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Sufficient Conditions for Uniform Global Asymptotic Stabilization of Discrete-Time Periodic Bilinear Systems ⋆

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Abstract: Sufficient conditions for uniform global asymptotic stabilization of the origin by state feedback are obtained for discrete-time bilinear systems with periodic coefficients. It is shown that for bilinear periodic control systems with Lyapunov stable free dynamics the property of consistency is sufficient for existence of stabilizing control.

Keywords: Asymptotic stabilization, Lyapunov methods, discrete-time systems, bilinear systems, periodic systems.

1. INTRODUCTION

Consider a bilinear control system with discrete-time dynamics

$$\begin{gathered} x(t+1) = A(t)x(t) + B(t,x(t))u(t), \\ B(t,x) = [B_1(t)x, \dots, B_r(t)x] \in M_{n,r}(\mathbb{R}), \\ (t,x,u) \in \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^r, \end{gathered} \tag{1}$$

where $A(t), B_i(t) \in M_n(\mathbb{R})$, $i = \overline{1,r}$; $M_n(\mathbb{K}) := M_{n,n}(\mathbb{K})$; $M_{n,m}(\mathbb{K})$ is the space of $n \times m$-matrices with elements of the field $\mathbb{K}$. We study the problem of asymptotic stabilization of the origin for the system (1) by state feedback. The well-known Jurdjevic–Quinn theorem (see Jurdjevic and Quinn (1978)) provides sufficient conditions for global asymptotic stabilization of the origin for continuous-time autonomous bilinear systems. Later, these sufficient conditions were generalized and extended to affine and general nonlinear systems (see Gauthier (1984), Kalouptsidis and Tsinias (1984), Lee and Arapostathis (1988), Byrnes and Isidori (1989), Byrnes et al. (1991), Outbib and Sallet (1992), Lin (1995b), Lin (1995a), Lin (1996b)).

Similar results were obtained for bilinear, affine, and general nonlinear autonomous systems with the discrete time (see Tsinias (1989), Byrnes et al. (1993), Byrnes and Lin (1994), Lin and Byrnes (1994), Lin (1996a), Grüne and Wirth (1999), Bacciotti and Biglio (2001), Navarro-Lopez et al. (2002), Monaco and Normand-Cyrot (2011), Zaitsev (2015b)).

The approach of damping control was extended to continuous-time time-varying bilinear (Zaitsev (2013a), Zaitsev (2013b)), affine (Zaitsev (2013c), Zaitsev (2013d)), and general nonlinear systems (Zaitsev (2016)) with periodic coefficients.

In this work, we obtain new sufficient conditions for uniform global asymptotic stabilization of the origin for bilinear periodic systems with discrete time dynamics.

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2. STABILITY OF DISCRETE-TIME PERIODIC LINEAR SYSTEMS

Consider a linear discrete-time system

$$x(t+1) = A(t)x(t), \quad t \in \mathbb{Z}, \quad x \in \mathbb{K}^n, \tag{2}$$

$A(t) \in M_n(\mathbb{K})$. Here $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Denote by $X(t,s)$ the transition matrix of the system (2). Then $X(t,s) = A(t-1)A(t-2)\cdot \ldots \cdot A(s)$ if $t > s$ and $X(t,s) = I$ if $t = s$. Recall that the system (2) is Lyapunov stable iff $X(t,0)$ is bounded on $\mathbb{N}$ and the system (2) is asymptotically stable iff $X(t,0) \to 0$ as $t \to +\infty$.

Denote by T the transposition, by $*$ the Hermitian conjugation of a vector or a matrix. Suppose $R \in M_n(\mathbb{C})$ is a Hermitian matrix, i.e., $R = R^*$. It defines the quadratic form $V(x) = x^*Rx$ on $\mathbb{C}^n$. For each $x \in \mathbb{C}^n$ we have $V(x) \in \mathbb{R}$. Indeed, let $R = P + iS$, $x = p + iq$, $P, S \in M_n(\mathbb{R})$, $p, q \in \mathbb{R}^n$. The equality $R = R^*$ implies that $P^T = P$, $S^T = -S$. It follows that $V(x) = x^*Rx = (p^T - iq^T)(P + iS)(p+iq) = p^TPp + q^TPq - 2p^TSq$. This equality implies, in particular, that if $x \in \mathbb{R}^n$, then

$$x^*Rx = x^*(\operatorname{Re} R)x. \tag{3}$$

We identify the quadratic form $V(x) = x^*Rx$ with the Hermitian matrix $R = R^*$ defining this form. For the matrices $R = R^*$, $Q = Q^*$, the inequality $R > Q$ ($R \le Q$) means that the quadratic form $V(x) = x^*(R - Q)x$ is positive definite (respectively, negative semi-definite), i.e., $\forall\, x \in \mathbb{C}^n \setminus \{0\}$ $V(x) > 0$ (respectively, $V(x) \le 0$).

Remark 1. It follows from (3) that if $R \in M_n(\mathbb{C})$, $R = R^* > 0$, and $P = \operatorname{Re} R \in M_n(\mathbb{R})$, then $P^T = P > 0$, i.e., $\forall\, x \in \mathbb{R}^n \setminus \{0\}$ $x^TPx > 0$. □

Consider a linear time-invariant discrete-time system

$$y(t+1) = Fy(t), \quad t \in \mathbb{Z}, \quad y \in \mathbb{K}^n, \tag{4}$$

$F \in M_n(\mathbb{K})$. The system (4) is Lyapunov stable iff F^t is bounded on $\mathbb{N}$ and the system (4) is asymptotically stable iff $F^t \to 0$ as $t \to +\infty$. Suppose that the eigenvalues $\lambda_j(F)$ of the matrix F satisfy conditions $|\lambda_j(F)| < 1$, $j = \overline{1,n}$ (i.e., the system (4) is asymptotically stable).

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Then for any matrix $H \in M_n(\mathbb{K})$ such that $H = H^* > 0$ there exists a $R \in M_n(\mathbb{K})$ such that $R^* = R > 0$ and $F^*RF - R = -H$. The solution of the last (discrete-time Lyapunov) equation is unique and is defined by the formula $R = H + \sum_{\nu=1}^{\infty}(F^*)^{\nu}HF^{\nu}$. This series converges.

Lemma 2. Let $\mathbb{K} = \mathbb{C}$ and the system (4) be Lyapunov stable. Then there exists a matrix $Q = Q^* > 0$ such that $F^*QF - Q \leq 0$.

Proof. Let the system (4) be Lyapunov stable. Then there exists a matrix $S \in M_n(\mathbb{C})$ such that $F = S^{-1}JS$, where $J = \operatorname{diag}\{J_1, J_2\}$; here $J_1 \in M_\alpha(\mathbb{C})$ is a Jordan matrix, $|\lambda_j(J_1)| = 1$, $j = \overline{1,\alpha}$, and the elementary divisors corresponding to the eigenvalues $\lambda_j(J_1)$ of the matrix J_1 are linear, i.e., $J_1 = \operatorname{diag}\{e^{\delta_1 i}, \ldots, e^{\delta_\alpha i}\}$, $\delta_j \in \mathbb{R}$, $j = \overline{1,\alpha}$; $J_2 \in M_{n-\alpha}(\mathbb{C})$, $|\lambda_j(J_2)| < 1$, $j = \overline{1, n-\alpha}$. By the structure of the matrix J_1, we have $J_1^*J_1 = I \in M_\alpha(\mathbb{C})$.

Let $H \in M_{n-\alpha}(\mathbb{C})$ be an arbitrary Hermitian positive definite matrix, i.e., $H = H^* > 0$. Set $T = \operatorname{diag}\{0, H\} \in M_n(\mathbb{C})$, where $0 \in M_\alpha(\mathbb{C})$. Then $T^* = T \geq 0$. Set $W = S^*TS$. Then $W^* = W \geq 0$ also. Since $|\lambda_j(J_2)| < 1$, there exists a matrix $R \in M_{n-\alpha}(\mathbb{C})$, $R^* = R > 0$, such that $J_2^*RJ_2 - R = -H < 0$. Set $L = \operatorname{diag}\{I, R\} \in M_n(\mathbb{C})$, where $I \in M_\alpha(\mathbb{C})$. Then $L^* = L > 0$. Set $Q = S^*LS$. Then we have $Q^* = Q > 0$ also. Thus,

$$F^*QF - Q = (S^{-1}JS)^*S^*LS(S^{-1}JS) - S^*LS =$$
$$S^*(J^*LJ - L)S = S^*\operatorname{diag}\{J_1^*J_1 - I, J_2^*RJ_2 - R\}S =$$
$$= S^*\operatorname{diag}\{0, -H\}S = S^*(-T)S = -W \leq 0.$$

The lemma is proved. □

Remark 3. Suppose that $F \in M_n(\mathbb{R})$, $y \in \mathbb{R}^n$ in (4), and (4) is Lyapunov stable. Then it follows from Lemma 2, Remark 1 and equality (3) that there exists a $P \in M_n(\mathbb{R})$, $P = P^T > 0$ such that $F^TPF - P \leq 0$. The matrix P is defined by the equality $P = \operatorname{Re} Q$, where Q is constructed by Lemma 2.

The converse to Lemma 2 is obviously true. □

Suppose that the system (2) is periodic, i.e., there exists an $\omega \in \mathbb{N}$ such that $A(t+\omega) = A(t)$ for all $t \in \mathbb{Z}$. Suppose that there exists a quadratic form $V(t,x) = x^*R(t)x$, $x \in \mathbb{K}^n$, $R(t) \in M_n(\mathbb{K})$, satisfying the following conditions for all $t \in \mathbb{Z}$:

$$R(t+\omega) = R(t), \qquad R^*(t) = R(t) > 0, \tag{5}$$
$$(W_AR)(t) := A^*(t)R(t+1)A(t) - R(t) \leq 0. \tag{6}$$

Conditions (5), (6) are sufficient for Lyapunov (non-asymptotic) stability of the system (2) due to Lyapunov Theorem for discrete-time systems (but are not sufficient for asymptotic stability of the system (2); example: $A(t) \equiv I$, $R(t) \equiv I$, $t \in \mathbb{Z}$). In fact, these conditions are also necessary if the periodic system (2) is reducible, i.e., can be transformed into a time-invariant one by a change of basis.

Lemma 4. Suppose $\mathbb{K} = \mathbb{C}$, the system (2) is ω-periodic and reducible. Then the implication $1 \Rightarrow 2$ holds for the following statements.

1. The system (2) is Lyapunov stable.

2. There exists a $R(t)$ satisfying conditions (5), (6) for all $t \in \mathbb{Z}$.

Proof. Suppose the system (2) is ω-periodic and reducible. Then there exists an ω-periodic nondegenerate (complex, in general) matrix $L(t)$ such that the state-space transformation $y(t) = L(t)x(t)$ reduces the system (2) into a system (4) with a constant matrix $F \in M_n(\mathbb{C})$. That transformation is the Lyapunov transformation, hence, it preserves the property of stability, i.e., the reduced system (4) is Lyapunov stable. The matrices of the systems (2) and (4) are related by the equality

$$A(t) = L^{-1}(t+1)FL(t). \tag{7}$$

By Lemma 2, there exists a $Q = Q^* > 0$ such that $F^*QF - Q \leq 0$. Hence, for all $t \in \mathbb{Z}$

$$L^*(t)(F^*QF - Q)L(t) \leq 0. \tag{8}$$

Let us construct the matrix

$$R(t) = L^*(t)QL(t). \tag{9}$$

Then $R(t) \in M_n(\mathbb{C})$, $R(t+\omega) \equiv R(t)$, $R(t) = R^*(t) > 0$, $t \in \mathbb{Z}$. Next, using (7), (9), and (8), we obtain $(W_AR)(t) =$

$$\begin{aligned} &= A^*(t)R(t+1)A(t) - R(t) = (L^{-1}(t+1)FL(t))^* \cdot \\ &\cdot L^*(t+1)QL(t+1)(L^{-1}(t+1)FL(t)) - \\ &- L^*(t)QL(t) = L^*(t)(F^*QF - Q)L(t) \leq 0. \end{aligned} \tag{10}$$

The lemma is proved. □

Lemma 5. Suppose $\mathbb{K} = \mathbb{R}$, the system (2) is ω-periodic and reducible. Then the implication $1 \Rightarrow 2$ holds for the following statements.

1. The system (2) is Lyapunov stable.

2. There exists a $P(t) \in M_n(\mathbb{R})$ satisfying the following conditions for all $t \in \mathbb{Z}$:

$$P(t+\omega) = P(t), \qquad P^T(t) = P(t) > 0, \tag{11}$$
$$(W_AP)(t) := A^T(t)P(t+1)A(t) - P(t) \leq 0. \tag{12}$$

Proof. Let us construct the matrix $R(t) \in M_n(\mathbb{C})$ for the matrix $A(t) \in M_n(\mathbb{R})$ by using Lemma 4. Set $P(t) = \operatorname{Re} R(t)$. Then $P(t+\omega) \equiv P(t)$, $P(t) = P^T(t) > 0$, $t \in \mathbb{Z}$. Next, $(W_AP)(t) = \operatorname{Re}(W_AR)(t)$. It follows from (3) that for any $x \in \mathbb{R}^n$ the equality

$$x^T(W_AP)(t)x = x^T(W_AR)(t)x \tag{13}$$

holds. Now, $(W_AP)(t) \leq 0$ follows from (10) and (13). □

Lemma 4 is similar to (Zaitsev, 2013c, Theorem 7) and (Zaitsev, 2013d, Theorem 4) for continuous-time systems.

Remark 6. In contrast to continuous-time systems, not every discrete-time periodic system (2) is reducible. The necessary and sufficient conditions for reducibility of a periodic discrete-time linear system (2) into a time-invariant one by a Lyapunov–Floquet transformation are given in Dooren and Sreedhar (1994). In particular, if $\det A(t) \neq 0$, $t \in \mathbb{Z}$, then the periodic system (2) is reducible. If the periodic system (2) is not reducible then the assertion of Lemma 4 and Lemma 5 is questionable.

3. UNIFORM GLOBAL ASYMPTOTIC STABILIZATION OF DISCRETE-TIME PERIODIC BILINEAR SYSTEMS

Suppose that the system (1) is periodic, i.e., there exists an $\omega \in \mathbb{N}$ such that $A(t+\omega) = A(t)$, $B_i(t+\omega) = B_i(t)$, $i = \overline{1,r}$, for all $t \in \mathbb{Z}$. For the corresponding free system (i.e., (1) with $u(t) \equiv 0$)

$$x(t+1) = A(t)x(t), \quad t \in \mathbb{Z}, \quad x \in \mathbb{R}^n, \tag{14}$$

we suppose that there exists a quadratic form

$$V(t,x) = x^T P(t)x, \quad x \in \mathbb{R}^n, \quad P(t) \in M_n(\mathbb{R}), \tag{15}$$

satisfying conditions (11), (12) for all $t \in \mathbb{Z}$. Set

$$G(t,x) = I + \frac{1}{2}B^T(t,x)P(t+1)B(t,x). \tag{16}$$

Then $G(t,x) \in M_r(\mathbb{R})$, $G(t,x) = G(t+\omega,x)$, $G(t,x) = G^T(t,x) \geq I > 0$, and $G^{-1}(t,x)$ is defined for all $t \in \mathbb{Z}$, $x \in \mathbb{R}^n$. Let us construct the control function

$$\widehat{u}(t,x) = -G^{-1}(t,x)B^T(t,x)P(t+1)A(t)x. \tag{17}$$

Then $\widehat{u}(t+\omega,x) = \widehat{u}(t,x)$, $t \in \mathbb{Z}$, $x \in \mathbb{R}^n$, and

$$\big(B^T(t,x)P(t+1)A(t)x\big)^T = -\widehat{u}^T(t,x)G(t,x). \tag{18}$$

Substituting

$$u(t) = \widehat{u}(t,x(t)) \tag{19}$$

into the system (1), we obtain the closed-loop system

$$x(t+1) = C(t,x(t)) = A(t)x(t) + B(t,x(t))\widehat{u}(t,x(t)). \tag{20}$$

We have $C(t+\omega,x) = C(t,x)$, $t \in \mathbb{Z}$, $x \in \mathbb{R}^n$. Considering the difference $\Delta V(t,x(t)) = V(t+1,x(t+1) - V(t,x(t))$ of the Lyapunov function (15) along the trajectory of (20), we obtain

$$\Delta V(t,x(t)) = x^T(t)\big(A^T(t)P(t+1)A(t) - \\ - P(t)\big)x(t) + 2\mu(t,x(t))\widehat{u}(t,x(t)),$$

where

$$\mu(t,x) = x^T A^T(t)P(t+1)B(t,x) + \\ + \frac{1}{2}\widehat{u}^T(t,x)B^T(t,x)P(t+1)B(t,x). \tag{21}$$

Substituting (18) for the first summand in (21) and taking into account (16), we obtain that $\mu(t,x) = -\widehat{u}^T(t,x)$. Hence,

$$\Delta V(t,x(t)) = x^T(t)\big(A^T(t)P(t+1)A(t) - \\ - P(t)\big)x(t) - 2\widehat{u}^T(t,x(t))\widehat{u}(t,x(t)). \tag{22}$$

Thus, by (12), we obtain that $\Delta V(t,x(t)) \leq 0$. Hence, the origin of the system (20) is Lyapunov stable. Consider the set

$$E(V) = \{(t,x) \in \mathbb{Z}\times\mathbb{R}^n : \Delta V(t,x) = 0\}.$$

By (22) and (12), the set $E(V)$ coincides with

$$\widehat{E}(V) = \{(t,x) \in \mathbb{Z}\times\mathbb{R}^n : x^T(W_AP)(t)x = |\widehat{u}(t,x)| = 0\}.$$

Set $\Omega_0(V) = \{(t,x) \in \mathbb{Z}\times\mathbb{R}^n : x^T(W_AP)(t)x = 0\}$, $S_0(V) = \{(t,x) \in \mathbb{Z}\times\mathbb{R}^n : x^TA^T(t)P(t+1)B(t,x) = 0\}$, $E_0(V) = \Omega_0(V) \cap S_0(V)$. By (17), $\widehat{E}(V) = E_0(V)$. Let $M(V)$ be the largest positive invariant set of the system (20) relative to $E(V)$, i.e., $M(V)$ is the union of all semi-trajectories $x(t)$, $t \geq t_0$ ($t_0 \in \mathbb{Z}$), of the system (20) such that $(t,x(t)) \in E(V)$ for all $t \geq t_0$. Then we have $0 \in M(V)$ because $\xi_0(t) \equiv 0$, $t \geq t_0$, is a solution of (20) and $\Delta V(t,0) \equiv 0$, $t \geq t_0$. If $M(V) = \{0\}$, then the origin of the system (20) is uniformly globally asymptotically (UGA) stable due to the Krasovsky–La Salle invariance principle for discrete-time periodic systems.

Denote by $\xi(t) = \xi(t,t_0,x_0)$, $t \geq t_0$ ($t_0 \in \mathbb{Z}$), a solution of the system (20) such that $\xi(t_0) = x_0$. Suppose that $\xi(t) \in M(V)$, $t \geq t_0$. Since $E(V) = \widehat{E}(V) = E_0(V)$, we obtain

$$\xi^T(t)(W_AP)(t)\xi(t) = 0, \quad t \geq t_0, \tag{23}$$

$$\xi^T(t)A^T(t)P(t+1)B(t,\xi(t)) = 0, \quad t \geq t_0, \tag{24}$$

and $\widehat{u}(t,\xi(t)) = 0$, $t \geq t_0$. By (20), $\xi(t+1) = A(t)\xi(t)$, $t \geq t_0$, i.e., $\xi(t)$ is a solution of the free system (14). Hence, $\xi(t) = X(t,t_0)x_0$. Substituting this equality in (23), (24), we obtain the equalities

$$x_0^TX^T(t,t_0)(W_AP)(t)X(t,t_0)x_0 \equiv 0, \quad t \geq t_0, \tag{25}$$

$$x_0^TX^T(t+1,t_0)P(t+1)B_i(t)X(t,t_0)x_0 \equiv 0, \\ t \geq t_0, \quad i = \overline{1,r}. \tag{26}$$

Let $M_0(V)$ be the largest positive invariant set of the free system (14) relative to $E_0(V)$. Thus, we obtain that $\xi(t) \in M_0(V)$, $t \geq t_0$. It follows that $M(V) \subset M_0(V)$. Therefore if $M_0(V) = \{0\}$, then $M(V) = \{0\}$. The condition $M_0(V) = \{0\}$ means that for any $t_0 \in \mathbb{Z}$ the identities (25), (26) hold only if $x_0 = 0$. In the last sentence, the phrase "for any $t_0 \in \mathbb{Z}$" can be replaced by the phrase "for some $t_0 \in \mathbb{Z}$". Let us prove this fact.

Lemma 7. The following statements are equivalent.

1. For any $t_0 \in \mathbb{Z}$ the identities (25), (26) hold only if $x_0 = 0$.
2. For some $t_0 \in \mathbb{Z}$ the identities (25), (26) hold only if $x_0 = 0$.

Proof. The implication $1 \Rightarrow 2$ is obvious. Let us prove the implication $2 \Rightarrow 1$. Suppose that for some $t_0 \in \mathbb{Z}$ the identities (25), (26) hold only if $x_0 = 0$. Consider an arbitrary $t_1 \in \mathbb{Z}$. There exists a $k \in \mathbb{N}$ such that $t_0 + k\omega \geq t_1$. Suppose that the identities

$$x_1^TX^T(t,t_1)(W_AP)(t)X(t,t_1)x_1 \equiv 0, \tag{27}$$

$$x_1^TX^T(t+1,t_1)P(t+1)B_i(t)X(t,t_1)x_1 \equiv 0, \ i = \overline{1,r}, \tag{28}$$

hold for $t \geq t_1$. Then (27), (28) hold for all $t \geq t_0 + k\omega$ as well. Denote $x_2 = X(t_0 + k\omega, t_1)x_1$. It follows from (27), (28) that the identities

$$x_2^TX^T(t,t_0+k\omega)(W_AP)(t)X(t,t_0+k\omega)x_2 \equiv 0, \tag{29}$$

$$x_2^TX^T(t+1,t_0+k\omega)P(t+1)\cdot \\ \cdot B_i(t)X(t,t_0+k\omega)x_2 \equiv 0, \quad i = \overline{1,r}, \tag{30}$$

hold for $t \geq t_0 + k\omega$. Denote $s = t - k\omega$. Then $t = s + k\omega$. The inequality $t \geq t_0 + k\omega$ is equivalent to $s \geq t_0$. We have $X(t,t_0+k\omega) = X(s+k\omega,t_0+k\omega) = X(s,t_0)$ by ω-periodicity of the system (14). Next, $(W_AP)(t) = (W_AP)(s+k\omega) = (W_AP)(s)$ due to ω-periodicity of $A(t)$ and $P(t)$. Thus, (29), (30) mean that

$$x_2^TX^T(s,t_0)(W_AP)(s)X(s,t_0)x_2 \equiv 0,$$

$$x_2^TX^T(s+1,t_0)P(s+1)B_i(s)X(s,t_0)x_2 \equiv 0, \quad i = \overline{1,r},$$

for $s \geq t_0$. By assumption, it follows that $x_2 = 0$, i.e.,

$$X(t_0 + k\omega, t_1)x_1 = 0. \tag{31}$$

Consider the equalities (27) at $t = t_1, t_1+1, \ldots, t_0+k\omega-1$. From these equalities, taking into account the definition of W_AP, we obtain

$$x_1^TP(t_1)x_1 = x_1^TX^T(t_1+1,t_1)P(t_1+1)X(t_1+1,t_1)x_1 = \\ = x_1^TX^T(t_1+2,t_1)P(t_1+2)X(t_1+2,t_1)x_1 = \ldots = \\ = x_1^TX^T(t_0+k\omega,t_1)P(t_0+k\omega)X(t_0+k\omega,t_1)x_1. \tag{32}$$

It follows from (32) and (31) that $x_1^TP(t_1)x_1 = 0$. Since $P(t) > 0$, we have $x_1 = 0$. The lemma is proved. $\square$

Thus, the following theorem is proved.

Theorem 8. Let the system (1) be ω-periodic. Suppose that there exists a matrix $P(t)$ satisfying conditions (11), (12) for all $t \in \mathbb{Z}$. Suppose that for some $t_0 \in \mathbb{Z}$ the identities (25), (26) hold only if $x_0 = 0$. Then the state feedback control (19) UGA stabilizes the origin of the system (1).

Theorem 8 is similar to (Zaitsev, 2013c, Theorem 8) and (Zaitsev, 2013d, Theorem 5) for continuous-time systems.

Corollary 9. Let the system (1) be ω-periodic, the system (14) be reducible and Lyapunov stable, and for some $t_0 \in \mathbb{Z}$ the identities (25), (26) hold only if $x_0 = 0$, where the matrix $P(t)$ is constructed by Lemma 4. Then the state feedback control (19) UGA stabilizes the origin of the system (1).

Corollary 9 follows from Lemma 4 and Theorem 8.

Let us construct the following matrices: $N_1(\tau,x) = B(\tau,x)$, $N_{i+1}(\tau,x) = [A(\tau+i)N_i(\tau,x), B(\tau+i, X(\tau+i,\tau)x]$, $i \geq 1$. Note that $N_i(\tau+\omega,x) = N_i(\tau,x) \in M_{n,ir}(\mathbb{R})$ for any $i \in \mathbb{N}$, $\tau \in \mathbb{Z}$, $x \in \mathbb{R}^n$.

Theorem 10. Let the system (1) be ω-periodic. Suppose that there exists a matrix $P(t)$ satisfying conditions (11), (12) for all $t \in \mathbb{Z}$. Suppose that the following condition holds:

$$\exists t_0 \in \mathbb{Z} \ \ \forall x \in \mathbb{R}^n \setminus \{0\} \ \ \exists \nu \geq 1 \ \ \operatorname{rank} N_\nu(t_0,x) = n. \quad (33)$$

Then the state feedback control (19) UGA stabilizes the origin of the system (1).

Proof. Let us prove that $M_0(V) = 0$ under the assumptions of the theorem. Then, by the proof of Theorem 8, the theorem will be proved. We prove the theorem by contradiction. Suppose $M_0(V) \neq 0$. Then there exist $t_1 \in \mathbb{Z}$ and $x_1 \in \mathbb{R}^n$, $x_1 \neq 0$, such that the solution $\xi(t) = \xi(t,t_1,x_1)$ of the free system (14) satisfies the condition $\xi(t) \in M_0(V)$ for all $t \geq t_1$. Hence, identities (27), (28) hold for $t \geq t_1$. Let us show that $\xi(t,t_1,x_1) \neq 0$ for any $t \geq t_1$. Suppose that $\xi(t_2,t_1,x_1) = 0$ for some $t_2 \geq t_1$. Hence, $X(t_2,t_1)x_1 = 0$. By (27) at $t = t_1, t_1+1, \ldots, t_2-1$, we obtain similarly to (32):

$$x_1^T P(t_1)x_1 = x_1^T X^T(t_1+1,t_1)P(t_1+1)X(t_1+1,t_1)x_1 = \\ = \ldots = x_1^T X^T(t_2,t_1)P(t_2)X(t_2,t_1)x_1 = 0.$$

Since $P(t) > 0$, we have $x_1 = 0$. This is contradiction. Hence, $\xi(t,t_1,x_1) \neq 0$, $t \geq t_1$. Let us construct the number $t_0 \in \mathbb{Z}$ from condition (33). By periodicity of $N_i(\tau,x)$, one can assume without loss of generality that $t_0 > t_1$. Set $x_0 = \xi(t_0,t_1,x_1)$. Hence,

$$\xi(t,t_0,x_0) \neq 0, \quad t \geq t_0. \quad (34)$$

Let us construct for $x_0 \neq 0$ the number $\nu \geq 1$ from condition (33) such that $\operatorname{rank} N_\nu(t_0,x_0) = n$. Since $\xi(t,t_0,x_0) \in M_0(V)$ for all $t \geq t_0$, the equalities (25), (26) hold.

Consider the function $\varphi(t,x) = x^T(W_A P)(t)x$. We have $\Omega_0(V) = \{(t,x) \in \mathbb{Z}\times\mathbb{R}^n : \varphi(t,x) = 0\}$. The function $\varphi(t,x)$ attains its maximum at any point $(\widetilde{t},\widetilde{x}) \in \Omega_0(V)$ because $\varphi(t,x) \leq 0$ for all $(t,x) \in \mathbb{Z}\times\mathbb{R}^n$, by (12). Consequently, $(\partial\varphi/\partial x)(\widetilde{t},\widetilde{x}) = 0$ for any $(\widetilde{t},\widetilde{x}) \in \Omega_0(V)$. We have $(\partial\varphi/\partial x)(t,x) = 2x^T(W_A P)(t)$. Therefore for any $m \in \mathbb{N}$ and for any function $(s,y) \to z(s,y) \in M_{n,m}(\mathbb{R})$ the equality

$$\widetilde{x}^T\big(A^T(\widetilde{t})P(\widetilde{t}+1)A(\widetilde{t}) - P(\widetilde{t})\big)z(s,y) = 0, \\ (\widetilde{t},\widetilde{x}) \in \Omega_0(V), \quad s \in \mathbb{Z}, \quad y \in \mathbb{R}^n, \quad (35)$$

holds. Equality (26) for $t = t_0$ implies that

$$x_0^T A^T(t_0)P(t_0+1)B_i(t_0)x_0 = 0, \quad i = \overline{1,r}.$$

This means that the row vector $x_0^T A^T(t_0)P(t_0+1)$ is orthogonal to the columns of the matrix $N_1(t_0,x_0)$.

Let us prove, by induction, the following assertion $(\mathcal{A})$: for all $k \in \mathbb{N}$ the row vector $x_0^T X^T(t_0+k,t_0)P(t_0+k)$ is orthogonal to the columns of the matrix $N_k(t_0,x_0)$. The basis for $k=1$ is proved. Assume $(\mathcal{A})$ holds for $k=i$, i.e.,

$$x_0^T X^T(t_0+i,t_0)P(t_0+i)N_i(t_0,x_0) = 0. \quad (36)$$

By (25), the inclusion $(t,\xi(t,t_0,x_0)) \in \Omega_0(V)$ holds for all $t \geq t_0$. Substituting $i\cdot r$ for m, t_0 for s, x_0 for y, t_0+i for $\widetilde{t}$, $X(t_0+i,t_0)x_0$ for $\widetilde{x}$, $N_i(t_0,x_0)$ for $z(s,y)$ in (35), we get

$$x_0^T X^T(t_0+i,t_0)\big(A^T(t_0+i)P(t_0+i+1)A(t_0+i) - \\ - P(t_0+i)\big)N_i(t_0,x_0) = 0.$$

Taking into account (36), we obtain

$$x_0^T X^T(t_0+i+1,t_0)P(t_0+i+1)\cdot \\ \cdot A(t_0+i)N_i(t_0,x_0) = 0. \quad (37)$$

It follows from (37) and (26) at $t = t_0+i$ that

$$x_0^T X^T(t_0+i+1,t_0)P(t_0+i+1)\cdot \\ \cdot [A(t_0+i)N_i(t_0,x_0), B(t_0+i, X(t_0+i,t_0)x_0] = 0$$

Thus, assertion $(\mathcal{A})$ holds for $k = i+1$. By induction, assertion $(\mathcal{A})$ holds for all $k \in \mathbb{N}$. In particular, for $k=\nu$, we have

$$x_0^T X^T(t_0+\nu,t_0)P(t_0+\nu)N_\nu(t_0,x_0) = 0.$$

Since $\operatorname{rank} N_\nu(t_0,x_0) = n$, we obtain

$$x_0^T X^T(t_0+\nu,t_0)P(t_0+\nu) = 0.$$

Since $P(t) > 0$, we have $X(t_0+\nu,t_0)x_0 = 0$. This contradicts (34). The theorem is proved. □

Let us construct the subspaces

$$\Gamma(t,t_0) = \operatorname{span}\{B_i(t)A(t-1)\cdot\ldots\cdot A(t_0), \\ A(t)B_i(t-1)A(t-2)\cdot\ldots\cdot A(t_0), \ldots, \\ A(t)A(t-1)\cdot\ldots\cdot A(t_0+1)B_i(t_0), \quad i=\overline{1,r}\} \subset M_n(\mathbb{R})$$

and

$$\Delta(t,t_0,x) = \Gamma(t,t_0)x \subset \mathbb{R}^n,$$

where $t_0, t \in \mathbb{Z}$, $t \geq t_0$, $x \in \mathbb{R}^n$. Obviously, the linear span of the column vectors of the matrix $N_\nu(t_0,x)$ coincides with the space $\Delta(t_0+\nu-1,t_0,x)$.

Theorem 11. Let the system (1) be ω-periodic. Suppose that there exists a matrix $P(t)$ satisfying conditions (11), (12) for all $t \in \mathbb{Z}$. Suppose that at least one of the following conditions holds:

(*i*) $\exists t_0 \in \mathbb{Z} \quad \forall x \neq 0 \quad \exists t \geq t_0 \quad \dim\Delta(t,t_0,x) = n$;

(*ii*) $\exists t_0 \in \mathbb{Z} \quad \exists t \geq t_0 \quad \Gamma(t,t_0) = M_n(\mathbb{R})$.

Then the state feedback control (19) UGA stabilizes the origin of the system (1).

Theorem 11 follows from Theorem 10 because condition (*i*) is equivalent to (33) and (*ii*) is sufficient for (*i*). Theorem 11 is similar to (Zaitsev, 2013d, Theorem 7.1) for continuous-time systems.

Corollary 12. Let the system (1) be ω-periodic, the system (14) be reducible and Lyapunov stable, and at least one of conditions (*i*), (*ii*) holds. Then the state feedback control (19) UGA stabilizes the origin of the system (1); here $P(t)$ is a matrix satisfying conditions (11), (12).

Corollary 12 follows from Lemma 4 and Theorem 11. Corollaries 9 and 12 are similar to (Zaitsev, 2013d, Corollary 1) for continuous-time systems.

4. CONSISTENT SYSTEMS

The system (1) is said to be *consistent on the interval* $[t_0, t_1)$ $(t_1 > t_0)$ if for every matrix $G \in M_n(\mathbb{R})$ there exists a control function $\widetilde{u}(t) = (\widetilde{u}_1(t), \ldots, \widetilde{u}_r(t)) \in \mathbb{R}^r$, $t \in [t_0, t_1)$, such that the solution of the matrix equation

$$Z(t+1) = A(t)Z(t) + (\widetilde{u}_1(t)B_1(t) + \ldots + \widetilde{u}_r(t)B_r(t))X(t, t_0), \quad t \in \mathbb{Z},$$

with the initial condition $Z(t_0) = 0$ satisfies the condition $Z(t_1) = G$.

The definition of consistency was introduced for the discrete-time system (1) and for a linear system closed-loop by a linear output feedback in Zaitsev (2014) similarly to the definition for continuous-time systems (see references in Zaitsev (2014)). This notion was studied in Zaitsev (2014), Zaitsev (2015a) in the connection with the eigenvalue assignment problem for time-invariant bilinear discrete-time systems.

Theorem 13. The system (1) is consistent on $[t_0, \vartheta) \iff \Gamma(\vartheta - 1, t_0) = M_n(\mathbb{R})$.

The proof is given in (Zaitsev, 2014, Equality (8)). Theorem 13 is similar to (Zaitsev, 2013c, Proposition 13) for continuous-time systems.

Thus, if the system (1) is consistent on some $[t_0, \vartheta)$ then condition (ii) of Theorem 11 holds, where $t = \vartheta - 1$. Therefore, we have the following statements.

Theorem 14. Let the system (1) be ω-periodic. Suppose that there exists a matrix $P(t)$ satisfying conditions (11), (12) for all $t \in \mathbb{Z}$. Suppose that the system (1) is consistent on some $[t_0, \vartheta)$. Then the state feedback control (19) UGA stabilizes the origin of the system (1).

Theorem 14 follows from Theorems 11 and 13.

Corollary 15. Let the system (1) be ω-periodic and consistent on some $[t_0, \vartheta)$, and the system (14) be reducible and Lyapunov stable. Then the state feedback control (19) UGA stabilizes the origin of the system (1); here $P(t)$ is a matrix satisfying conditions (11), (12).

Corollary 15 follows from Lemma 4 and Theorem 14. Corollary 15 is similar to (Zaitsev, 2013c, Corollary 14) for continuous-time systems.

5. TIME-INVARIANT SYSTEMS

Suppose the system (1) is time-invariant, i.e. $A(t) \equiv A$, $B_i(t) \equiv B_i$, $i = \overline{1,r}$. Then the free system (14) is time-invariant (hence, reducible) and Lyapunov stability of the free system (14) is equivalent to existence of a stationary matrix $P(t) \equiv P$ satisfying conditions (11), (12).

We obtain that Theorem 8 and Corollary 9 coincides with (Byrnes et al., 1993, Theorem 2.1), where $g(x) = B(x)$; Theorem 11 and Corollary 12 with (i) coincides with (Zaitsev, 2015b, Theorem 7); Theorem 11 and Corollary 12 with (ii) coincides with (Zaitsev, 2015b, Theorem 8); Theorem 14 and Corollary 15 coincides with (Zaitsev, 2015b, Theorem 9).

6. EXAMPLE

Consider the discrete-time bilinear system

$$x(t+1) = (A(t) + u(t)B(t))x(t), \quad (t, x, u) \in \mathbb{Z} \times \mathbb{R}^2 \times \mathbb{R}; \tag{38}$$

$$A(t) := \begin{Vmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{Vmatrix}, \quad t \in \mathbb{Z}; \tag{39}$$

$$\begin{gathered} B(0) := \begin{Vmatrix} 1 & -2 \\ 1 & -1 \end{Vmatrix}, \quad B(1) := \begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix}, \\ B(2) := \begin{Vmatrix} -1 & 1 \\ -2 & 1 \end{Vmatrix}, \quad B(3) := \begin{Vmatrix} 1 & 1 \\ 1 & 2 \end{Vmatrix}; \\ B(t+4) := B(t), \quad t \in \mathbb{Z}, \quad t \geq 0; \\ B(t) := B(t+4), \quad t \in \mathbb{Z}, \quad t < 0. \end{gathered} \tag{40}$$

This system is ω-periodic with the period $\omega = 4$. The matrix $P = I$ satisfies conditions (11), (12). Suppose $t_0 = 0, t_1 = 2$. Let us construct $\Delta(t_1, t_0, x_0) \subset \mathbb{R}^2$. We have $B(2)A(1)A(0) = \begin{Vmatrix} 1 & 1 \\ 1 & 2 \end{Vmatrix}$, $A(2)B(1)A(0) = \begin{Vmatrix} -2 & -1 \\ -1 & -1 \end{Vmatrix}$, $A(2)A(1)B(0) = \begin{Vmatrix} 1 & -1 \\ -1 & 2 \end{Vmatrix}$. Suppose $x_0 = \mathrm{col}\,(\alpha, \beta)$. Then $\Delta(t_1, t_0, x_0) = \mathrm{span}\,\{z_1, z_2, z_3\} \in \mathbb{R}^2$, where

$$\begin{gathered} z_1 = \mathrm{col}\,(\alpha + \beta, \alpha + 2\beta), \\ z_2 = \mathrm{col}\,(-2\alpha - \beta, -\alpha - \beta), \\ z_3 = \mathrm{col}\,(\alpha - \beta, -\alpha + 2\beta). \end{gathered}$$

We have

$$\begin{gathered} d_1 := \det[z_1, z_2] = \alpha^2 + 3\alpha\beta + \beta^2, \\ d_2 := \det[z_1, z_3] = -2\alpha^2 + 4\beta^2. \end{gathered}$$

The equalities $d_1 = d_2 = 0$ holds only if $\alpha = \beta = 0$. Hence, if $x_0 \neq 0$ then $\mathrm{span}\,\Delta(2, 0, x_0) = \mathbb{R}^2$. Thus, condition (i) of Theorem 11 is satisfied. Therefore, the system (38), (39), (40) is UGA stabilizable by the feedback control $u(t) = \widehat{u}(t, x(t))$. Here the feedback control law has the form (17):

$$\begin{gathered} \widehat{u}(0, x) = -2(x_1^2 - x_1x_2 - x_2^2)/(2 + 2x_1^2 - 6x_1x_2 + 5x_2^2), \\ \widehat{u}(1, x) = -2(x_1^2 - x_1x_2 - x_2^2)/(2 + 5x_1^2 + 6x_1x_2 + 2x_2^2), \\ \widehat{u}(2, x) = -2(x_1^2 + x_1x_2 - x_2^2)/(2 + 5x_1^2 - 6x_1x_2 + 2x_2^2), \\ \widehat{u}(3, x) = -2(-x_1^2 - x_1x_2 + x_2^2)/(2 + 2x_1^2 + 6x_1x_2 + 5x_2^2), \\ \widehat{u}(t+4, x) = \widehat{u}(t, x), \quad t \in \mathbb{Z}, \quad t \geq 0; \\ \widehat{u}(t, x) = \widehat{u}(t+4, x), \quad t \in \mathbb{Z}, \quad t < 0. \end{gathered}$$

Remark 16. Note that the system (38), (39), (40) cannot be stabilizable by any non-feedback control $u(t) = \widetilde{u}(t)$ depending only on t. In fact, the system (38) with $u(t) = \widetilde{u}(t)$ is linear in x. One can check that $\det(A(t) + \widetilde{u}(t)B(t)) = 1 + (\widetilde{u}(t))^2$ for all $t \in \mathbb{Z}$. Hence, for the transition matrix $X_{\widetilde{u}}(t, s)$ of the system (38) with $u(t) = \widetilde{u}(t)$, we have

$$\det X_{\widetilde{u}}(t, 0) = \prod_{i=0}^{t-1} (1 + (\widetilde{u}(i))^2) \geq 1$$

for all $t > 0$, i.e., $\det X_{\widetilde{u}}(t, 0) \not\to 0$ as $t \to +\infty$. Thus the system (38) with $u(t) = \widetilde{u}(t)$ is not asymptotically stable.

7. CONCLUSION

We have extended the results on UGA stabilization of discrete-time autonomous bilinear homogeneous systems obtained in Byrnes et al. (1993), Byrnes and Lin (1994), Lin and Byrnes (1994), Zaitsev (2015b) to non-autonomous periodic discrete-time bilinear homogeneous systems. It seems true that one can extend the results of Byrnes et al. (1993), Byrnes and Lin (1994), Lin and Byrnes (1994) on UGA stabilization of discrete-time affine and general nonlinear autonomous systems to periodic systems.

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