Evasion from Pursuers in the Problem of Group Pursuit with Fractional Derivatives and Phase Constraints

A. S. Bannikov
Udmurt State University
Izhevsk, Russia
asbannikov@gmail.com

***Abstract*—In this paper we consider the evasion problem from the group of pursuers in the finite-dimensional Euclidean space. The motion is describe by the linear system of fractional order**

$$\left({}^C D^\alpha_{0+} z_i\right) = A z_i + u_i - v,$$

where ${}^C D^\alpha_{0+} f$ is the Caputo derivative of order $\alpha \in (0,1)$ of the function f and A is a simple matrix. The initial positions are given at the initial time. The set of admissible controls of all players is a convex compact. It is further assumed that the evader does not leave the convex polyhedron with nonempty interior. In terms of the initial positions and the parameters of the game, sufficient conditions for the solvability of the evasion problem are obtained.

I. Introduction

An important direction in the development of the modern theory of differential games is associated with the development of methods for solving game problems of pursuit-evasion with the participation of several objects [1], [2], [6], moreover, in addition to deepening the classical methods of solving, actively searching for new tasks to which the developed methods. In particular, the problems of the pursuit of two persons, described by equations with fractional derivatives, where sufficient capture conditions were obtained, were considered in papers [3], [7]–[9].

In this paper we consider one task of pursuing a group of pursuers by one evader, provided that the motion of all participants is described by linear equations with derivatives fractional in the Caputo sense, the matrix of the system is simple, and the runaway does not leave the limits of a convex polyhedral set.

Sufficient conditions for evasion are obtained, expressed in terms of the initial positions and parameters of the game. The work continues the research [4].

Definition 1 (see [10, p. 97]): Let $f\colon [0,\infty) \to \mathbb{R}^k$ — be an absolutely continuous function and $\alpha \in (0,1)$. *Caputo derivative of order* α of a function f is a function ${}^C D^\alpha_{0+} f$ of the form

$$\left({}^C D^\alpha_{0+} f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha}\, d\tau.$$

In the space $\mathbb{R}^k$ ($k \geqslant 2$) $n+1$ persons the differential game is considered: n pursuers P_i, $i = 1, \ldots, n$, and the evader E. The law of motion of each of the pursuers P_i has the form

$$\left({}^C D^\alpha_{0+} x_i\right)(t) = a x_i(t) + u_i(t), \quad x_i(0) = x_i^0, \quad u_i \in Q. \quad (1)$$

The law of motion of the evader E has the form

$$\left({}^C D^\alpha_{0+} y\right)(t) = a y(t) + v(t), \quad y(0) = y^0, \quad v \in Q. \quad (2)$$

Here $\alpha \in (0,1)$, x_i, y, u_i, $v \in \mathbb{R}^k$, Q is a convex compactum in $\mathbb{R}^k$, a is a real number. We assume that $x_i^0 \neq y^0$ for all i. In addition, it is assumed that the evader E does not leave a convex polyhedral set with a nonempty interior Ω of the form

$$\Omega = \left\{\xi \in \mathbb{R}^k \,\middle|\, \langle p_j, \xi\rangle \leqslant \mu_j, \quad j = 1, \ldots, r, \quad r \geqslant 0\right\},$$

where p_j, $j = 1, \ldots, r$, are the unit vectors of $\mathbb{R}^k$, μ_j, $j = 1, \ldots, r$, are real numbers such that $\operatorname{int}\Omega \neq 0$; for $r = 0$ we assume that $\Omega = \mathbb{R}^k$.

Instead of the systems (1), (2) we consider systems

$$\begin{aligned} \left({}^C D^\alpha_{0+} z_i\right)(t) &= a z_i(t) + u_i(t) - v(t), \\ z_i(0) = z_i^0 &= x_i^0 - y^0, \quad u_i, v \in Q. \end{aligned} \quad (3)$$

Let $T > 0$ be an arbitrary positive number, $\sigma = \{0 = t_0 < t_1 < \ldots < t_{s+1=T}\}$ is a finite partition of the interval $[0, T]$.

Definition 2: The piecewise-program strategy S_E of the evader E, given on $[0, T]$, corresponding to a partition σ is a family of mappings b_l, $l = 0, \ldots, s$, that correspond the values $(t_l, z_1(t_l), \ldots, z_n(t_l))$ to a measurable function $v_l(t)$, defined for $t \in [t_l, t_{l+1})$ and such that $v_l(t) \in Q$, $y(t) \in \Omega$ for all $t \in [t_l, t_{l+1}]$.

Definition 3: The piecewise-program counterstrategy CS_{P_i} of the pursuer P_i, given on $[0, T]$, corresponding to the partition σ is the family of mappings c_l, $l = 0, \ldots, s$, that correspond the values $(t_l, z_1(t_l), \ldots, z_n(t_l))$ and control $v_l(t)$, $t \in [t_l, t_{l+1})$, to a measurable function $u_l^i(t)$, defined for $t \in [t_l, t_{l+1})$ and such that $u_l^i(t) \in Q$ for all $t \in [t_l, t_{l+1}]$.

Definition 4: In the differential game *evasion from the meeting* occurs, if for any $T > 0$ there are partition σ of the interval $[0, T]$ and the strategy S_E of the evader E such that for any trajectories $x_i(t)$ of pursuers P_i the following holds

$$x_i(t) \neq y(t), \quad t \in [0, T]. \quad (4)$$

978-1-5386-4556-7/18/$31.00 ©2018 IEEE

Let $\operatorname{Int} A$ and $\operatorname{co} A$ be the interior and convex hull of set A respectively, $I(l) = \{1, 2, \ldots, n+l\}$, $E_\rho(z;\mu) = \sum_{k=0}^{\infty} \dfrac{z^k}{\Gamma(\mu + k/\rho)}$ —is a generalized Mittag-Leffler function,

$$\lambda_i(v) = \max\left\{\lambda \geqslant 0 \,\middle|\, -\lambda z_i^0 \in Q - v\right\}, \quad i \in I(0),$$
$$\lambda_{n+j}(v) = \langle p_j, v\rangle, \quad j = 1, \ldots, r,$$
$$\delta_r = \min_{v \in Q} \max_{l \in I(r)} \lambda_l(v).$$

II. Sufficient conditions for the evasion problem

Theorem 1: Let $r = 0$ ($\Omega = \mathbb{R}^k$), $\delta_0 = 0$. Then in a differential game an evasion occurs.

Proof: From the hypothesis of the theorem it follows that there exist $v_0 \in Q$ such that

$$\lambda_i(v_0) = 0 \text{ for all } i \in I(0).$$

Let $T > 0$ be an arbitrary positive number. We define the strategy of the evader E, assuming

$$\sigma = \{0, T\}, \quad v(t) = v_0 \text{ for all } t \in [0, T].$$

Let $u_i(t)$ be an arbitrary admissible control of the pursuer P_i. Then the solutions of the systems (3) are representable in the form

$$z_i(t) = E_{1/\alpha}(at^\alpha; 1) z_i^0 +$$
$$+ \int_0^t (t-\tau)^{\alpha-1} E_{1/\alpha}(a(t-\tau)^\alpha; \alpha)(u_i(\tau) - v_0)\, d\tau =$$
$$= E_{1/\alpha}(at^\alpha; 1) z_i^0 + t^\alpha E_{1/\alpha}(at^\alpha; \alpha+1)(\bar{u_i}^t - v_0),$$

where

$$\bar{u_i}^t = \frac{\int_0^t (t-\tau)^{\alpha-1} E_{1/\alpha}(a(t-\tau)^\alpha; \alpha) u_i(\tau)\, d\tau}{t^\alpha E_{1/\alpha}(at^\alpha; \alpha+1)} \in Q.$$

Indeed, from Theorem 4.1.1 [5, p. 101] it follows that for $0 < \alpha < 1$ the functions $E_{1/\alpha}(z; 1)$ and $E_{1/\alpha}(z; \alpha+1)$ are positive for all $z \in \mathbb{R}^1$, therefore for any value of a and for all $t > 0$ $\bar{u_i}^t \in Q$ by the convexity of Q.

We show that $z_i(t) \neq 0$ for all $i = 1, \ldots, n$ for all $0 < t \leqslant T$. We argue by contradiction. If for some i_0 there was a moment of time t^* such that the equality $z_{i_0}(t^*) = 0$ holds, this would mean that $\bar{u_{i_0}}^{t^*} = v_0 - \dfrac{E_{1/\alpha}(at^\alpha; 1)}{t^\alpha E_{1/\alpha}(at^\alpha; \alpha+1)} z_i^0$. By the definition of λ_{i_0} this would mean that $\lambda_{i_0}(v_0) \geqslant \dfrac{E_{1/\alpha}(a(t^*)^\alpha; 1)}{t^\alpha E_{1/\alpha}(a(t^*)^\alpha; \alpha+1)} > 0$, which contradicts the hypothesis of the theorem. Thus the theorem 1 is proved. ■

Corollary 1: Let $r = 0$ ($\Omega = \mathbb{R}^k$), V be a convex strictly convex compact set with a smooth boundary and

$$0 \notin \operatorname{Int} \operatorname{co}\left\{z_1^0, \ldots, z_n^0\right\}.$$

Then in a differential game, an evasion occurs.

Proof: It follows from the conditions of the corollary and [1, p. 15] that $\delta_0 = 0$, that is, in the differential game, an evasion occurs due to the theorem 1. ■

Theorem 2: Let $r = 1$, $\delta_1 = 0$ and $a\mu_1 \leqslant 0$. Then in a differential game, an evasion occurs.

Proof: From the hypothesis of the theorem it follows that there exists $v_0 \in Q$ such that

$$\lambda_i(v_0) = 0 \text{ for all } i \in I(0), \qquad \lambda_{n+1}(v_0) = \langle p_1, v_0\rangle \leqslant 0.$$

It follows from the theorem 1 that if for any $T > 0$ we take the partition $\sigma = \{0, T\}$ and $v(t) = v_0$, then neither one of the pursuers P_i does not catch up with the evader E, since the equality $z_i(t) = 0$ is impossible because $\lambda_i(v_0) = 0$, $i \in I(0)$. It remains only to make sure that the evader does not leave the set Ω. Consider the scalar product $\langle p_1, y(t)\rangle$:

$$\langle p_1, y(t)\rangle = E_{1/\alpha}(at^\alpha; 1)\langle p_1, y^0\rangle +$$
$$+ t^\alpha E_{1/\alpha}(at^\alpha; \alpha+1)\langle p_1, v_0\rangle \leqslant$$
$$\leqslant E_{1/\alpha}(at^\alpha; 1)\langle p_1, y^0\rangle.$$

For $a < 0$ the function $E_{1/\alpha}(at^\alpha; 1)$ decreases monotonically on the interval $[0, T]$ from $E_{1/\alpha}(0; 1) = 1$ to $E_{1/\alpha}(aT^\alpha; 1) > 0$, since its derivative $\frac{d}{dt} E_{1/\alpha}(at^\alpha; 1) = at^{\alpha-1} E_{1/\alpha}(at^\alpha; \alpha) < 0$ for all $t > 0$. Therefore, for $\mu_1 \geqslant 0$ and for all $t > 0$

$$\langle p_1, y(t)\rangle \leqslant E_{1/\alpha}(at^\alpha; 1)\langle p_1, y^0\rangle \leqslant \mu_1.$$

For $a = 0$ $E_{1/\alpha}(at^\alpha; 1) = E_{1/\alpha}(0; 1) = 1$, therefore for any μ_1 and for all $t > 0$

$$\langle p_1, y(t)\rangle \leqslant E_{1/\alpha}(0; 1)\langle p_1, y^0\rangle \leqslant \mu_1.$$

For $a > 0$ the function $E_{1/\alpha}(at^\alpha; 1)$ increases monotonically on the interval $[0, T]$ from $E_{1/\alpha}(0; 1) = 1$ to $E_{1/\alpha}(aT^\alpha; 1) > 1$. Therefore, for $\mu_1 \leqslant 0$ and for all $t > 0$

$$\langle p_1, y(t)\rangle \leqslant E_{1/\alpha}(at^\alpha; 1)\langle p_1, y^0\rangle \leqslant \mu_1.$$

Hence, under the conditions of the theorem, the control $v(t) = v_0$ leaves the evader E in the set Ω. The theorem is proved. ■

Theorem 3 (evasion in a cone): Let $r \geqslant 1$, $\delta_r = 0$ and $\mu_j = 0$, $j = 1, \ldots, r$ (Ω be a convex cone). Then in a differential game, an evasion occurs.

Proof: From the hypothesis of the theorem it follows that there exists $v_0 \in Q$ such that

$$\lambda_i(v_0) = 0 \text{ for all } i \in I(0),$$
$$\lambda_{n+j}(v_0) = \langle p_j, v_0\rangle \leqslant 0, \quad j = 1, \ldots, r.$$

It follows from the theorem 1 that if for any $T > 0$ we take the partition $\sigma = \{0, T\}$ and $v(t) = v_0$, then neither one of the pursuers P_i does not catch up with the evader E, since the equality $z_i(t) = 0$ is impossible because $\lambda_i(v_0) = 0$, $i \in I(0)$. It follows from the theorem 2 that the evader does not leave the set Ω, since for any value a and for any $j = 1, \ldots, r$ the following inequalities holds:

$$\langle p_j, y(t)\rangle = E_{1/\alpha}(at^\alpha; 1)\langle p_j, y^0\rangle +$$
$$+ t^\alpha E_{1/\alpha}(at^\alpha; \alpha+1)\langle p_j, v_0\rangle \leqslant$$
$$\leqslant E_{1/\alpha}(at^\alpha; 1)\langle p_j, y^0\rangle \leqslant \mu_j = 0. \quad (5)$$

Hence, under the conditions of the theorem, the control $v(t) = v_0$ leaves the evader E in the convex cone Ω. The theorem is proved. ■

Corollary 2: Let $r > 0$, $\delta_r = 0$ and $a\mu_j \leqslant 0$ for all $j = 1, \dots, r$. Then in a differential game, an evasion occurs.

Proof: From the condition $\delta_r = 0$ and the theorem 1 it follows that if for an arbitrary $T > 0$ we take the partition $\sigma = \{0, T\}$ and control $v(t) = v_0$, then none of the pursuers P_i catches up with the evader E, since the equality $z_i(t) = 0$ is impossible because $\lambda_i(v_0) = 0$, $i \in I(0)$. The conditions $a\mu_j \leqslant 0$, $j = 1, \dots, r$, ensure the inequalities (5) holds, , which in turn mean that the evader does not leave the set Ω for all $t > 0$. Hence, in the differential game, an evasion occurs. ■

ACKNOWLEDGMENT

This work is financially supported by the Russian Foundation for Basic Research, projects no. 16–01–000346-a, no. 18–51–41005.

REFERENCES

[1] A. I. Blagodatskikh and N. N. Petrov, *Conflict interaction of groups of controlled objects (in Russian)*, Izhevsk, Russia: Udmurt State University, 2009.

[2] N. N. Krasovskii and A. I. Subbotin, *Positional differential games (in Russian)*, Moscow, Russia: Nauka, 1974.

[3] M. Sh. Mamatov, "The pursuit problem, described by differential equations of fractional order (in Russian)," *Aktual'nye Problemy Gumanitarnykh i Estestvennykh Nauk*, no. 4-1, pp. 28–32, 2016.

[4] N. N. Petrov, "One problem of group pursuit with fractional derivatives and phase constraints (in Russian)," *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, vol. 27, issue 1, pp. 54–59, 2017.

[5] A. Yu. Popov and A. M. Sedletskii. "Distribution of roots of Mittag-Leffler functions," *Journal of Mathematical Sciences*, vol. 190, issue 2, pp. 209–409, 2013.

[6] A. A. Chikrii, *Conflict-controlled processes*, Dordrecht, Netherlands: Springer, 1997.

[7] A. A. Chikrii and I. I. Matichin, "Game problems for fractional-order linear systems," *Proceedings of the Steklov Institute of Mathematics*, vol. 268, suppl. 1, pp. 54–70, 2010.

[8] A. A. Chikrii and I. I. Matichin, "On linear conflict-controlled processes with fractional derivatives (in Russian)," *Trudy Instituta Matematiki i Mekhaniki URO RAN*, vol. 17, no. 2, pp. 256–270, 2011.

[9] S. D. Éidel'man and A. A. Chikrii, "Dynamic game problems of approach for fractional-order equations," *Ukrainian Mathematical Journal*, vol. 52, issue 11, pp. 1787–1806, 2000.

[10] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam, Netherlands: Elsevier, 2006.