman guaranteed equilibrium, *Model Assisted Statistics and Applications*, 2014, vol. 9, no. 3, pp. 223–239. DOI: 10.3233/MAS-140295
72. Zhukovskiy V.I., Vaisman K.S. To problems about Berge equilibrium, *Vestnik Pskovskogo vol'nogo universiteta. Ser. Matematika i Informatika,* 1997, no. 1, pp. 49–70.
73. Zhukovskiy V.I., Zhiteneva Yu.M. Bertrand difference-differential game model in the framework of Berge equilibrium, *Analiz, modelirovanie, upravlenie, razvitie sotsial'no-ekonomicheskikh system: sbornik nauchnykh trudov IX Mezhdunarodnoi shkoly-symposiuma AMUR-2015* (Analysis, modeling, management, development of socio-economic systems: proceedings of IX international school-symposium AMUR-2015), Crimean Federal University, Simferopol, 2015, pp. 109–115 (in Russian).

74. Zolotarev V.V. *Hybrid equilibrium in games involving uncertainty*, Abstract of Cand. Sci. (Phys.-Math.) Dissertation, Moscow, 2002, 25 p. (In Russian).

Received 01.02.2017

Larbani Moussa[3], Professor, Department of Business Administration, International Islamic University Malaysia, Kuala Lumpur, 53100, Malaysia.
E-mail: larbani61@hotmail.com

Zhukovskii Vladislav Iosifovich[4], Doctor of Physics and Mathematics, Professor, Lomonosov Moscow State University, GSP-1, Leninskie Gory, Moscow, 119991, Russia.
E-mail: zhkvlad@yandex.ru

[3]Moussa Larbani is an Algerian scholar, Professor at the Department of Business Administration of the International Islamic University Malaysia. His research activities cover many areas, including game theory, multiple criteria optimization, decision making under uncertainty, artificial intelligence, finance and economics. He is a co-founder of Second Order Games and Optimization in Changeable Space with P. L. Yu. He completed his PhD in ordinary differential games and optimal control at the faculty of Mechanics and Mathematics of Metchnikov Odessa State University, Ukraine in 1991.

[4]Vladislav Iosifovich Zhukovskiy is a Russian scholar, Professor at the Department of Optimal Control of the Faculty of Computational Mathematics and Cybernetics of Lomonosov Moscow State University. He completed his PhD thesis at Lomonosov Moscow State University in 1965. In 1988, he completed his doctoral thesis at Sverdlovsk Mathematics and Mechanics Institute. He is a foreign member of the Georgian National Academy of Science, Merited Scientist of Russia and honorary academician of the Academy of Nonlinear Sciences. He published more than 200 articles, including 16 monographs. His research and contributions cover many areas, including game theory (differential and static), optimal control, multiple criteria optimization, dynamic systems, economics and finance. He has supervised 26 PhD students.

М. Ларбани, В. И. Жуковский

Равновесие по Бержу в играх в нормальной форме: литературный обзор

Цитата: *Известия Института математики и информатики Удмуртского государственного университета.* 2017. Т. 49. С. 80–110.

Ключевые слова: равновесие по Бержу, взаимная поддержка, кооперация, Золотое Правило, альтруизм, определение равновесия по Бержу, существование равновесия по Бержу, вычисление равновесия по Бержу.

УДК: 517.958, 530.145.6

DOI: 10.20537/2226-3594-2017-49-04

Представлен обзор литературы, посвященной равновесию по Бержу в играх в нормальной форме. Обзор показывает, что исследования равновесия по Бержу за последние несколько лет набирают обороты, поскольку в настоящее время это равновесие основано на теории игр, философии и социальном взаимодействии. Оно охватывает взаимную поддержку, сотрудничество, координацию и моделирует альтруизм и моральное Золотое правило в играх в нормальной форме. Математическое исследование равновесия по Бержу продвигается, но не является полным; требуются дополнительные исследования в областях, связанных с его существованием и вычислением. Применение равновесия по Бержу в реальных социально-экономических взаимодействиях, где игроки взаимно дополняют друг друга, является почти неизученной областью исследований.

Список литературы

1. Abalo K.Y., Kostreva M.M. Equi-well-posed games // Journal of Optimization Theory and Applications. 1996. Vol. 89. Issue 1. P. 89–99. DOI: 10.1007/BF02192642
2. Abalo K.Y., Kostreva M.M. Fixed points, Nash games and their organization // Topological Methods in Nonlinear Analysis. 1996. Vol. 8. No. 1. P. 205–215.
3. Abalo K.Y., Kostreva M.M. Intersection theorems and their applications to Berge equilibria // Applied Mathematics and Computation. 2006. Vol. 182. Issue 2. P. 1840–1848. DOI: 10.1016/j.amc.2006.06.021
4. Abalo K.Y., Kostreva M.M. Berge equilibrium: some recent results from fixed-point theorems // Applied Mathematics and Computation. 2005. Vol. 169. Issue 1. P. 624–638. DOI: 10.1016/j.amc.2004.09.080
5. Abalo K.Y., Kostreva M.M. Some existence theorems of Nash and Berge equilibria // Applied Mathematics Letters. Vol. 17. Issue 5. P. 569–573. DOI: 10.1016/S0893-9659(04)90127-9
6. Бельских Ю.А., Жуковский В.И., Самсонов С.П. Альтруистическое равновесие (по Бержу) в модели дуополии Бертрана // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2016. Т. 26. Вып. 1. С. 27–45. DOI: 10.20537/vm160103
7. Berge C. Theorie generale des jeux a n personnes. Paris: Gauthier-Villars, 1957. 114 p.
8. Борибекова К.А., Жаркынбаев С. Равновесие по Бержу в одной дифференциально-разностной игре // Многокритериальные динамические задачи при неопределенности: тезисы докладов Всероссийской конференции. Орехово-Зуево, 1991. С. 83–86.
9. Colman A.M. Game theory and its applications in the social and biological sciences. Oxford: Butterworth-Heinemann, 1995. 375 p.
10. Colman A.M., Körner T.W., Musy O., Tazdaït T. Mutual support in games: some properties of Berge equilibria // Journal of Mathematical Psychology. 2011. Vol. 55. Issue 2. P. 166–175. DOI: 10.1016/j.jmp.2011.02.001
11. Corley H.W., Kwain P. An algorithm for computing all Berge equilibria // Game Theory. 2015. Vol. 2015. Article ID 862842. 2 p. DOI: 10.1155/2015/862842
12. Courtois P., Nessah R., Tazdaït T. How to play games? Nash versus Berge behaviour rules // Economics and Philosophy. 2015. Vol. 31. Issue 1. P. 123–139. DOI: 10.1017/S026626711400042X
13. Deghdak M. On the existence of Berge equilibrium with pseudocontinuous payoffs // ROMAI J. 2014. Vol. 10. No. 1. P. 25–37.
14. Демьянов В.Ф., Малоземов В.И. Введение в минимакс. М.: Наука, 1972.
15. Dinovsky V.D. Existence d'un point d'equilibre de Berge? // Многокритериальные динамические задачи при неопределенности: тезисы докладов Всероссийской конференции. Орехово-Зуево, 1991. С. 75–77.
16. Dumitrescu D., Lung R., Gasko N., Nagy R. Equilibria detection in non-cooperative game theory: an evolutionary approach // Proceedings of the 4th World Congress of the Game Theory Society. Istanbul, Turkey. July 22–26, 2012.
17. Gaidov S.D. Optimal strategies in two-player stochastic differential games // C. R. Acad. Bulg. Sci. 1986. Vol. 39. P. 33–36.

18. Gaidov S.D. Berge-equilibrium in stochastic differential games // Mathematica Balkanica. New Series. 1987. Vol. 1. P. 25–32.
19. Гинчев И.А. Метод достижения равновесия по Бержу в биматричных играх // Многокритериальные динамические задачи при неопределенности: тезисы докладов Всероссийской конференции. Орехово-Зуево, 1991. С. 78–82.
20. Glicksberg I.L. A further generalization of the Kakutani fixed theorem, with application to Nash equilibrium points // Proc. Amer. Math. Soc. 1952. Vol. 3. Issue 1. P. 170–174. DOI: 10.1090/S0002-9939-1952-0046638-5
21. Горбатов А.С., Жуковский В.И. Равновесие по Бержу в олигополии Бертрана при учете импорта // Таврический вестник информатики и математики. 2015. № 2 (27). С. 55–64.
22. Гусейнов А.Н., Жуковский В.И., Кудрявцев К.Н. Математические основы Золотого правила нравственности. М.: Едиториал УРСС, 2016. 280 с.
23. Kakutani S.A. A generalization of Brouwer's fixed point theorem // Duke Math. J. 1941. Vol. 8. No. 3. P. 457–459. DOI: 10.1215/S0012-7094-41-00838-4
24. Keskin K., Çagri Sağlam H. On the existence of Berge equilibrium: an order theoretic approach // Int. Game Theory Rev. 2015. Vol. 17. Issue 3. 1550007. 9 p. DOI: 10.1142/S0219198915500073
25. Krim F. An in depth study of Berge equilibrium. Master's Dissertation. Faculty of Exact Sciences, University of Tizi-Ouzou. Algeria, 2001.
26. Larbani M. Sur l'existence de l'equilibre de Berge pour un jeu a n-personnes // Optimisation et decision: Actes des Deuxiemes journees francophones de recherche operationnelle. Francoro'98 / F. Ben Abdelaziz, M. Haouari, K. Mellouli. Centre de Publication Universitaire de Tunisie, 2000. P. 291–300.
27. Larbani M., Krim F. Berge Equilibrium // An in depth study of Berge equilibrium. Master's Dissertation. Faculty of Exact Sciences, University of Tizi-Ouzou. Algeria, 2001. P. 14–44.
28. Larbani M., Nessah R. A note on the existence of Berge and Berge–Nash equilibria // Mathematical Social Sciences. 2008. Vol. 55. Issue 2. P. 258–271. DOI: 10.1016/j.mathsocsci.2007.07.004
29. Ledyard J.O. Public goods: a survey of experimental research // The Handbook of Experimental Economics / J.H. Kagel, A.E. Roth. Princeton: Princeton University Press, 1995. P. 111–194.
30. Lung R.I., Suciu M., Gaskó N., Dumitrescu D. Characterization and detection of ε-Berge–Zhukovskii equilibria // PLOS ONE. 2015. Vol. 10. No. 7. P. e0131983. DOI: 10.1371/journal.pone.0131983
31. Mashchenko S.O. Individually-optimal equilibria of non-cooperative games in preference relations // Cybernetics and Systems Analysis. 2009. Vol. 45. Issue 1. P. 153–159. DOI: 10.1007/s10559-009-9086-4
32. Musy O., Pottier A., Tazdait T. A new theorem to find Berge equilibria // Int. Game Theory Rev. 2012. Vol. 14. Issue 1. 1250005. 10 p. DOI: 10.1142/S0219198912500053
33. Nash J. Non-cooperative games // Annals of Mathematics. 1951. Vol. 54. No. 2. P. 286–295. DOI: 10.2307/1969529
34. Nash J.F. Equilibrium points in n-person games // Proc. Natl. Acad. Sci. USA. 1950. Vol. 36. No. 1. P. 48–49. DOI: 10.1073/pnas.36.1.48
35. Nessah R., Larbani M. g-maximum equality // Proceedings of the 3rd International Conference on Nonlinear Analysis and Convex Analysis. Tokyo, 2003 / W. Takahashi, T. Tanaka. Yokohama: Yokohama Publishers, 2004. P. 391–400.
36. Nessah R., Larbani M., Tazdait T. A note on Berge equilibrium // Appl. Math. Lett. 2007. Vol. 20. Issue 8. P. 926-932. DOI: 10.1016/j.aml.2006.09.005
37. Nessah R., Larbani M., Tazdait T. On Berge equilibrium // CIRED Working Papers. 2007. HAL Id: halshs-00271452. https://halshs.archives-ouvertes.fr/halshs-00271452
38. Pottier A., Nessah R. Berge–Vaisman and Nash equilibria: transformation of games // Int. Game Theory Rev. 2014. Vol. 16. Issue 4. 1450009. 8 p. DOI: 10.1142/S0219198914500091
39. Radjef M. Sur l'existence d'un equilibre de Berge pour un jeu différentiel à n-personnes // Cahiers Mathématiques de l'Université d'Oran. 1988. Vol. 1. P. 89–93.
40. Sally D. Conversation and cooperation in social dilemmas: a meta-analysis of experiments from 1958 to 1992 // Rationality and Society. 1995. Vol. 7. Issue 1. P. 58–92. DOI: 10.1177/1043463195007001004
41. Sen A. Goals, commitment and identity // The Journal of Law, Economics, and Organization. 1985. Vol. 1. Issue 2. P. 341–355. DOI: 10.1093/oxfordjournals.jleo.a036895
42. Sen A. On ethics and economics. Oxford: Basil Blackwell, 1987. 146 p.
43. Shubik M. Review: the general theory of n-person games by Claude Berge // Econometrica. 1961. Vol. 29. No. 4. P. 821. DOI: 10.2307/1911828
44. Вайсман К.С. Равновесие по Бержу // Линейно-квадратичные дифференциальные игры / В.И. Жуковский, А.А. Чикрий. Киев: Наукова Думка, 1994. Раздел 3.2. С. 119–142.
45. Вайсман К.С. Равновесие по Бержу: дис. ... канд. физ.-матем. наук / СПбГУ. Санкт-Петербург, 1995. 110 с.
46. Вайсман К.С. Арбитражная схема Нэша при неопределенности // Кооперативные игры при неопре-

деленности / В.И. Жуковский. М.: Едиториал УРСС, 1999. С. 231–249.
47. Вайсман К.С., Жуковский В.И. Равновесие по Бержу при неопределенности // Многокритериальные динамические задачи при неопределенности: тезисы докладов Всероссийской конференции. Орехово-Зуево, 1994. С. 97–98.
48. Вайсман К.С., Жуковский В.И. Свойства равновесия по Бержу // Математические проблемы экологии: тезисы докладов Всероссийской конференции. Институт природных ресурсов, экологии и криологии Сибирского отделения Российской академии наук. Чита, 1994. С. 27–28.
49. van Dam E. Rationality and the golden rule. Working Paper. Tilburg: Tilburg University, 2014. 18 p.
50. Высокос М.И., Жуковский В.И. Золотое правило в модели дуополии Курно // Таврический вестник информатики и математики. 2015. № 2 (27). С. 46–54.
51. Житомирский Г.И., Вайсман К.С. Равновесие по Бержу // Сложные управляемые системы. Псков: Псковский педагогический институт, 1994. С. 52–57.
52. Жуковский В.И. Некоторые задачи в неантагонистических дифференциальных играх // Математические методы в исследовании операций. София: Болг. академия наук, 1985. С. 103–195.
53. Жуковский В.И. Введение в дифференциальные игры при неопределенности. Равновесие по Бержу–Вайсману. Москва: Едиториал УРСС, 2010. 176 с.
54. Жуковский В.И., Чикрий А.А. Линейно-квадратичные дифференциальные игры. Киев: Наукова думка, 1994. 320 с.
55. Жуковский В.И., Чикрий А.А., Солдатова Н.Г. Равновесие по Бержу в конфликтах при неопределенности // XII Всероссийское совещание по проблемам управления (ВСПУ-2014): Труды. М.: ИПУ РАН, 2014. С. 8290–8302.
56. Жуковский В.И., Горбатов А.С. Связь золотого правила с равновесием по Бержу // Теория управления и теория обобщенных решений уравнений Гамильтона–Якоби: Тезисы докладов II Международного семинара, посвященного 70-летию со дня рождения академика А.И. Субботина. Екатеринбург: ИММ УрО РАН, УрФУ, 2015. С. 58–60.
57. Жуковский В.И., Кудрявцев К.Н. Математические основы Золотого правила. I. Статический вариант // Математическая теория игр и ее приложения. 2015. Т. 7. Вып. 3. С. 16–47.
58. Жуковский В.И., Кудрявцев К.Н. Парето-равновесная ситуация: достаточные условия и существование в смешанных стратегиях // Математическая теория игр и ее приложения. 2015. Т. 7. Вып. 1. С. 74–91.
59. Жуковский В.И., Кудрявцев К.Н. Уравновешивание конфликтов при неопределенности. I. Аналог седловой точки // Математическая теория игр и ее приложения. 2013. Т. 5. Вып. 1. С. 27–44.
60. Жуковский В.И., Кудрявцев К.Н. Уравновешивание конфликтов при неопределенности. II. Аналог максимина // Математическая теория игр и ее приложения. 2013. Т. 5. Вып. 2. С. 3–45.
61. Жуковский В.И., Кудрявцев К.Н., Горбатов А.С. Равновесие по Бержу в модели олигополии Курно // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2015. Т. 25. Вып. 2. С. 147–156. DOI: 10.20537/vm150201
62. Жуковский В.И., Сачков С.Н. Об одном необычном способе улаживания конфликтов // Современные технологии управления–2014. Москва, 2014. С. 337–355.
63. Zhukovskii V.I., Sachkov S.N. Bilanciamento conflitti friendly // Italian Science Review. 2014. Issue 9 (18). P. 169–179. http://www.ias-journal.org/archive/2014/september/Zhukovsky.pdf
64. Zhukovskiy V.I., Sachkov S.N., Smirnova L.V. Existence of Berge equilibrium in mixed strategies // Ученые записки Таврического национального университета им. В.И. Вернадского. 2014. Т. 27 (66). № 1. С. 261–279.
65. Zhukovskiy V.I., Sachkov S.N., Smirnova L.V. Berge Equilibrium // Анализ, моделирование, управление, развитие экономических систем: материалы научных трудов VIII Международной школы-симпозиума АМУР-2014. Крымский федеральный университет. Симферополь, 2014. С. 124–133.
66. Zhukovskii V.I., Salukvadze M.E. The vector-valued maximin. New York: Academic Press, 1994. 407 p.
67. Жуковский В.И., Смирнова Л.Н. Существование равновесия по Бержу в дифференциальной игре с «разделенной» динамикой // Таврический вестник информатики и математики. 2015. № 2 (27). С. 77–81.
68. Жуковский В.И., Смирнова Н.Г., Горбатов А.С. Математические основы Золотого правила. II. Динамический вариант // Математическая теория игр и ее приложения. 2016. Т. 8. Вып. 1. С. 27–62.
69. Zhukovskiy V.I., Topchishvili A.L. The explicit form of Berge strongly guaranteed equilibrium in linear-quadratic non-cooperative games // International Journal of Operations and Quantitative Management. 2015. Vol. 21. No. 4. P. 265–273.
70. Zhukovskiy V.I., Topchishvili A.L. Mathematical model of golden rule in the form of differential positional game of many persons // International Journal of Operations and Quantitative Management. 2016. Vol. 22. No. 3. P. 203–229.

71. Zhukovskii V., Topchishvili A., Sachkov S. Application of probability measures to the existence problem of Berge–Vaisman guaranteed equilibrium // Model Assisted Statistics and Applications. 2014. Vol. 9. No. 3. P. 223–239. DOI: 10.3233/MAS-140295
72. Zhukovskiy V.I., Vaisman K.S. To problems about Berge equilibrium // Вестник Псковского вольного университета. Математика и информатика. 1997. № 1. С. 49–70.
73. Жуковский В.И., Житенева Ю.М. Одношаговая модель олигополии Бертрана в рамках равновесия по Бержу // Анализ, моделирование, управление, развитие социально-экономических систем: сборник научных трудов IX Международной школы-симпозиума АМУР-2015. Крымский федеральный университет. Симферополь, 2015. С. 109–115.
74. Золотарев В.В. Гибридное равновесие в играх с неопределенностью: автореферат дисс. ... канд. физ.-матем. наук. Москва, 2002. 25 с.

Поступила в редакцию 01.02.2017

Ларбани Муса, профессор, Университет IIUM, Куала Лумпур, 53100, Малайзия.
E-mail: larbani61@hotmail.com

Жуковский Владислав Иосифович, д. ф.-м. н., профессор, кафедра оптимального управления, факультет вычислительной математики и кибернетики, Московский государственный университет им. М. В. Ломоносова, 119991, Россия, г. Москва, ГСП-1, Ленинские горы.
E-mail: zhkvlad@yandex.ru