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SPECTRAL SET OF A LINEAR SYSTEM WITH DISCRETE TIME

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Abstract. Fix a certain class of perturbations of the coefficient matrix $A(\cdot)$ of a discrete linear homogeneous system of the form

$$x(m+1) = A(m)x(m), \quad m \in \mathbb{N}, \quad x \in \mathbb{R}^n,$$

where the matrix $A(\cdot)$ is completely bounded on $\mathbb{N}$. The spectral set of this system corresponding to a given class of perturbations is the collection of complete spectra of the Lyapunov exponents of perturbed systems when perturbations runs over the whole class considered. In this paper, we examine the class $\mathcal{R}$ of multiplicative perturbations of the form

$$y(m+1) = A(m)R(m)x(m), \quad m \in \mathbb{N}, \quad y \in \mathbb{R}^n,$$

where the matrix $R(\cdot)$ is completely bounded on $\mathbb{N}$. We obtain conditions that guarantee the coincidence of the spectral set $\lambda(\mathcal{R})$ corresponding to the class $\mathcal{R}$ with the set of all nondecreasing n-tuples of n numbers.

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Let $\mathbb{R}^n$ be the Euclidean space of dimension n with the standard norm $\|\cdot\|$. We denote by $M_n(\mathbb{R})$ the space of all real $(n \times n)$-matrices with the spectral norm, i.e., the operator norm induced in $M_n(\mathbb{R})$ by the Euclidean norm in $\mathbb{R}^n$. Let $E \in M_n(\mathbb{R})$ be the identity matrix. We also denote by $\mathbb{R}^n_{\leq}$ the set of all nondecreasing n-tuples of real numbers. For a fixed tuple $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n_{\leq}$ and arbitrary $\delta > 0$, we denote by $\mathcal{O}_\delta(\mu)$ the set of all tuples $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n_{\leq}$ such that

$$\max_{j=1,\dots,n} |\nu_j - \mu_j| < \delta.$$

Thus, $\mathcal{O}_\delta(\mu)$ is the δ-neighborhood of the tuple μ in the set $\mathbb{R}^n_{\leq}$.

Consider the linear homogeneous system with discrete time

$$x(m+1) = A(m)x(m), \tag{1}$$

where the argument m runs over the set $\mathbb{N}$ of all natural numbers. Assume that the unknown function x takes its values in $\mathbb{R}^n$ and for each m the coefficient $A(m)$ belongs to the space $M_n(\mathbb{R})$. In the sequel, we also assume that the function $A(\cdot)$ is *completely bounded* on $\mathbb{N}$ (see [3]), i.e., for each m, there exists $A^{-1}(m)$, and, moreover, there exists a such that

$$\sup_{m \in \mathbb{N}} \big(\|A(m)\| + \|A^{-1}(m)\|\big) \leq a.$$

For an arbitrary nontrivial solution $x(\cdot)$ of the system (1), we defined its *Lyapunov exponent* by the equality

$$\lambda[x] \doteq \varlimsup_{m \to \infty} \frac{1}{m} \ln \|x(m)\|$$

and denote by Λ the *spectrum of Lyapunov exponents* of the system (1), i.e., the set of all $\lambda \in \mathbb{R}$ for each of which there exists a nontrivial solution $x(\cdot)$ of the system (1) with exponent λ. It is known (see [4, pp. 51–52]) that the set Λ consists of no more than n different numbers. Let $\Lambda = \{\Lambda_1, \dots, \Lambda_p\}$,

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where $\Lambda_1 < \cdots < \Lambda_p$, $p \leq n$. Assume that the Lyapunov exponent of the trivial solution of the system (1) is equal to $-\infty$.

For each $j \in \{1, \ldots, p\}$, consider the set E_j of all solutions of the system (1) whose exponents do not exceed Λ_j. Assume that the set E_0 consists of the trivial solution of the system (1). Then (see [4, p. 54]) the sets E_j are linear subspaces satisfying the following strict inclusions and inequalities:

$$E_0 \subset E_1 \subset \cdots \subset E_p, \quad 0 = \dim E_0 < \dim E_1 < \cdots < \dim E_p = n.$$

We set

$$n_j = \dim E_j - \dim E_{j-1}, \quad j = 1, \ldots, p;$$

n_j is called the *multiplicity* of the exponent Λ_j. Note that $n_1 + \cdots + n_p = n$. The set of n numbers $\Lambda_1, \ldots, \Lambda_1, \ldots, \Lambda_p, \ldots, \Lambda_p$, where each Λ_j is listed n_j times, is called the *complete spectrum of the Lyapunov exponents* of the system (1). Below we denote it by

$$\lambda(A) = \big(\lambda_1(A), \ldots, \lambda_n(A)\big),$$

assuming that $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$. Thus, $\lambda(A) \in \mathbb{R}^n_{\leq}$.

We examine the problem on the behavior of the complete spectrum of the Lyapunov exponents of the system (1) under various perturbations of its coefficients.

Definition 1. We fix a class of perturbations of the coefficient matrix $A(\cdot)$ of the system (1). The *spectral set* of the system (1) corresponding to the class of perturbations considered is the collection of complete spectra of the Lyapunov exponents of perturbed systems where perturbations run over the whole class.

First, we consider the perturbed system of the form

$$y(m+1) = \big(A(m) + Q(m)\big)y(m), \quad m \in \mathbb{N}, \quad y \in \mathbb{R}^n. \tag{2}$$

The system (2) is said to be *additively perturbed* for the system (1) and perturbations $Q(\cdot)$ themselves are called *additive.* In order for the perturbed system (2) to possess the complete spectrum of Lyapunov exponents consisting of n numbers, it suffices to require that the additively perturbed matrix $A(\cdot)+Q(\cdot)$ be completely bounded. Thus, we arrive at the notion of an admissible additive perturbation.

Definition 2. An additive perturbation $Q(\cdot)$ is said to be *admissible* for the system (1) if the matrix $A(\cdot) + Q(\cdot)$ is completely bounded on $\mathbb{N}$.

We also consider *multiplicative perturbations* of the system (1):

$$y(m+1) = A(m)R(m)y(m), \quad m \in \mathbb{N}, \quad y \in \mathbb{R}^n. \tag{3}$$

In this case, the matrix $R(\cdot)$ is called the *multiplicative perturbation* of the system (1). If the matrix $A(\cdot)R(\cdot)$ is completely bounded, then the complete spectrum of Lyapunov exponents of the perturbed system (3) consists of n numbers. Since, by the condition, the matrix $A(\cdot)$ is completely bounded, we obtain the following definition.

Definition 3. A multiplicative perturbation $R(\cdot)$ is said to be *admissible* if the matrix $R(\cdot)$ is completely bounded on $\mathbb{N}$.

Let $\mathcal{Q}$ be the set of all admissibly additively perturbed systems of the form (2) and $\mathcal{R}$ be the set of all admissibly multiplicatively perturbed systems of the form (3). For an arbitrary $\delta > 0$, we denote by $\mathcal{Q}_\delta$ the set of all admissibly additively perturbed systems of the form (2) whose perturbations $Q(\cdot)$ satisfy the inequality

$$\sup_{m \in \mathbb{N}} \|Q(m)\| < \delta.$$

Similarly, by $\mathcal{R}_\delta$ we denote the set of all admissibly multiplicatively perturbed systems of the form (3) whose perturbations $R(\cdot)$ satisfy the inequality

$$\sup_{m\in\mathbb{N}} \|R(m) - E\| < \delta.$$

For an arbitrary admissible additive perturbation $Q(\cdot)$, we denote by

$$\lambda(A+Q) = \big(\lambda_1(A+Q),\ \ldots,\ \lambda_n(A+Q)\big) \in \mathbb{R}^n_{\le}$$

the complete spectrum of the Lyapunov exponents of the system (2) and by $\lambda(\mathcal{Q})$ the spectral set of the system (1) corresponding to the class of all admissible additive perturbations. Similarly, for arbitrary $\delta > 0$, we denote by $\lambda(\mathcal{Q}_\delta)$ the spectral set of the system (1) corresponding to the class of admissible additive perturbations $Q(\cdot)$ satisfying the estimate

$$\sup_{m\in\mathbb{N}} \|Q(m)\| < \delta.$$

Further, let

$$\lambda(AR) = \big(\lambda_1(AR),\ \ldots,\ \lambda_n(AR)\big) \in \mathbb{R}^n_{\le}$$

be the complete spectrum of Lyapunov exponents of the system (3) for an arbitrary admissible multiplicative perturbation $R(\cdot)$. We denote by $\lambda(\mathcal{R})$ the spectral set of the system (1) corresponding to the class of all multiplicative perturbations, and for arbitrary $\delta > 0$ denote by $\lambda(\mathcal{R}_\delta)$ the spectral set of the system (1) corresponding to the class of admissible multiplicative perturbations $R(\cdot)$ satisfying the estimate

$$\sup_{m\in\mathbb{N}} \|R(m) - E\| < \delta.$$

Theorem 1 (see [1])**.** *The sets $\lambda(\mathcal{Q})$ and $\lambda(\mathcal{R})$ coincide. For any $\delta > 0$, the following inclusions hold*:

$$\lambda(\mathcal{Q}_\delta) \subset \lambda(\mathcal{R}_{a\delta}), \quad \lambda(\mathcal{R}_\delta) \subset \lambda(\mathcal{Q}_{a\delta}).$$

Note that in the theory of linear systems with continuous time perturbations of coefficients, systems are usually taken in the additive form (see, e.g., [2]). In the case of systems with discrete time, multiplicative perturbations are more convenient, and Theorem 1 allows one to obtain important consequences for additively perturbed systems.

Below, we present certain results on the spectral set of multiplicatively perturbed systems expressed in terms of the asymptotic theory of linear systems. We recall the corresponding definitions and comment on them.

Definition 4 (see [4])**.** A linear transform

$$y = L(m)x, \quad m \in \mathbb{N}, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n,$$

where the function $L : \mathbb{N} \to M_n(\mathbb{R})$ is completely bounded, is called the *Lyapunov transform*. The systems (1) and

$$y(m+1) = C(m)y(m), \quad m \in \mathbb{N}, \quad y \in \mathbb{R}^n, \tag{4}$$

are said to be *asymptotically equivalent* if there exists a Lyapunov transform that maps one of them to the other.

It is known (see [4]) that asymptotically equivalent systems have the same complete spectra of Lyapunov exponents. Moreover, Definition 4 immediately implies that the systems (1) and (4) are asymptotically equivalent if and only if there exists a completely bounded matrix-valued function $L : \mathbb{N} \to M_n(\mathbb{R})$ such that for each $m \in \mathbb{N}$ the equality

$$C(m) = L(m+1)A(m)L^{-1}(m)$$

holds.

Definition 5 (see [4]). A system (1) is said to be *good* if

$$\sum_{i=1}^{n} \lambda_i(A) = \varliminf_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m-1} \ln|\det A(j)|.$$

Each stationary and periodic system is good.

A system (1) is good if and only if it is asymptotically equivalent to a system (4) with an upper triangular matrix $C(m) = \{c_{ij}(m)\}_{i,j=1}^{n}$ for which the following exact limits exist:

$$\lambda_i = \lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m-1} \ln|c_{ii}(j)|, \quad i = 1, \ldots, n;$$

in this case, $\lambda_i(A) = \lambda_i$.

Definition 6. A system (1) is called *diagonalizable* if it is asymptotically equivalent to a system (4) with a diagonal matrix $C(m) = \operatorname{diag}\bigl(c_1(m), \ldots, c_n(m)\bigr)$. In this case,

$$\lambda_i(A) = \varlimsup_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m-1} \ln|c_i(j)|, \quad i = 1, \ldots, n.$$

Definition 7 (see [1]). The Lyapunov exponents of the system (1) are said to be *stable* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any multiplicatively perturbed system of the form (3) from the set $\mathcal{R}_\delta$ we have

$$\Bigl|\lambda_i(A) - \lambda_i(AR)\Bigr| < \varepsilon, \quad i = 1, \ldots, n,$$

i.e., the following inclusion holds:

$$\lambda\bigl(\mathcal{R}_\delta\bigr) \subset \mathcal{O}_\varepsilon\bigl(\lambda(A)\bigr).$$

Theorem 2. *Assume that at least one of the following conditions holds*:

(1) *the system* (1) *is good*;

(2) *the system* (1) *is diagonalizable*;

(3) *the Lyapunov exponents of the system* (1) *are stable.*

Then the spectral set $\lambda\bigl(\mathcal{R}\bigr)$ *coincides with the set* $\mathbb{R}^n_{\leq}$ *os all nondecreasing* n*-tuples of real numbers. Moreover, for each* $\delta > 0$ *there exists* $\ell > 0$ *such that for an arbitrary tuple* $\mu = (\mu_1, \ldots, \mu_n) \in \mathcal{O}_\delta\bigl(\lambda(A)\bigr)$, *the exists an admissible multiplicative perturbation* $R(\cdot)$ *such that* $\lambda(AR) = \mu$ *and*

$$\sup_{m\in\mathbb{N}} \|R(m) - E\| \leq \ell \max_{j=1,\ldots,n} |\mu_j - \lambda_j(A)|.$$

Corollary 1. *Assume that at least one of the following conditions hold*:

(1) *the system* (1) *is good*;

(2) *the system* (1) *is diagonalizable*;

(3) *the Lyapunov exponents of the system* (1) *are stable.*

Then for each $\delta > 0$ *there exists* $\ell > 0$ *such that*

$$\mathcal{O}_\delta\bigl(\lambda(A)\bigr) \subset \lambda\bigl(\mathcal{R}_{\ell\delta}\bigr).$$

Corollary 2. *If the Lyapunov exponents of the system* (1) *are stable, then for each* $\varepsilon > 0$ *there exist* $\delta_1 > 0$ *and* $\delta_2 > 0$ *such that*

$$\mathcal{O}_{\delta_1}\bigl(\lambda(A)\bigr) \subset \lambda\bigl(\mathcal{R}_{\delta_2}\bigr) \subset \mathcal{O}_\varepsilon\bigl(\lambda(A)\bigr).$$

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