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On dense subsets of Tychonoff products of T_1-spaces ☆

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ABSTRACT

We consider dense subsets of $Z^{\mathfrak{c}} = \prod_{\alpha \in 2^\omega} Z\alpha$, where $Z = Z_\alpha$ is a separable not single point T_1-space.

We construct in $Z^{\mathfrak{c}}$, $|Z| \geq \omega$, (Theorem 4.1) a countable dense set $Q \subseteq Z$ such that every countable subset of Q contains a countable subset, which can be project on "many" subsets of Z.

From this theorem follow some facts.

Theorem 4.2 states that in the product $Z^{\mathfrak{c}}$ of a not single point separable T_1-space Z there is a countable dense set which contains no non-trivial convergent in $Z^{\mathfrak{c}}$ sequences.

The existence of such set in the product $I^{\mathfrak{c}}$, where $I = [0, 1]$ was proved [13] by W. H. Priestley.

In [9] we proved that such set exists in a product of 2^ω separable not single point T_2-spaces.

Theorem 4.4 states that if Z is a separable not countably compact T_1-space, then there is a countable dense subset $Q \subseteq Z^{\mathfrak{c}}$, satisfying the following condition: if $E \subseteq Q$ is a countable set and E converges to a set $F \subseteq Z^c$, then $|E \setminus F| < \omega$.

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1. Introduction

One of the most interesting problem in products of topological spaces is the problem of existence of dense subsets. It is not so hard to prove that in a countable product of separable spaces there is a countable dense subset. But in a case of continuum product the problem becomes much more difficult. The most well-known result is Hewitt–Marczewski–Pondiczery theorem (see [2,3]): the Tychonoff product $\prod_{\alpha \in 2^\omega} X\alpha$ of 2^ω many separable spaces is separable.

We consider the problem of the existence in the Tychonoff product of 2^ω many separable spaces a dense countable subset with additional properties.

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In [13] W.H.Priestley proved that in I^{2^ω} there is a countable dense set, which contains no non-trivial convergent sequences in I^{2^ω}, where I is closed unit interval.

In [14] P.Simon proved that such countable dense set exists in D^{2^ω}, where D is a two-point discrete space. He proved that in D^{2^ω} there is a countable dense set such that the closure of every countable subset of it has a cardinality 2^{2^ω}.

In [8] we also considered this problem. We proved that in a product of countable discrete spaces Z^{2^ω} there is a dense set Q such that every sequence of Q has a subsequence, which is discrete and closed in Z^{2^ω}. Therefore no sequence from Q is convergent.

In [9] we proved that such countable dense sets, which contain no convergent sequences are not only in products of discrete spaces: we proved that such set exists in a product of 2^ω separable not single point T_2-spaces.

Now we consider a product $Z^\mathfrak{c}$ of separable T_1-space Z.

We prove Theorem 4.1, which states the following.

Let Z be a separable space, $|Z| \geq \omega$, and $\eta \subseteq \operatorname{Exp} Z$ be a family of countable subsets of Z, $|\eta| \leq 2^\omega$.

Then there is a countable dense subset $Q \subseteq Z^\mathfrak{c} = \prod\limits_{X \in \exp\omega} ZX$, satisfying the following condition:

(*) every countable set $E \subseteq Q$ contains a countable subset $E' \subseteq E$ such that for every $G \in \eta$ there is $X \in \exp\omega$ such that $\pi_X(E') = G$ and $\pi_{X|_{E'}} : E' \to G$ is a finite-to-one mapping.

Using Theorem 4.1, we prove some theorems.

Theorem 4.2 states that in the Tychonoff product $Z^\mathfrak{c}$, where Z is a not single point separable T_1-space, there is a countable dense set Q which contains no non-trivial convergent in $Z^\mathfrak{c}$ sequences.

From this it follows, in particular, that in the Tychonoff product $Z_m^\mathfrak{c}$, where Z_m is an infinite space with minimal T_1-topology, there is a countable dense set which contains no non-trivial convergent sequences.

Consider some other consequences from Theorem 4.1. Consider a countable product $Z^\omega = \prod\limits_{i\in\omega} Z_i$, where $Z = Z_i$ $(i \in \omega)$ is an uncountable space.

Then for every countable dense subset $Q \subseteq Z^\omega$ there is a point $z \in Z$ such that for a set $Z_z = Z \setminus \{z\}$ the following holds: $Z_z^\omega \supseteq Q$.

In the case of $Z^\mathfrak{c}$ we have the Theorem 4.3, which states:

let Z be a separable T_1-space, $\omega \leq |Z| \leq \mathfrak{c}$, then in $Z^\mathfrak{c}$ there is a countable dense subset Q, which satisfies the following condition:

for all $z \in Z$ the following holds: $|Z_z^\mathfrak{c} \cap Q| < \omega$ where $Z_z = Z \setminus \{z\}$.

For a space $I^\mathfrak{c}$ in every its dense subset, in particular in Q, which contains no convergent sequences, there is a sequence $E \subseteq Q$, which converges to a set F such that $E \cap F = \emptyset$. For $R^\mathfrak{c}$ we have the following.

Theorem 4.4 states:

let Z be a separable not countably compact T_1-space, then there is a countable dense subset $Q \subseteq Z^\mathfrak{c}$, which satisfies the following condition: if $E \subseteq Q$ is a countable set and E converges to a set $F \subseteq Z^c$, then $|E \setminus F| < \omega$.

The proofs we will construct and use independent matrices.

In [11] K.Kunen defined a notion of an independent linked family on ω, which generalizes notion of the independent family of sets [10,5]. J. van Mill in [12] defined the notion of independent matrix of clopen subsets of ω^*. This notion naturally corresponds to the notion of independent matrix of ω, which is a subfamily of the independent linked family on ω.

R. Engelking and M. Karlowicz [4] used a notion of this type for their proof of Hewitt–Marczewski–Pondiczery theorem.

Some independent matrices were constructed in [6–9].

2. Preliminaries

Definitions and notions used in the paper can be found in [1–3].

$d(X)$ denotes the density of a space X, by $[A]$ we denote the closure of a set A, $\exp A$ denotes the set of all subsets of A and by $\operatorname{Exp} A$ we denote the set of all non-empty subsets of a set A. By $\exp_\omega A$ we denote the set of all countable subsets of A, $\mathfrak{c} = 2^\omega$.

We say that a set A is countable if $|A| = \omega$.

By Y^X we denote the set of all mappings from X to Y. By π_α we denote an α-projection of $\prod\limits_{\alpha\in A} X\alpha$ on X_α.

We say that a set A converges to a set B if $|A \setminus U| < \omega$ for every neighborhood U of B.

We say that the character of a set A in a space X is countable ($\chi(A, X) = \omega$) if there is a countable base for A in X.

A sequence $\{x_n\}_{n=1}^{\infty}$ is called trivial if there is $n_0 \in \omega$ such that $x_n = x_{n_0}$ for all $n \geq n_0$.

A subset $A \subseteq X$ is called sequentially closed if A contains limits of all its convergent in X sequences.

By Z_m we denote a T_1-space with minimal topology, its topology consists of complements of finite subsets of Z_m.

The proofs we will use the notion of the independent matrix of subsets of a countable set.

Definition 2.1 *([12]).* For a countable set an indexed family $\{A_{ij} : i \in I, j \in J\}$ of its subsets is called the J by I independent matrix if

- whenever $j_0, j_1 \in J$ are distinct and $i \in I$, then $|A_{ij_0} \cap A_{ij_1}| < \omega$;
- if $i_1, \ldots, i_n \in I$ are distinct and $j_1, \ldots, j_n \in J$, then

$$|\bigcap\{A_{i_k j_k} : k = 1, \ldots, n\}| = \omega.$$

The following construction of the 2^ω by 2^ω independent matrix of subsets of a countable set can be found in [11], [12].

Consider a countable set

$$H' = \{< k, u >: k \in \omega, u \in (\exp k)^{\exp k}\}.$$

For $X, Y \in \exp\omega$ let

$$A_{XY} = \{< k, u >\in H' : u(X \cap k) = Y \cap k\}.$$

The family $\mathfrak{M} = \{A_{XY} : X, Y \in \exp\omega\}$ is the 2^ω by 2^ω independent matrix.

We will use the following notation (see [6]).

Consider the set $(\operatorname{Exp} k)^{\exp k}$ for $k \in \omega$. Elements of this set we will denote by u, v, *ect.*

For $k \in \omega$ denote

$$H_k = \{u \in (\operatorname{Exp} k)^{\exp k} : \{n\} \in u(\exp k) \text{ for all } n < k\},$$
$$H = \bigcup\{H_k : k \in \omega\}.$$

For $X \in \exp\omega$ and $Y \in \operatorname{Exp}\omega$ denote

$$A_k(X, Y) = \{u \in H_k : u(X \cap k) = Y \cap k\}$$

and

$$A(X,Y) = \bigcup\{A_k(X,Y)\colon k \in \omega\}.$$

Define the matrix

$$\mathfrak{M}_1 = \{A(X,Y) : X \in \exp\omega, Y \in \operatorname{Exp}\omega\}.$$

$\mathfrak{M}_1$ is an independent matrix and satisfies the following conditions:

(i) if Y_1, $Y_2 \in \operatorname{Exp}\omega$ are distinct and $X \in \exp\omega$, then $|A(X,Y_1) \cap A(X,Y_2)| < \omega$;
(ii) if $X_1,\dots,X_n \in \exp\omega$ are distinct and $Y_1,\dots,Y_n \in \operatorname{Exp}\omega$, then there is $k_0 \in \omega$ such that $(\bigcap\{A(X_i,Y_i)\colon i = 1,\dots,n\}) \cap H_k \neq \emptyset$ for all $k > k_0$;
(iii) $\bigcup\{A(X,Y)\colon Y \in \operatorname{Exp}\omega\} = H$ for every $X \in \exp\omega$.

We will consider properties of the matrix $\mathfrak{M}_1$ and some other matrices and will use them for a construction of dense sets of products.

3. Construction of the matrix $\mathfrak{M}_2$

Consider the matrix $\mathfrak{M}_1 = \{A(X,Y) : X \in \exp\omega, Y \in \operatorname{Exp}\omega\}$ and its properties.

Lemma 3.1 *([6]). Let $u \in H_{k_0} \subseteq H$ for some $k_0 \in \omega$ and a set $F \subseteq H$ be such that*

– *$|F \cap H_k| \le 1$ for all $k \in \omega$;*
– *$|F \cap H_k| = \emptyset$ for $k \le k_0$.*

Then for every set $X \in \exp\omega$ there is a set $Y \in \operatorname{Exp}\omega$ such that $u \in A(X,Y)$ and $A(X,Y) \cap F = \emptyset$.

Lemma 3.2 *([6]). Let $u, v \in H$, $u \neq v$. For every $B \subseteq \exp\omega$, $|B| < 2^\omega$, there is $X \in \exp\omega \setminus B$ and $Y \in \operatorname{Exp}\omega$ such that $u \in A(X,Y)$ and $v \notin A(X,Y)$.*

Lemma 3.3 *([8]). Let $X \in \exp\omega$ and $F \subseteq H$ be such that $|F \cap H_k| \le 1$ for all $k \in \omega$. Then there is a family $T'(X,F) \subseteq \operatorname{Exp}\omega$ such that*

(1) *$|T'(X,F)| = \omega$;*
(2) *$|A(X,Y) \cap F| < \omega$ for all $Y \in T'(X,F)$;*
(3) *$\bigcup\{A(X,Y)\colon Y \in T'(X,F)\} = H$.*

Let $\widetilde{\mathfrak{F}}$ be the family of countable subsets of H such that for every $F \in \widetilde{\mathfrak{F}}$ the following holds:
1) $|F \cap H_i| \le 1$ for all $i \in \omega$;
2) $|\{Y \in \operatorname{Exp}\omega : A(X,Y) \cap F = \emptyset\}| \ge \omega$ for every $X \in \exp\omega$.
Let us prove for the set $\widetilde{\mathfrak{F}}$ the following.

Lemma 3.4. *For every countable set $E \subseteq H$ there are 2^ω many $F \in \widetilde{\mathfrak{F}}$ such that $F \subseteq E$.*

Proof. Let $E' \subseteq E$ be a countable set such that $|E' \cap H_k| = 1$ for all $k \in D$ for a countable set $D \subseteq \omega$.

Let λ be a family of countable subsets of D, $|\lambda| = 2^\omega$, such that every set $B = \{k_i : i \in \omega\} \in \lambda$ satisfies the following:

– $k_0 \geq 2$;
– $k_{i+1} \geq k_i + 4$ for all $i \in \omega$.

For every $B \in \lambda$ define $F(B) = E' \cap (\bigcup\{H_{k_i} : k_i \in B\} = \bigcup\{E' \cap H_{k_i} : k_i \in B\}$. Since $|\lambda| = 2^\omega$, we have $|\{F(B) : B \in \lambda\}| = 2^\omega$.

Let $B \in \lambda$. We will prove that $F(B) \in \widetilde{\mathfrak{F}}$.

Indeed, $F(B)$ satisfies condition 1). Let us prove that $F(B)$ satisfies condition 2).

Let $X \in \exp\omega$. We will construct 2^ω many sets $Y \in \operatorname{Exp}\omega$ such that $A(X,Y) \cap F(B) = \emptyset$.

Let v_{k_i} be such that $\{v_{k_i}\} = E' \cap H_{k_i}$ for $k_i \in B$.

Construct a family $\{Y_{k_i} : i \in \omega\}$ of subsets of $\operatorname{Exp}\omega$ such that for every $k_i \in B$ the following holds:

– $Y_{k_i} \subseteq k_i$;
– $v_{k_i}(X \cap k_i) \neq Y_{k_i}$;
– $Y_{k_i} \subseteq Y_{k_{i+1}}$;
– $Y_{k_i} \cap k_{i'} = Y_{k_{i'}}$ if $i' < i$.

Let $Y_{k_0} \subseteq k_0$ be such that $Y_{k_0} \neq v_{k_0}\ (X \cap k_0)$ (it is possible because $k_0 \geq 2$).

Assume $\{Y_{k_{i'}} : i' < i\}$ has been chosen. Let $Y'_{k_i} = \bigcup\{Y_{k_{i'}} : i' < i\}$.

Consider $\{k \in \omega : k_{i-1} < k < k_i\} = (k_{i-1}, k_i)$.

Let $(k_{i-1}, k_i)^*$ be a set of $k \in (k_{i-1}, k_i)$ such that $v_{k_i}(X \cap k_i) \neq Y'_{k_i} \cup \{k\}$. Since $|(k_{i-1}, k_i)| \geq 3$, we have $|(k_{i-1}, k_i)^*| \geq 2$.

Let $k^* \in (k_{i-1}, k_i)^*$. We define a set $Y_{k_i} = Y'_{k_i} \cup \{k^*\}$.

Let $Y = \cup\{Y_{k_i} : k_i \in A\}$.

Note that the set Y is countable.

We have $A(X,Y) \cap F(B) = \emptyset$.

Since $|(k_{i-1}, k_i)^*| \geq 2$ for all $k_i \in B$, there are 2^ω sets $Y \in \operatorname{Exp}\omega$ such that $A(X,Y) \cap F(B) = \emptyset$. So $F(B) \in \widetilde{\mathfrak{F}}$.

Since $|\lambda| = 2^\omega$, we have $|\{F(B) : B \in \lambda\}| = 2^\omega$.

Note, that from Lemma 3.4 we have $|\widetilde{\mathfrak{F}}| = 2^\omega$. □

Lemma 3.5. *Let $F \in \widetilde{\mathfrak{F}}$ and $X \in \exp\omega$. Then there is a family $T(X,F) \subseteq \operatorname{Exp}\omega$ such that:*

– *$|T(X,F)| = \omega$;*
– *$\bigcup\{A(X,Y) : Y \in T(X,F)\} = H$;*
– *$|A(X,Y) \cap F| < \omega$ for all $Y \in T(X,F)$;*
– *$|\{Y \in T(X,F) : A(X,Y) \cap F\} = \emptyset| \geq \omega$.*

Proof. Let $F \in \widetilde{\mathfrak{F}}$ and $X \in \exp\omega$. By Lemma 3.3 there is a family $T'(X,F) \subseteq \operatorname{Exp}\omega$ such that

– $|T'(X,F)| = \omega$;
– $|A(X,Y) \cap F| < \omega$ for all $Y \in T'(X,F)$;
– $\bigcup\{A(X,Y) : Y \in T'(X,F)\} = H$.

Since $F \in \widetilde{\mathfrak{F}}$, there is $T'' \subseteq \operatorname{Exp}\omega$, $|T''| = \omega$, such that $A(X,Y) \cap F = \emptyset$ for $Y \in T''$. Denote $T(X,F) = T' \cup T''$. □

Now we will define a matrix $\mathfrak{M}_2$.

I Let P be the set of all ordered pairs (u,v) of elements $u, v \in H$.

By Lemma 3.2 there is a countable family

$$\mathfrak{L} = \{X_{(u,v)} : (u,v) \in P\}$$

of sets $X_{(u,v)} \in \exp\omega$ such that for every $X_{(u,v)} \in \mathcal{L}$ there is $Y_{(u,v)} \in \operatorname{Exp}\omega$ such that $u \in A(X_{(u,v)}, Y_{(u,v)})$, $v \notin A(X_{(u,v)}, Y_{(u,v)})$ and $X_{(u,v)} \neq X_{(u',v')}$ if $(u,v) \neq (u',v')$.

Denote

$$\mathcal{R} = \exp\omega \setminus \mathcal{L}.$$

Let η be a family of countable subsets of H such that $|\eta| \leq \mathfrak{c}$.

Consider the family $\widetilde{\mathfrak{F}}$ of subsets of H.

II Let $\Theta\colon \mathcal{R} \to \widetilde{\mathfrak{F}}$ be a mapping such that $|\Theta^{-1}(F)| = |\eta|$ for every $F \in \widetilde{\mathfrak{F}}$.

Consider $F \in \widetilde{\mathfrak{F}}$. Let $X \in \Theta^{-1}(F)$. Let

$$T(X,F) \subseteq \operatorname{Exp}\omega$$

be a countable set from Lemma 3.5 for F and X.

III Let $X_{(u,v)} \in \mathcal{L}$ and $Y_{(u,v)} \in \operatorname{Exp}\omega$ be such that $u \in A(X_{(u,v)}, Y_{(u,v)})$ and $v \notin A(X_{(u,v)}, Y_{(u,v)})$.

Let

$$T_{(u,v)} \subseteq \operatorname{Exp}\omega$$

be a countable family such that $Y_{(u,v)} \in T_{(u,v)}$ and $\bigcup\{A(X_{(u,v)}, Y)\colon Y \in T_{(u,v)}\} = H$.

IV Define

$$T_X = \begin{cases} T_{(u,v)} & \text{for } X = X_{(u,v)} \in \mathcal{L}; \\ T(X,F) & \text{for } F \in \widetilde{\mathfrak{F}} \text{ and } X \in \Theta^{-1}(F). \end{cases}$$

By the similar way as in [8] we define a matrix

$$\mathfrak{M}_2 = \{\widetilde{A}(X,Y)\colon X \in \exp\omega, Y \in T_X\},$$

which satisfies the following.

Lemma 3.6. *The matrix $\mathfrak{M}_2$ satisfies the following conditions:*

(1) *for every $(u,v) \in P$ there is $X = X_{(u,v)} \in \mathcal{L}$ and $Y \in T_{(u,v)}$ such that $u \in \widetilde{A}(X_{(u,v)}, Y_{(u,v)})$ and $v \notin \widetilde{A}(X_{(u,v)} Y_{(u,v)})$;*
(2) *if $F \in \widetilde{\mathfrak{F}}$ and $X \in \Theta^{-1}(F)$ then $|\widetilde{A}(X,Y) \cap F| < \omega$ for all $Y \in T(X,F)$;*
(3) *if $F \in \widetilde{\mathfrak{F}}$ and $X \in \Theta^{-1}(F)$ then $|\{Y \in T(X,F)\colon \widetilde{A}(X,Y) \cap F = \emptyset\}| = \omega$;*
(4) *$\widetilde{A}(X,Y) \cap \widetilde{A}(X,Y') = \emptyset$ for all $X \in \exp\omega$ and $Y, Y' \in T_X$, $Y \neq Y'$;*
(5) *$\bigcup\{\widetilde{A}(X,Y)\colon Y \in T_X\} = H$ for all $X \in \exp\omega$;*
(6) *if $X_1, \ldots, X_n \in \exp\omega$ are distinct and $Y_i \in T_{X_i}$ $(i = 1, \ldots, n)$ then there is $k_0 \in \omega$ such that $(\bigcap\{\widetilde{A}(X_i, Y_i)\colon i = 1, \ldots, n\}) \cap H_k \neq \emptyset$ for all $k > k_0$.*

The matrix $\mathfrak{M}_2$ generates a space

$$\Sigma = \prod_{X \in \exp\omega} TX.$$

For $\xi \in \Sigma$ let ξ_X be a X-coordinate of ξ.

The following Lemma 3.7 is similar to Lemma 3.5 [8].

Lemma 3.7. *The set Σ satisfies the following conditions:*

(1) *if $\xi_1, \xi_2 \in \Sigma$, $\xi_1 \neq \xi_2$, then there is $X \in \exp\omega$ such that $\widetilde{A}(X, {\xi_1}_X) \cap \widetilde{A}(X, {\xi_2}_X) = \emptyset$;*
(2) *$|\bigcap\{\widetilde{A}(X, \xi_X) \colon X \in \exp\omega\}| \leq 1$ for all $\xi \in \Sigma$;*
(3) *for every $u \in H$ there is the only*

$$\xi^u \in \Sigma$$

such that $\cap\{\widetilde{A}(X, \xi^u_X) \colon X \in \exp\omega\} = \{u\}$;
(4) *if $u_1, u_2 \in H$, $u_1 \neq u_2$, then $\xi^{u_1} \neq \xi^{u_2}$.*

Denote

$$\mu \colon H \to \Sigma$$

a mapping from H into Σ defined by the rule: $\mu(u) = \xi^u$ for every $u \in H$.

Lemma 3.8. *The set $\mu(H)$ is dense in the space Σ.*

Proof. For the set H the set $\mu(H)$ is dense in Σ.

Indeed, let $O(X, \dots, X_n, Y_1, \dots, Y_n)$ be a standard basic open set of the base of Σ, where $Y_i \in T_{X_i}$ $(i = 1, \dots, n)$.

By (6) of Lemma 3.6 we have

$$\left(\bigcap\{\widetilde{A}(X_i, Y_i) \colon i = 1, \dots, n\}\right) \cap H \neq \emptyset$$

For $u \in \cap\{\widetilde{A}(X_i, Y_i) \colon i = 1, \dots, n\}$ we have $\xi^u_{X_i} = Y_i$ $(i = 1, \dots, n)$ and therefore $\xi^u \in O(X, \dots, X_n, Y_1, \dots, Y_n)$, so $O(X, \dots, X_n, Y_1, \dots, Y_n) \cap \mu(H) \neq \emptyset$. $\square$

4. The main theorems

Consider a separable space Z and

$$\text{let } Z^{\mathfrak{c}} = \prod_{X \in \exp\omega} ZX, \ Z_X = Z.$$

Theorem 4.1. *Let Z be a separable space, $|Z| \geq \omega$, and $\eta \subseteq \operatorname{Exp} Z$ be a family of countable subsets of Z, $|\eta| \leq 2^\omega$.*

Then there is a countable dense subset $Q \subseteq Z^{\mathfrak{c}} = \prod_{X \in \exp\omega} ZX$, satisfying the following condition:

(*) every countable set $E \subseteq Q$ contains a countable subset $E' \subseteq E$ such that for every $G \in \eta$ there is $X \in \exp\omega$ such that $\pi_X(E') = G$ and $\pi_{X|_{E'}} \colon E' \to G$ is a finite-to-one mapping.

Proof. Consider two cases.

(I) At first we will consider the case $|Z| > \omega$.

(A) We will construct a mapping $\Psi\colon \Sigma \to \prod\limits_{X\in\exp\omega} ZX$ as follows.

The construction of Ψ consists of parts (A1)–(A4).

(A1) Consider $Z^{\mathfrak{c}} = \prod\limits_{X\in\exp\omega} ZX$ and the set $\mathcal{R} \subseteq \exp\omega$.

Let η be the family from the formulation of the theorem, and $\Theta\colon \mathcal{R} \to \widetilde{\mathcal{F}}$ be a mapping such that $|\Theta^{-1}(F)| = |\eta|$ for every $F \in \widetilde{\mathcal{F}}$ (see *II* of the construction of the matrix $\mathfrak{M}_2$).

For every $X \in \Theta^{-1}(F)$ fix countable subsets $G(X)$, $D(X)$ of the space $Z_X = Z$ such that:

- $G(X) \in \eta$;
- $G(X) \neq G(X')$ if $X \neq X'$;
- $\{G(X)\colon X \in \Theta^{-1}(F)\} = \eta$ for every $F \in \widetilde{\mathcal{F}}$;
- $G(X) \subseteq D(X)$;
- $D(X)$ is dense in $Z_X = Z$;
- $|D(X) \setminus G(X)| = \omega$.

We can do this since $|Z| > \omega$.

(A2) For every $X \in \exp\omega$ we will define a mapping $\psi_X\colon T_X \to Z_X$.

Let $X \in \mathcal{R}$. We will define a mapping

$$\psi_X\colon T_X \to Z_X$$

as follows. Since $X \in \mathcal{R}$, we have $X \in \Theta^{-1}(F)$ for some $F \in \widetilde{\mathcal{F}}$. Consider $T_X = T(X, F)$ for this X and F (see *IV* of the construction of $\mathfrak{M}_2$).

Let $T'_X = \{Y \in T_X\colon \widetilde{A}(X,Y) \cap F \neq \emptyset\}$ and $T''_X = T_X \setminus T'_X$.

By (2), (3) and (5) of Lemma 3.6 sets T'_X and T''_X are countable.

We will define $\psi_X\colon T_X \to Z_X$ as some one-to-one mapping T_X into Z_X such that:

$$\psi_X(T'_X) = G(X),$$
$$\psi_X(T_X) = D(X).$$

(A3) Let $X \in \mathcal{L}$, i.e. $X = X_{(u,v)}$ for some ordered pair (u, v). In this case we have $T_X = T_{(u,v)}$ (see *IV* of the construction of $\mathfrak{M}_2$). Let $D(X) = D(X_{(u,v)})$ be some countable dense subset of $Z_{X_{(u,v)}}$.

We will define

$$\psi_X\colon T_X \to D_X$$

as some one-to-one mapping ψ_X from $T_X = T_{X_{(u,v)}}$ onto $D_X = D_{X_{(u,v)}}$.

(A4) We define the mapping

$$\Psi\colon \Sigma \to \prod_{X\in\exp\omega} ZX$$

as follows.

For $\xi \in \Sigma$ let $\Psi(\xi) = z \in \prod\limits_{X\in\exp\omega} ZX$ be such that $\pi_X(z) = \pi_X(\Psi(\xi)) = \psi_X(\xi_X)$ (recall that ξ_X is a X-coordinate of $\xi \in \Sigma$ and $\xi_X \in T_X$).

The mapping Ψ is one-to-one mapping from Σ onto $\prod\limits_{X\in\exp\omega} D(X) \subseteq \prod\limits_{X\in\exp\omega} ZX$.

The mapping μ is a one-to-one mapping from H into Σ. Then

$$\mu \circ \Psi\colon H \to \prod_{X \in \exp\omega} D(X)$$

is a one-to-one mapping from H into $\prod\limits_{X \in \exp\omega} D(X) \subseteq Z^{\mathfrak{c}}$.

For every $u \in H$ we denote $z^u = \Psi(\mu(u)) = \Psi(\xi^u)$. By the definition of Ψ we have $\pi_X(z^u) = \pi_X(\Psi(\xi^u)) = \psi_X(\xi^u_X)$.

(B) Define

$$Q = \Psi(\mu(H)) \text{ and } Q_k = \Psi(\mu(H_k)).$$

The set Q is dense in $Z^{\mathfrak{c}}$.

Consider spaces $\Sigma = \prod\limits_{X \in \exp\omega} TX$ and $\prod\limits_{X \in \exp\omega} D(X)$. The mapping $\Psi : \Sigma \to \prod\limits_{X \in \exp\omega} D(X)$ is continuous.

By Lemma 3.7 the set $\mu(H)$ is a dense subset of Σ. Then $Q = \Psi(\mu(H))$ is dense in $\prod\limits_{X \in \exp\omega} D(X)$. Since $D(X)$ is dense subset of Z_X for all $X \in \exp\omega$, Q is dense in $Z^{\mathfrak{c}}$.

(C) Consider $F \in \widetilde{\mathfrak{F}}$ and let $X \in \Theta^{-1}(F)$. We will prove the following:

- $\pi_X(\Psi(\mu(F))) = G(X)$;
- $\pi_X|_{\Psi(\mu(F))}$ is a finite-to-one mapping from $\Psi(\mu(F))$ onto $G(X)$.

We have $\Psi(\mu(F)) = \Psi(\{\xi^u\colon u \in F\})$, $\pi_X(\Psi(\mu(F))) = \psi_X(\{\xi^u_X\colon u \in F\})$, and $\{\xi^u_X\colon u \in F\} \subseteq T_X = T(X, F)$.

Prove that $\{\xi^u_X\colon u \in F\} = T'_X$ for $F \in \widetilde{\mathfrak{F}}$.

Indeed, since $\xi^u_X \ni u$ for every $u \in F$, we have $\{\xi^u_X\colon u \in F\} \subseteq T'_X$ and $F \subseteq \bigcup\{\widetilde{A}(X, \xi^u_X)\colon u \in F\}$.

From (4) of Lemma 3.6 we have $\{\xi^u_X\colon u \in F\} = T'_X$.

Therefore $\pi_X(\Psi(\mu(F))) = \psi_X(\{\xi^u_X\colon u \in F\}) = \psi_X(T'_X)$.

By the definition of the mapping ψ_X we have $\psi_X(T'_X) = G(X)$, so

$$\pi_X(\Psi(\mu(F))) = G(X).$$

Let us prove that

$$\pi_X|_{\Psi(\mu(F))}\colon \Psi(\mu(F)) \to G(X)$$

is a finite-to-one mapping from $\Psi(\mu(F))$ onto $G(X)$.

Let $Y \in T_X$. Consider $\psi_X(Y)$.

If for some $z^u \in \Psi_X(\mu(F))$ we have $\psi_X(Y) = \psi_X(\xi^u_X) = \pi_X(z^u)$, then, since $\psi_X\colon T_X \to Z_X$ is a one-to-one mapping, we have $\xi^u_X = Y$. Therefore $u \in Y$.

Since $|F \cap Y| < \omega$ for every $Y \in T_X$, we $|\{u \in F\colon \pi_X(z^u) = \psi_X(Y)\}| < \omega$. Therefore

$$\pi_X|_{\Psi(\mu(F))}\colon \Psi(\mu(F)) \to G(X)$$

is a finite-to-one mapping.

(D) Let us prove the property (*) of Q, declared in the theorem. Use (C).

Let $E \subseteq Q$ be a countable subset. Let $\widetilde{E} \subseteq H$ be such that

$$\Psi(\mu(\widetilde{E})) = E.$$

There is $F \in \widetilde{\mathfrak{F}}$ such that $F \subseteq \widetilde{E}$ (see Lemma 3.4).

Let $G \in \eta$. Consider $X \in \Theta^{-1}(F)$ such that $G(X) = G$. Then for $\Psi(\mu(F)) \subseteq E$ we have

$$\pi_X(\Psi(\mu(F))) = G(X) = G$$

and $\pi_X|_{\Psi(\mu(F))}\colon \Psi(\mu(F)) \to G(X)$ is a finite-to-one mapping (see (C)).

Define

$$E' = \Psi(\mu(F).$$

II. Consider the case $|Z| = \omega$.

Let A be a countable set.

Denote $Y = \prod\limits_{\alpha\in A} Z\alpha$, where $Z_\alpha = Z$ for all $\alpha \in A$.

Y is an uncountable separable space.

Consider $\prod\limits_{X\in\exp\omega} YX$, where $Y_X = Y$ and $Y_X = \prod\limits_{\alpha\in A} Z_{\alpha,X}$, $Z_{\alpha,X} = Z_\alpha = Z$ for all $X \in \exp\omega$.

We have $Z^{\mathfrak{c}} = \prod\limits_{X\in\exp\omega} YX$. Fix a point $y_0 \in Y$ and $\widetilde{\alpha} \in A$.

Let $\eta \subseteq \operatorname{Exp} Z$ be a family of countable subsets of Z, $|\eta| \le 2^\omega$.

For every element $G \subseteq Z$ of family η we define a set $G' \in Y$:

$$G' = \{y \in Y\colon \pi_{\widetilde{\alpha}}(y) \in G \text{ and } \pi_\alpha(y) = \pi_\alpha(y_0) \text{ for } \alpha \ne \widetilde{\alpha}\}.$$

Let $\eta' = \{G'\colon G \in \eta\}$.

Consider $Z^{\mathfrak{c}} = \prod\limits_{X\in\exp\omega} YX = \prod\limits_{X\in\exp\omega} \Big(\prod\limits_{\alpha\in A} Z_{\alpha,X} \Big)$.

In $Z^{\mathfrak{c}} = \prod\limits_{X\in\exp\omega} YX$ there is a countable set $Q \subseteq Z^{\mathfrak{c}}$, satisfying (*).

Let $E \subseteq Q \subseteq Z^{\mathfrak{c}} = \prod\limits_{X\in\exp\omega} YX$ be a countable set.

By (*) there is a countable subset $E' \subseteq E$ such that for every $G' \in \eta'$ there is $X \in \exp\omega$ such that $\pi_X(E') = G'$ and $\pi_X(E')\colon E' \to G'$ is a finite-to-one mapping.

Let $G \in \eta$ and consider corresponding $G' \in \eta'$. Then there is $\widetilde{X} \in \exp\omega$ such that

$$\pi_{\widetilde{X}}(E') = G'$$

where $\pi_{\widetilde{X}}\colon \prod\limits_{X\in\exp\omega} YX \to Y_{\widetilde{X}}$ is a $\widetilde{X}$-projection from $\prod\limits_{X\in\exp\omega} YX$ on $Y_{\widetilde{X}} = \prod\limits_{\alpha\in A} Z_{\alpha,\widetilde{X}}$ and $\pi_{\widetilde{X}}|_{E'}\colon E' \to G'$ is a finite-to-one mapping.

Consider $\pi_{\widetilde{\alpha},\widetilde{X}}\colon \prod\limits_{\alpha\in A} Z_{\alpha,\widetilde{X}} \to Z_{\widetilde{\alpha},\widetilde{X}}$.

Then $\pi_{\widetilde{\alpha},\widetilde{X}}(\pi_{\widetilde{X}}(E')) = \pi_{\widetilde{\alpha},\widetilde{X}}(G')) = G$.

The mapping $\pi_{\widetilde{\alpha},\widetilde{X}} \circ \pi_{\widetilde{X}}$ is a projection of $Z^{\mathfrak{c}} = \prod\limits_{X\in\exp\omega} YX = \prod\limits_{X\in\exp\omega} \Big(\prod\limits_{\alpha\in A} Z_{\alpha,X} \Big)$ on $Z_{\widetilde{\alpha},\widetilde{X}}$.

And $\pi_{\widetilde{\alpha},\widetilde{X}} \circ \pi_{\widetilde{X}}|_{E'}$ is a finite-to-one mapping. $\square$

Consider some facts, which follow from Theorem 4.1.

Theorem 4.2. *Let Z be a not single point separable T_1-space. Then there is a countable dense subset $Q \subseteq Z^{\mathfrak{c}}$ which contains no convergent in $Z^{\mathfrak{c}}$ nontrivial sequences.*

Proof. Let Z be a separable not single point T_1-space.

Consider Z^ω.

There are two countable subsets of Z^ω with disjoint closures.

Indeed, Z is not single point T_1-space, and let $x, y \in Z$, $x \neq y$. Consider $\pi^{-1}(x)$ and $\pi^{-1}(y)$ where $\pi\colon Z^\omega \to Z$ is a projection on Z. Then $\pi^{-1}(x)$ and $\pi^{-1}(y)$ are closed infinite sets, $\pi^{-1}(x) \cap \pi^{-1}(y) = \emptyset$.

Let $\Phi_x \subseteq \pi^{-1}(x)$ and $\Phi_y \subseteq \pi^{-1}(y)$ be countable subsets of Z^ω, then $[\Phi_x] \cap [\Phi_y] \neq \emptyset$.

Consider a subset $G = \Phi_x \cup \Phi_y$ of the space Z^ω and let $\eta = \{G\}$ be a family consisting of the only one element G.

Consider $\prod\limits_{X \in \exp\omega} YX$, where $Y_X = Z^\omega$ for all $X \in \exp\omega$.

$\prod\limits_{X \in \exp\omega} YX$ is naturally homeomorphic to $Z^{\mathfrak{c}}$.

By Theorem 4.1 in $\prod\limits_{X \in \exp\omega} YX$ there is a countable dense set $Q \subseteq \prod\limits_{X \in \exp\omega} YX$, satisfying (*) of the Theorem 4.1 for η.

Suppose, there is a non-trivial convergent sequence $\xi = \{x_n \colon n \in \omega\} \subseteq Q$.

Then there is a countable subset E' of $\{x_n \colon n \in \omega\}$ and X such that for $\pi_X : \prod\limits_{X \in \exp\omega} YX \to Y_X$ we have $\pi_X(E') = \Phi_x \cup \Phi_y = G$ and $\pi_{X|_{E'}} \colon E' \to \Phi_x \cup \Phi_y$ is an "on" and finite-to-one mapping. Since $[\Phi_x] \cap [\Phi_y] = \emptyset$, we get that ξ contains two subsequences with disjoint closures. This contradicts the fact that ξ is a convergent sequence. □

From this it follows, in particular, that in the Tychonoff product $Z_m^{\mathfrak{c}}$, where Z_m is an infinite space with minimal T_1-topology, there is a countable dense set which contains no non-trivial convergent sequences.

Consider a countable product $Z^\omega = \prod\limits_{i \in \omega} Z_i$, where $Z = Z_i$ $(i \in \omega)$ is an uncountable space.

Then for every countable dense subset $Q \subseteq Z^\omega$ there is a point $z \in Z$ such that for a set $Z_z = Z \setminus \{z\}$ the following holds: $Z_z^\omega \supseteq Q$.

In the case of $Z^{\mathfrak{c}}$ we have the following.

Theorem 4.3. *Let Z be a separable space, $\omega \leq |Z| \leq \mathfrak{c}$. Then in $Z^{\mathfrak{c}}$ there is a countable dense subset Q, which satisfies the following condition:*

for all $z \in Z$ the following holds: $|Z_z^{\mathfrak{c}} \cap Q| < \omega$ where $Z_z = Z \setminus \{z\}$.

Proof. For a space Z let η be the family of all countable subsets of Z. Let Q be a countable dense set of $Z^{\mathfrak{c}}$, which satisfies (*) of Theorem 4.1 for η.

Suppose that for some $z \in Z$ we have that a set $E = Z_z^{\mathfrak{c}} \cap Q$ is countable. Let $z \in G$ for some $G \in \eta$. By (*) of Theorem 4.1, there is $E' \subseteq E$ and $X \in \exp\omega$ such that $\pi_X(E') = G$. Then $\pi_X(E') \ni z$. But since $E' \subseteq Z_z^{\mathfrak{c}} \cap Q$ we have that $\pi_X(E') \subseteq Z_z$ and therefore $z \notin \pi_X(E')$. Contradiction. □

For a space $I^{\mathfrak{c}}$ in every dense subset, in particular in Q, which has no convergent sequences, there is a sequence $E \subseteq Q$, which converges to a set F such that $E \cap F = \emptyset$. For $R^{\mathfrak{c}}$ we have the following.

Theorem 4.4. *Let Z be a separable not countably compact T_1-space. Then there is a countable dense subset $Q \subseteq Z^{\mathfrak{c}}$ satisfying the following condition: if $E \subseteq Q$ is a countable set and E converges to a set $F \subseteq Z^c$, then $|E \setminus F| < \omega$.*

Proof. Let $G \subseteq Z$ be a countable discrete closed set and the family $\eta = \{G\}$ consists of the only set G.

Then for η there is a countable dense set of $Q \subseteq Z^{\mathfrak{c}}$ satisfying (*) of the Theorem 4.1.

Let us prove the that for every countable subset $E \subseteq Q$ there is a countable discrete and closed subset $E' \subseteq E$.

Let $E \subseteq Q$.

By (*) of Theorem 4.1 for E there is a countable E' such that for some $X \in \exp \omega$ we have $\pi_X(E') = G$ and π_X on E' is a finite-to-one mapping of E'.

Let us prove that E' is discrete and closed.

Let us prove that E' discrete.

Let $z \in E'$. Consider $\pi_X(z)$. Since G is discrete, there is an open set $U \subseteq Z$ such that $U \cap G = \pi_X(z)$. Then $E' \cap \pi_X^{-1}(U) = \pi_X^{-1}(\pi_X(z))$.

Since π_X on E' is a finite-to-one mapping of E', a set $E' \cap \pi_X^{-1}(U) = \pi_X^{-1}(\pi_X(z))$ is finite.

Therefore there is an open in $Z^{\mathfrak{c}}$ neighborhood Oz of z such that $Oz \cap E' = \{z\}$.

Let us prove that E' is closed. Let $z \in Z^{\mathfrak{c}} \setminus E'$.

Let $\pi_X(z) \in G$. By the same arguments as before, we can prove that there is a neighborhood Oz such that $Oz \cap E' = \emptyset$.

If $\pi_X(z) \notin G$ then, since G is closed, there is a neighborhood U of $\pi_X(z)$ such that $U \cap G(X) = \emptyset$. Then $\pi^{-1}(U)$ is a neighborhood of z and $\pi^{-1}(U) \cap E' = \emptyset$.

Now let E be a countable subset of Q and $F \subseteq Z^c$ be such that E converges to F. Suppose $|E \setminus F| = \omega$. Then there is a countable $E' \subseteq E \setminus F$ which is discrete and closed in $Z^{\mathfrak{c}}$.

Then $U = Z^{\mathfrak{c}} \setminus E'$ is a neighborhood of F and $|E \setminus U| = \omega$. Contradiction. □

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