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PROPORTIONAL LOCAL ASSIGNABILITY OF LYAPUNOV SPECTRUM OF LINEAR DISCRETE TIME-VARYING SYSTEMS*

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Abstract. We consider a local version of the pole assignment problem for linear discrete time-varying systems. Our aim is to obtain sufficient conditions to place the Lyapunov spectrum of the closed-loop system in an arbitrary position within some neighborhood of the Lyapunov spectrum of the free system using an appropriate time-varying linear feedback. Moreover, we assume that the norm of the matrix of linear feedback should be bounded from above by the distance between these two spectra with some constant multiplier. We prove that diagonalizability, Lyapunov regularity, and stability of the Lyapunov spectrum each separately are the required sufficient conditions provided that the open-loop system is uniformly completely controllable.

Key words. pole assignment problem, proportional local assignability, Lyapunov spectrum, linear discrete time-varying systems

AMS subject classifications. 93B55, 37J40, 39A06, 39A22

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1. Introduction.

It is well known [34] that for a linear time-invariant system the controllability is equivalent to the possibility of assigning an arbitrary set of the eigenvalues of the closed-loop system by choosing a suitable time-invariant feedback.

The search for an analogue of this property for linear time-varying systems has a long history and has not been completed yet [1, 17, 18, 22, 29, 35]. Even the formulation for the problem encountered many difficulties. First, because for time-varying systems we have many nonequivalent concepts of controllability [21]. Second, because we have no proper replacement for the concept of poles, but their role, to a

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certain extent, is taken by some numerical characteristics such as the Lyapunov and Bohl exponents [2].

The Lyapunov exponents describe exponential growth or decay of system trajectory and, in particular, characterize exponential stability. In turn, the Bohl exponents are related to the uniform growth or decay rate of the trajectory and characterize the uniform asymptotic stability. For linear time-invariant systems, the Bohl and the Lyapunov exponents coincide and, for discrete-time systems, they are equal to the logarithm of absolute value of the poles.

Research on the pole placement problem for time-varying systems may be divided into three groups. Historically, the first group is composed of works in which the relationship between different types of controllability and the possibility of finding a feedback loop, for which the closed-loop system is time-invariant with in-advance fixed poles, is studied. This approach is presented in [29, 35]. The second group is formed by the works where the relationship between controllability and various types of stabilizability, or more broadly the existence of feedback, which guarantees that all trajectories tend to zero at a rate faster than an in-advance given rate, is studied in [1, 17, 18]. Finally, the third group contains studies concerning the possibility of the placement of some numerical characteristics (including the Lyapunov and the Bohl exponents) through a feedback in given locations [3, 26, 27, 31, 32].

This paper belongs to the third group and it contains a continuation of the research presented in [3]. Here, we consider a local version of the pole assignment problem for linear discrete time-varying systems, whereas in [3] a global version was investigated. Our aim is to obtain sufficient conditions to place the Lyapunov spectrum of the closed-loop system in an arbitrary position within some neighborhood of the Lyapunov spectrum of the free system using some time-varying linear feedback. Moreover, we assume that the norm of the feedback matrix should be bounded from above by the distance between these two spectra, with some constant coefficient. We say that the Lyapunov spectrum is proportionally locally assignable if all of these assumptions are valid. Our main result is that diagonalizability, Lyapunov regularity, and stability of the Lyapunov spectrum each separately are sufficient conditions when the open-loop system is uniformly completely controllable. In [3] we have shown that uniform complete controllability is a sufficient condition for global assignability of the Lyapunov spectrum. The question of the necessity of this condition is a subject of our current research.

The paper is organized as follows. In section 2, we introduce the basic notation and definitions, including the concept of proportional local assignability of the Lyapunov spectrum. In section 3, we prove some auxiliary results concerning the uniform complete controllability. We consider the concept of the uniform complete controllability as the most suitable one for the pole placement problem for time-varying systems. Section 4 is devoted to the concept of dynamic equivalence of linear systems. The main object of section 5 is multiplicatively perturbed systems. Section 6 contains the main result of the paper. In section 7 we discuss our results and present some open questions. The work is ended with conclusions.

2. Basic notation and formulation of the problem.

Let $\mathbb{R}^s$ be the s-dimensional Euclidean space with a fixed orthonormal basis $e_1, \ldots, e_s$ and the standard norm $\|\cdot\|$. By $\mathbb{R}^{s\times t}$ we shall denote the space of all real matrices of the size $s\times t$ with the spectral norm, i.e., with the operator norm generated in $\mathbb{R}^{s\times t}$ by Euclidean norms in $\mathbb{R}^s$ and $\mathbb{R}^t$, respectively. By $[a_1, \ldots, a_t] \in \mathbb{R}^{s\times t}$ we denote a matrix with sequential columns $a_1, \ldots, a_t \in \mathbb{R}^s$; $I \in \mathbb{R}^{s\times s}$ is the identity matrix. For any sequence

$F = \big(F(n)\big)_{n\in\ \mathbb{N}} \subset \mathbb{R}^{s\times t}$ we define

$$\|F\|_\infty = \sup_{n\in\mathbb{N}} \|F(n)\|.$$

A bounded sequence $\big(L(n)\big)_{n\in\mathbb{N}} \subset \mathbb{R}^{s\times s}$ of invertible matrices such that $\big(L^{-1}(n)\big)_{n\in\mathbb{N}}$ is bounded will be called the Lyapunov sequence.

By $\mathbb{R}^s_{\leq}$ we denote the set of all nondecreasing sequences of s real numbers. For a fixed sequence $\mu = (\mu_1,\ldots,\mu_s) \in \mathbb{R}^s_{\leq}$ and any $\delta > 0$ let us denote by $O_\delta(\mu)$ the set of all sequences $\nu = (\nu_1,\ldots,\nu_s) \in \mathbb{R}^s_{\leq}$ such that $\max_{j=1,\ldots,s} |\nu_j - \mu_j| < \delta$. In other words, $O_\delta(\mu)$ is a δ-neighborhood of the sequence $\mu \in \mathbb{R}^s_{\leq}$ with respect to the metric generated by the vector l_∞ norm of the space $\mathbb{R}^s$ [16, p. 265] on its subset $\mathbb{R}^s_{\leq}$.

We consider a discrete linear time-varying system

$$x(n+1) = A(n)x(n) + B(n)u(n), \quad n \in \mathbb{N}, \tag{2.1}$$

with a Lyapunov sequence $A = (A(n))_{n\in\mathbb{N}} \subset \mathbb{R}^{s\times s}$, a bounded sequence $B = (B(n))_{n\in\mathbb{N}} \subset \mathbb{R}^{s\times t}$, and a control sequence $u = (u(n))_{n\in\mathbb{N}} \subset \mathbb{R}^t$.

Let

$$a \doteq \|A\|_\infty + \|A^{-1}\|_\infty + \|B\|_\infty < \infty.$$

Note that

$$\|A\|_\infty + \|A^{-1}\|_\infty \geq \|A(1)\| + \|A^{-1}(1)\| \geq \|A(1)\| + \|A(1)\|^{-1} \geq 2;$$

hence $a \geq 2$.

We shall denote the transition matrix of the free system

$$x(n+1) = A(n)x(n) \tag{2.2}$$

by $\Phi_A(n,m)$, $n,m \in \mathbb{N}$, and the Lyapunov spectrum of the system (2.2) by

$$\lambda\,(A) = \big(\lambda_1(A), \lambda_2(A), \ldots, \lambda_s(A)\big) \in \mathbb{R}^s_{\leq}$$

(see [3] for definitions of these concepts).

The solution of (2.1) corresponding to a control $u = \big(u(n)\big)_{n\in\mathbb{N}}$ and the initial condition $x(k_0) = x_0$, where $k_0 \in \mathbb{N}$, $x_0 \in \mathbb{R}^s$, is denoted by $(x\,(n, k_0, x_0, u))_{n\geq k_0}$. Using the variation of constants formula this solution can be written in the form (see [13, p. 130], [14, p. 20], [21, p. 92])

$$x\,(n, k_0, x_0, u) = \Phi_A\,(n, k_0)\,x_0 + \sum_{j=k_0}^{n-1} \Phi_A\,(n, j+1)\,B(j)u(j), \quad n > k_0. \tag{2.3}$$

For a sequence $U = (U(n))_{n\in\mathbb{N}} \subset \mathbb{R}^{t\times s}$ we consider a linear feedback control for system (2.1),

$$u(n) = U(n)x(n), \quad n \in \mathbb{N}.$$

We identify the control u with the sequence U. The sequence U is called a feedback control for the system (2.1).

Definition 2.1 (see [3]). *A bounded sequence*

$$U = (U\,(n))_{n\in\mathbb{N}} \subset \mathbb{R}^{t\times s}$$

is said to be an admissible feedback control for system (2.1) *if* $(A\,(n) + B\,(n)\,U\,(n))_{n\in\mathbb{N}}$ *is a Lyapunov sequence.*

Let $U = (U(n))_{n\in\mathbb{N}}$ be any admissible feedback control for the system (2.1). Then for a closed-loop system

$$x(n+1) = (A(n) + B(n)U(n))\, x(n) \tag{2.4}$$

we can define the Lyapunov spectrum $\lambda(A + BU) \in \mathbb{R}^s_{\leq}$.

DEFINITION 2.2. *The Lyapunov spectrum of the system* (2.4) *is called proportionally locally assignable if there exist $\ell > 0$ and $\delta > 0$ such that for all*

$$\mu = (\mu_1, \dots, \mu_s) \in O_\delta\big(\lambda(A)\big)$$

there exists an admissible feedback control $U = (U(n))_{n\in\mathbb{N}}$ for the system (2.1)*, satisfying the estimate*

$$\|U\|_\infty \leq \ell \max_{j=1,\dots,s} |\lambda_j(A) - \mu_j| \tag{2.5}$$

and providing the validity of the relation

$$\lambda(A + BU) = \mu. \tag{2.6}$$

In this paper we obtain sufficient conditions for the local proportional assignability of the Lyapunov spectrum of the system (2.4). We notice that in [3] a problem of the global assignability of the spectrum (2.4) was considered. The Lyapunov spectrum of the system (2.4) is called globally assignable if for any $\mu \in \mathbb{R}^s_{\leq}$ there exists an admissible feedback control $U = \big(U(n)\big)_{n\in\mathbb{N}}$ for the system (2.1), such that $\lambda(A+BU) = \mu$. In [3] it was shown that the uniform complete controllability of the system (2.1) is a sufficient condition for global assignability of the Lyapunov spectrum of (2.4). However, it is not a sufficient condition for proportional local assignability since a small shift of the Lyapunov spectrum may require a feedback with a large norm.

3. Controllability. In the literature several nonequivalent definitions of controllability of the system (2.1) are considered. There are global controllability, controllability from zero, to zero, and others (see [21]). In the paper we shall use the concept of uniform complete controllability [36]. Its formal definition is as follows.

DEFINITION 3.1. *The system* (2.1) *is uniformly completely controllable if there exist a positive p_0 and a natural K such that for all $x_0 \in \mathbb{R}^s$ and $k_0 \in \mathbb{N}$ there exists a control sequence $u(n)$, $n = k_0, k_0 + 1, \dots, k_0 + K - 1$, such that*

$$x\left(k_0 + K, k_0, x_0, u\right) = 0$$

and

$$\|u(n)\| \leq p_0 \left\|x_0\right\| \tag{3.1}$$

for all $n = k_0, k_0 + 1, \dots, k_0 + K - 1$. In such a situation we shall also say that the system (2.1) *is K-uniformly completely controllable.*

In controllability investigation a crucial role is played by the Kalman controllability matrix

$$W\left(k, n\right) = \sum_{j=k}^{n-1} \Phi_A\left(k,\, j+1\right) B(\,j) B^T(\,j) \Phi_A^T\left(k,\, j+1\right),$$

where $n > k$, $n, k \in \mathbb{N}$. The next theorem gives, in the terms of the Kalman controllability matrix, necessary and sufficient conditions for K-uniform complete controllability.

THEOREM 3.2 (see [15, Proposition 3, p. 34]). *Suppose that A is a Lyapunov sequence and B is bounded. Then the system* (2.1) *is uniformly completely controllable if and only if there exist a positive γ and a natural K such that*

$$W(k_0, k_0 + K) \geq \gamma I \tag{3.2}$$

for all $k_0 \in \mathbb{N}$.

Before we discuss the main problem of the local assignability of the Lyapunov spectrum we shall prove some auxiliary results concerning the uniform complete controllability. Similar results for continuous-time systems have been proven in article [25] (see also [27, Lemma 12.1, Theorem 12.1]).

LEMMA 3.3. *The system* (2.1) *is K-uniformly completely controllable if and only if there exists $p > 0$ such that for all $k_0 \in \mathbb{N}$ and each matrix $H \in \mathbb{R}^{s\times s}$ there exists a sequence $V(n)$, $n = k_0, \ldots, k_0 + K - 1$, of elements of $\mathbb{R}^{t\times s}$ such that*

$$\|V(n)\| \leq p \|H - I\|$$

and the sequence $Z(n)$, $n = k_0, k_0 + 1, \ldots, k_0 + K$, given by

$$Z(n + 1) = Z(n) + \Phi_A (k_0, n + 1) B(n)V(n) \tag{3.3}$$

and the initial condition

$$Z(k_0) = I, \tag{3.4}$$

satisfies

$$Z(k_0 + K) = H. \tag{3.5}$$

Proof. Suppose that (2.1) is K-uniformly completely controllable. Consider $k_0 \in \mathbb{N}$ and $H \in \mathbb{R}^{s\times s}$. From Definition 3.1 for each vector $z_l = (I - H) e_l$, $l = 1, 2, \ldots, s$, there exists a control $u_l(n)$, $n = k_0, k_0 + 1, \ldots, k_0 + K - 1$, such that

$$\|u_l(n)\| \leq p_0 \|z_l\|$$

and

$$x (k_0 + K, k_0, z_l, u_l) = 0.$$

The last equality together with (2.3) gives

$$\begin{aligned} 0 &= \Phi_A (k_0 + K, k_0) z_l + \sum_{j=k_0}^{k_0+K-1} \Phi_A (k_0 + K, j + 1) B(j)u_l(j) \\ &= \Phi_A(k_0 + K, k_0)\left(z_l + \sum_{j=k_0}^{k_0+K-1} \Phi_A(k_0, j + 1)B(j)u_l(j)\right); \end{aligned}$$

therefore, by the nonsingularity of the matrix $\Phi_A(k_0 + K, k_0)$ we have the equality

$$(I - H) e_l + \sum_{j=k_0}^{k_0+K-1} \Phi_A (k_0, j + 1) B(j)u_l(j) = 0.$$

Denoting

$$V(n) = [u_1(n), \ldots, u_s(n)]$$

we have

$$(I-H)\,e_l+\sum_{j=k_0}^{k_0+K-1}\Phi_A\left(k_0,j+1\right)B(j)V(j)e_l=0$$

for every $l\in\{1,\ldots,s\}$; therefore,

$$(I-H)+\sum_{j=k_0}^{k_0+K-1}\Phi_A\left(k_0,j+1\right)B(j)V(j)=0$$

and

$$H=I+\sum_{j=k_0}^{k_0+K-1}\Phi_A\left(k_0,j+1\right)B(j)V(j).$$

But the solution of (3.3), (3.4) has the form

$$Z(n)=I+\sum_{j=k_0}^{n-1}\Phi_A\left(k_0,j+1\right)B(j)V(j); \tag{3.6}$$

hence $Z(k_0+K)=H$. Moreover, for all $n\in\{k_0,\ldots,k_0+K-1\}$ and $x=\sum_{j=1}^{s}x_je_j\in\mathbb{R}^s$ we have

$$\begin{aligned}\|V(n)x\|&=\left\|\sum_{j=1}^{s}x_jV(n)e_j\right\|=\left\|\sum_{j=1}^{s}x_ju_j(n)\right\|\\&\le\sum_{j=1}^{s}|x_j|\,\|u_j(n)\|\le\|x\|\sum_{j=1}^{s}\|u_j(n)\|\\&\le\|x\|\,p_0\sum_{j=1}^{s}\|z_j\|\le p_0s\,\|x\|\,\|H-I\|\,.\end{aligned}$$

Defining $p=p_0s$ we obtain

$$\|V(n)\|=\sup_{x\neq 0}\frac{\|V(n)x\|}{\|x\|}\le p\,\|H-I\|\,.$$

Conversely, suppose that for all $k_0\in\mathbb{N}$ and $H\in\mathbb{R}^{s\times s}$ there exists a sequence $V(n)\in\mathbb{R}^{t\times s}$, $n=k_0,k_0+1,\ldots,k_0+K-1$, such that the sequence $Z(n)$, $n=k_0,\ldots,k_0+K-1$, given by (3.3) and (3.4) satisfies (3.5) and the inequality $\|V(n)\|\le p\|H-I\|$ holds for certain positive p and all $n=k_0,k_0+1,\ldots,k_0+K-1$. Consider any $k_0\in\mathbb{N}$ and $x_0\in\mathbb{R}^s$. Adding up (3.3) from $n=k_0$ to $n=k_0+K-1$ we obtain

$$\sum_{j=k_0}^{k_0+K-1}\Phi_A\left(k_0,j+1\right)B(j)V(j)=H-I.$$

Let $H=I+[-x_0,0,\ldots,0]$. We define the control as $u(n)=V(n)e_1$. Then we have

$$\begin{aligned}&x\left(k_0+K,k_0,x_0,u\right)\\&\quad=\Phi_A\left(k_0+K,k_0\right)\left(x_0+\sum_{j=k_0}^{k_0+K-1}\Phi_A\left(k_0,j+1\right)B(j)V(j)e_1\right)\\&\quad=\Phi_A\left(k_0+K,k_0\right)\left(-H+I+\sum_{j=k_0}^{k_0+K-1}\Phi_A\left(k_0,j+1\right)B(j)V(j)\right)e_1=0.\end{aligned}$$

Moreover,

$$\|u(n)\| \leq \|V(n)\| \leq p\,\|H - I\| = p\,\|(H - I)\,e_1\| = p\,\|x_0\|\,.$$

It means that the system (2.1) is K-uniformly completely controllable. $\square$

The next theorem will play a fundamental role in our further considerations.

THEOREM 3.4. *If the system* (2.1) *is K-uniformly completely controllable, then there exist $\alpha > 0$ and $r > 0$ such that for all $k_0 \in \mathbb{N}$ and each matrix $H \in \mathbb{R}^{s\times s}$, $\|H - I\| < r$, there exists a sequence $U(n)$, $n = k_0, \ldots, k_0 + K - 1$, of elements of $\mathbb{R}^{t\times s}$ such that*

$$\|U(n)\| \leq \alpha\,\|H - I\|\,,$$
$$\left\|\big(A(n) + B(n)U(n)\big)^{-1}\right\| < 3a^{2K},$$

and

$$\Phi_{A+BU}\,(k_0 + K, k_0) = \Phi_A\,(k_0 + K, k_0)\,H,$$

where Φ_{A+BU} is the transition matrix of the system (2.4).

Proof. Consider any $H \in \mathbb{R}^{s\times s}$ and $k_0 \in \mathbb{N}$. Let $V(n)$, $n = k_0, \ldots, k_0 + K - 1$, and $Z(n)$, $n = k_0, \ldots, k_0 + K$, be the sequences from Lemma 3.3. From (3.6) we deduce that for $n = k_0 + 1, \ldots, k_0 + K$

$$\begin{aligned}\|Z(n) - I\| &\leq \left\|\sum_{j=k_0}^{n-1} \Phi_A\,(k_0, j+1)\,B(j)V(j)\right\| \leq \sum_{j=k_0}^{n-1} \|\Phi_A\,(k_0, j+1)\|\,\|B(j)\|\,\|V(j)\| \\ &\leq p\,\|H - I\| \sum_{j=k_0}^{n-1} \|\Phi_A\,(k_0, j+1)\|\,\|B(j)\| \leq p\|H - I\| \sum_{j=k_0}^{n-1} a^{j-k_0+1} \\ &\leq p\|H - I\| \sum_{j=k_0}^{k_0+K-1} a^{j-k_0+1} = p\|H - I\| \sum_{i=1}^{K} a^i = p\|H - I\| \frac{a(a^K - 1)}{a - 1}.\end{aligned}$$

Define

$$r = \frac{a - 1}{2a(a^K - 1)p},$$

and assume that $\|H - I\| < r$, then $\|Z(n) - I\| < 1/2$. The last inequality implies [16, p. 301] that $Z(n)$ is invertible and

$$\begin{aligned}\|Z^{-1}(n)\| &= \|Z^{-1}(n) - I + I\| \leq \|Z^{-1}(n) - I\| + 1 = \|Z^{-1}(n)(I - Z(n))\| + 1 \\ &\leq \|Z^{-1}(n)\|\,\|I - Z(n)\| + 1 < \frac{\|Z^{-1}(n)\|}{2} + 1,\end{aligned}$$

and therefore, $\|Z^{-1}(n)\| < 2$. Consider the sequence $V_1(n)$, $n = k_0, \ldots, k_0 + K - 1$, given by

$$V_1(n) = V(n)Z^{-1}(n).$$

Then

$$\|V_1(n)\| \leq 2p\,\|H - I\|$$

and

$$Z(n+1) = \big(I + \Phi_A\,(k_0, n+1)\,B(n)V_1(n)\big)Z(n),$$
$$Z(k_0) = I, \quad Z(k_0 + K) = H.$$

Denoting

$$Y(n) = \Phi_A(n, k_0)Z(n), \quad U(n) = V_1(n)\Phi_A(k_0, n),$$

we have

$$\begin{aligned}
Y(n+1) &= \Phi_A(n+1, k_0)\, Z(n+1) \\
&= A(n)\Phi_A(n, k_0)\big(I + \Phi_A(k_0, n+1)B(n)V_1(n)\big)Z(n) \\
&= A(n)\Phi_A(n, k_0)\, Z(n) + A(n)A^{-1}(n)B(n)V_1(n)Z(n) \\
&= A(n)Y(n) + B(n)V_1(n)\Phi_A(k_0, n)\, \Phi_A(n, k_0)\, Z(n) \\
&= A(n)Y(n) + B(n)U(n)Y(n) = \big(A(n) + B(n)U(n)\big)Y(n).
\end{aligned}$$

It means that the sequence $Y(n)$, $n = k_0, \ldots, k_0 + K - 1$, is the solution of the matrix equation

$$Y(n+1) = (A(n) + B(n)U(n))\, Y(n) \tag{3.7}$$

with the above defined sequence $U(n)$. Moreover,

$$Y(k_0) = I.$$

Consequently,

$$Y(n) = \Phi_{A+BU}(n, k_0), \quad n = k_0, \ldots, k_0 + K,$$

and

$$\Phi_{A+BU}(k_0 + K, k_0) = Y(k_0 + K) = \Phi_A(k_0 + K, k_0)\, Z(k_0 + K) = \Phi_A(k_0 + K, k_0)\, H.$$

Observe that

$$\|U(n)\| \le \|V_1(n)\|\, \|\Phi_A(k_0, n)\| \le 2pa^K\, \|H - I\|$$

and defining $\alpha = 2pa^K$ we obtain the estimate $\|U(n)\| < \alpha\|H - I\|$. Finally, from (3.7) we get the invertibility of the matrix $A(n) + B(n)U(n)$ and the estimates

$$\begin{aligned}
\left\|\big(A(n) + B(n)U(n)\big)^{-1}\right\| &\le \|Y(n)\|\, \|Y^{-1}(n+1)\| \\
&\le \|\Phi_A(n, k_0)\|\, \|Z(n)\|\, \|Z^{-1}(n+1)\|\, \|\Phi_A(k_0, n+1)\| \\
&< 2a^{2K}\|Z(n)\| \le 2a^{2K}\big(\|Z(n) - I\| + 1\big) < 2a^{2K}3/2 = 3a^{2K},
\end{aligned}$$

which complete the proof. ∎

4. Dynamic equivalence. In our further considerations we shall use the concept of dynamically equivalent systems.

Definition 4.1 (see [15, p. 15], [14, p. 100]). *Let $\big(L(n)\big)_{n\in\mathbb{N}} \subset \mathbb{R}^{s\times s}$ be a Lyapunov sequence. A linear transformation*

$$y = L(n)x, \quad n \in \mathbb{N}, \tag{4.1}$$

of the space $\mathbb{R}^s$ is called a Lyapunov transformation.

Definition 4.2 (see [15, p. 15]). *We say that the system* (2.2) *is dynamically equivalent to the system*

$$y(n+1) = C(n)y(n), \quad n \in \mathbb{N},\ y \in \mathbb{R}^s, \tag{4.2}$$

if there exists a Lyapunov transformation (4.1) *which connects these systems, i.e., for every solution $x(n)$ of the system* (2.2) *the function $y(n) = L(n)x(n)$ is a solution of system* (4.2) *and for every solution $y(n)$ of the system* (4.2) *the function $x(n) = L^{-1}(n)y(n)$ is a solution of the system* (2.2).

Let us note that if a Lyapunov transformation (4.1) establishes the dynamic equivalence between systems (2.2) and (4.2), then

$$y(n+1) = L(n+1)x(n+1) = L(n+1)A(n)x(n) = L(n+1)A(n)L^{-1}(n)y(n), \quad n \in \mathbb{N};$$

hence

$$C(n) = L(n+1)A(n)L^{-1}(n), \quad n \in \mathbb{N}. \tag{4.3}$$

Thus, systems (2.2) and (4.2) are dynamically equivalent if and only if there exists a Lyapunov sequence $\big(L(n)\big)_{n\in\mathbb{N}} \subset \mathbb{R}^{s\times s}$ such that the equality (4.3) is satisfied.

It is well known (see, for example, [6], [14, p. 125]) that dynamically equivalent systems have the same Lyapunov spectrum. We shall use the following criterion of dynamic equivalence [3].

THEOREM 4.3 (see [3]). *Suppose that $A = (A(n))_{n\in\mathbb{N}}$ and $C = (C(n))_{n\in\mathbb{N}}$ are Lyapunov sequences. Assume that*

$$\Phi_C(n_{k+1}, n_k) = \Phi_A(n_{k+1}, n_k)$$

for all $k \in \mathbb{N}$, where $(n_k)_{k\in\mathbb{N}}$ is a sequence of natural numbers such that $0 < n_{k+1} - n_k \le c < \infty$ for all $k \in \mathbb{N}$. Then the systems (2.2) *and* (4.2) *are dynamically equivalent.*

5. Perturbed systems. Together with the system (2.2) we consider the perturbed system

$$z(n+1) = A(n)R(n)z(n), \quad n \in \mathbb{N}. \tag{5.1}$$

The perturbation $\big(R(n)\big)_{n\in\mathbb{N}}$ will be called the multiplicative perturbation of the system (2.2). Let $\Phi_{AR}(n,k)$ be the transition matrix of the system (5.1).

LEMMA 5.1. *For any natural numbers $n > k$, the equality*

$$\Phi_{AR}(n,k) = \Phi_A(n,k) + \sum_{j=k}^{n-1} \Phi_A(n,j)\big(R(j) - I\big)\Phi_{AR}(j,k) \tag{5.2}$$

holds.

Proof. Let us introduce the notation

$$Q(n) = A(n)\big(R(n) - I\big), \quad n \in \mathbb{N}.$$

With this notation we have $A(n) + Q(n) = A(n)R(n)$, and therefore, the system (5.1) coincides with the system

$$z(n+1) = \big(A(n) + Q(n)\big)z(n), \quad n \in \mathbb{N},$$

and the transition matrices of these systems are equal:

$$\Phi_{AR}(n,k) = \Phi_{A+Q}(n,k), \quad n > k.$$

Let us fix $k \in \mathbb{N}$. We consider the matrix $\Phi_{A+Q}(n,k) = [z_1(n), \dots, z_s(n)]$ for $n \ge k$. As $\Phi_{A+Q}(k,k) = I$ we obtain $z_l(k) = e_l$, $l = 1, \dots, s$. For any $l \in \{1, \dots, s\}$ and all natural numbers $n > k$ we have the formula

$$\begin{aligned}\Phi_{A+Q}(n,k)e_l = z_l(n) &= \Phi_A(n,k)e_l + \sum_{j=k}^{n-1} \Phi_A(n,j+1)Q(j)z_l(j)\\ &= \Phi_A(n,k)e_l + \sum_{j=k}^{n-1} \Phi_A(n,j+1)Q(j)\Phi_{A+Q}(j,k)e_l\\ &= \Phi_A(n,k)e_l + \sum_{j=k}^{n-1} \Phi_A(n,j+1)Q(j)\Phi_{A+Q}(j,k)e_l.\end{aligned}$$

Therefore,

$$\Phi_{A+Q}(n,k) = \Phi_A(n,k) + \sum_{j=k}^{n-1} \Phi_A(n,j+1)Q(j)\Phi_{A+Q}(j,k)$$

and

$$\begin{aligned}\Phi_{AR}(n,k) = \Phi_{A+Q}(n,k) &= \Phi_A(n,k) + \sum_{j=k}^{n-1} \Phi_A(n,j+1)Q(j)\Phi_{A+Q}(j,k)\\ &= \Phi_A(n,k) + \sum_{j=k}^{n-1} \Phi_A(n,j+1)A(j)\big(R(j)-I\big)\Phi_{AR}(j,k)\\ &= \Phi_A(n,k) + \sum_{j=k}^{n-1} \Phi_A(n,j)\big(R(j)-I\big)\Phi_{AR}(j,k).\end{aligned}$$ □

Below we shall examine the problem of the behavior of the Lyapunov exponents of the perturbed system (5.1) for different types of perturbations R. According to our assumptions the sequence $\big(A(n)\big)_{n\in\mathbb{N}}$ is a Lyapunov sequence. For this reason, the Lyapunov spectrum $\lambda(AR)$ of the perturbed system (5.1) will consist of s numbers if the sequence $\big(R(n)\big)_{n\in\mathbb{N}}$ is a Lyapunov sequence. Therefore, it is natural to introduce the following definition.

DEFINITION 5.2 (see [5]). *A multiplicative perturbation $\big(R(n)\big)_{n\in\mathbb{N}}$ is called admissible if the sequence $\big(R(n)\big)_{n\in\mathbb{N}}$ is a Lyapunov sequence.*

The set of all admissible multiplicative perturbations will be denoted by $\mathcal{R}$, and the subset of $\mathcal{R}$ consisting of perturbations satisfying the condition $\|R-I\|_\infty < \delta$ will be denoted by $\mathcal{R}_\delta$.

THEOREM 5.3. *If the system* (2.1) *is uniformly completely controllable, then there exist $\widehat{\delta} > 0$ and $l > 0$ such that for each $\big(R(n)\big)_{n\in\mathbb{N}} \in \mathcal{R}_{\widehat{\delta}}$ there exists an admissible feedback control $U = \big(U(n)\big)_{n\in\mathbb{N}}$ for the system* (2.1) *such that*

$$\|U\|_\infty \le l\|R-I\|_\infty$$

and the system (5.1) *is dynamically equivalent to the system* (2.4).

Proof. Let $K \in \mathbb{N}$ be such that the system (2.1) is K-uniformly completely controllable. According to Theorem 3.4 there exist α, $r > 0$ such that for all

$m \in \mathbb{N}$ and any matrix $H_m \in \mathbb{R}^{s\times s}$, $\|H_m - I\| < r$, there exists a sequence $U_m(n)$, $n = (m-1)K + 1, \dots, mK$ of elements of $\mathbb{R}^{t\times s}$ such that

$$\|U_m(n)\| \le \alpha \|H_m - I\|,$$
$$\left\|\big(A(n) + B(n)U_m(n)\big)^{-1}\right\| < 3a^{2K},$$

and

$$\Phi_{A+BU_m}\left(mK+1, (m-1)\,K+1\right) = \Phi_A\left(mK+1, (m-1)\,K+1\right) H_m. \tag{5.3}$$

Let $\widehat{\delta}$ be such that

$$\widehat{\delta} K a^{2K} (\widehat{\delta}+1)^K < r. \tag{5.4}$$

Consider any sequence $R = (R(n))_{n\in\mathbb{N}} \subset \mathcal{R}_{\widehat{\delta}}$. Let us observe that

$$\|R\|_\infty \le \|R - I\|_\infty + 1 < \widehat{\delta} + 1.$$

From Lemma 5.1 we have

$$\begin{aligned}
&\Phi_{AR}\big(mK+1, (m-1)K+1\big)\\
&\quad = \Phi_A\big(mK+1, (m-1)K+1\big)\\
&\quad\quad + \sum_{j=(m-1)K+1}^{mK} \Phi_A(mK+1, j)\big(R(j) - I\big)\Phi_{AR}\big(j, (m-1)K+1\big)\\
&\quad = \Phi_A\big(mK+1, (m-1)K+1\big) H_m,
\end{aligned}$$

where

$$H_m = I + \sum_{j=(m-1)K+1}^{mK} \Phi_A\big((m-1)K+1, j\big)\big(R(j) - I\big)\Phi_{AR}\big(j, (m-1)K+1\big).$$

Moreover,

$$\begin{aligned}
\|H_m - I\| &\le \|R - I\|_\infty \sum_{j=(m-1)K+1}^{mK} \left\|\Phi_A\big((m-1)K+1, j\big)\right\| \left\|\Phi_{AR}\big(j, (m-1)K+1\big)\right\|\\
&\le \|R - I\|_\infty K a^K \big(a\|R\|_\infty\big)^K\\
&< \|R - I\|_\infty K a^{2K} (\widehat{\delta}+1)^K < \widehat{\delta} K a^{2K} (\widehat{\delta}+1)^K < r.
\end{aligned}$$

The last inequality together with (5.3) implies that there exists a bounded sequence $(U(n))_{n\in\mathbb{N}}$,

$$U(n) = U_m(n), \quad m \in \mathbb{N}, \quad n \in \{(m-1)K+1, \dots, mK\},$$

such that

$$\begin{aligned}
\Phi_{A+BU}\big(mK+1, (m-1)K+1\big) &= \Phi_{A+BU_m}\big(mK+1, (m-1)K+1\big)\\
&= \Phi_A\big(mK+1, (m-1)K+1\big) H_m\\
&= \Phi_{AR}\big(mK+1, (m-1)K+1\big).
\end{aligned}$$

Observe that

$$\begin{aligned}\|U\|_\infty &= \sup_{m\in\mathbb{N}}\left(\max_{n=(m-1)K+1,\dots,mK}\|U_m(n)\|\right)\\ &\le \alpha \sup_{m\in\mathbb{N}}\|H_m - I\|\\ &\le \alpha K a^{2K}(\widehat{\delta}+1)^K\|R-I\|_\infty \doteq l\|R-I\|_\infty.\end{aligned}$$

The constructed sequence $\big(U(n)\big)_{n\in\mathbb{N}}$ is a Lyapunov sequence, since

$$\|U\|_\infty \le l\|R-I\|_\infty < l\widehat{\delta},$$

and therefore,

$$\|A+BU\|_\infty \le \|A\|_\infty + \|B\|_\infty\|U\|_\infty < a(1+l\widehat{\delta}).$$

Moreover, the matrix $A(n)+B(n)U(n)$ is invertible for all $n\in\mathbb{N}$ and

$$\left\|\big(A+BU\big)^{-1}\right\|_\infty < 3a^{2K}.$$

From Theorem 4.3 we conclude that the systems (2.4) and (5.1) are dynamically equivalent. ∎

6. Main result. We start with the following definition.

DEFINITION 6.1. *The Lyapunov spectrum of the system* (5.1) *is called proportionally globally assignable if for all* $\Delta > 0$ *there exists* $\widehat{\ell} = \widehat{\ell}(\Delta) > 0$ *such that for any sequence*

$$\mu = (\mu_1,\dots,\mu_s) \in O_\Delta\big(\lambda(A)\big)$$

there exists a sequence $R = (R(n))_{n\in\mathbb{N}} \in \mathcal{R}$ *satisfying the estimate*

$$\|R-I\|_\infty \le \widehat{\ell}\max_{j=1,\dots,s}|\lambda_j(A)-\mu_j| \tag{6.1}$$

and providing the validity of the relation

$$\lambda(AR) = \mu. \tag{6.2}$$

THEOREM 6.2. *Suppose that system* (2.1) *is uniformly completely controllable. If the Lyapunov spectrum of* (5.1) *is globally proportionally assignable, then the Lyapunov spectrum of* (2.4) *is locally proportionally assignable.*

Proof. From the global proportional assignability of the spectrum of (5.1) it follows that for $\Delta = 1$ there exists $\widehat{\ell} = \widehat{\ell}(1) > 0$ such that for any sequence

$$\mu = \big(\mu_1,\dots,\mu_s\big) \in O_1\big(\lambda(A)\big)$$

there exists a sequence $R = \big(R(n)\big)_{n\in\mathbb{N}} \in \mathcal{R}$ satisfying the estimate (6.1) and providing the validity of relation (6.2). Since (2.1) is uniformly completely controllable, then according to Theorem 5.3, there exist $\widehat{\delta} > 0$ and $l > 0$ such that for each system (5.1) with $R \in \mathcal{R}_{\widehat{\delta}}$ there exists an admissible feedback control $U = (U(n))_{n\in\mathbb{N}}$ for the system (2.1), such that $\|U\|_\infty \le l\|R-I\|_\infty$ and the system (5.1) is dynamically equivalent to the system (2.4). Let $\delta_1 = \min\{1, \widehat{\delta}/\widehat{\ell}\}$. Consider any sequence

$$\mu = \big(\mu_1, \dots, \mu_s\big) \in O_{\delta_1}\big(\lambda(A)\big) \subset O_1\big(\lambda(A)\big).$$

From the global proportional assignability of the spectrum of (5.1) it follows that there exists a sequence $R \in \mathcal{R}$ such that

$$\|R-I\|_\infty \le \widehat{\ell} \max_{j=1,\dots,s} |\lambda_j(A) - \mu_j| \le \widehat{\ell}\delta_1 \le \widehat{\delta}$$

and (6.2) is satisfied. By the uniform complete controllability of the system (2.1) for this sequence R there exists an admissible feedback control U for the system (2.1) such that

$$\|U\|_\infty \le l\|R-I\|_\infty \le l\widehat{\ell} \max_{j=1,\dots,s} |\lambda_j(A) - \mu_j|$$

and such that the systems (2.4) and (5.1) are dynamically equivalent. Since equivalent systems have the same spectrum the proof is completed. ∎

In the remaining part of this section we shall present results about local proportional assignability of the spectrum of the system (2.4). They will be expressed in the forms of certain concepts from the asymptotic theory of linear systems, which are defined below.

Definition 6.3. *The system* (2.2) *is called diagonalizable if it is dynamically equivalent to the system* (4.2) *with diagonal matrix* $C(n)$.

Definition 6.4 (see [14, p. 63]). *The system* (2.2) *is called regular (in the Lyapunov sense) if the equality*

$$\sum_{i=1}^{s} \lambda_i(A) = \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} \ln|\det A(j)|$$

holds.

The notion of regularity of linear differential systems was introduced in the famous paper of Lyapunov [24], and this concept was further developed in [33]. Some facts about regularity of discrete equations may be found in the works [7, 10, 11, 12, 14]. Let us notice that all time-invariant or all periodic systems are regular.

Definition 6.5 (see [5]). *The Lyapunov spectrum of the system* (2.2) *is called stable if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\lambda\big(\mathcal{R}_\delta\big) \subset O_\varepsilon\big(\lambda(A)\big),$$

where

$$\lambda\big(\mathcal{R}_\delta\big) = \big\{\lambda(AR) \,|\, R \in \mathcal{R}_\delta\big\}.$$

The effect of instability of the Lyapunov spectrum under the influence of small coefficient perturbations for linear continuous-time systems was discovered by Perron [30]. Later, the stability property of the Lyapunov spectrum for these systems was investigated in [8, 9, 19, 20, 28]. The study of this property for discrete-time systems was started in [5].

Now we show global proportional assignability of the Lyapunov spectrum of the system (5.1) under one of the following assumptions: stability of the Lyapunov spectrum or diagonalizability or regularity of the free system (2.2).

THEOREM 6.6. *If the system* (2.2) *is diagonalizable, then the Lyapunov spectrum of the system* (5.1) *is proportionally globally assignable.*

Proof. Suppose that the Lyapunov transformation (4.1) transforms the system (2.2) into the system (4.2) with a diagonal matrix

$$C(n) = \operatorname{diag}\bigl(c_1(n), \ldots, c_s(n)\bigr), \quad n \in \mathbb{N}.$$

Then [14, p. 55] the Lyapunov spectrum of the diagonal system (4.2) consists of the numbers

$$\lambda_j = \limsup_{n\to\infty} n^{-1} \sum_{k=1}^{n} \ln |c_j(k)|, \quad j = 1, \ldots, s.$$

Without loss of generality, we may assume that these numbers are numbered in increasing order. Since the Lyapunov transformation preserves the Lyapunov spectrum, it follows that $\lambda_j(A) = \lambda_j$, $j = 1, \ldots, s$.

Let us fix $\Delta > 0$ and a sequence $\mu = (\mu_1, \ldots, \mu_s) \in O_\Delta\bigl(\lambda(A)\bigr)$. Denote $\nu_j = \mu_j - \lambda_j(A)$, $j \in \{1, \ldots, s\}$. Then $|\nu_j| < \Delta$ for all $j \in \{1, \ldots, s\}$. Consider an admissible multiplicative perturbation $H = \operatorname{diag}\bigl(e^{\nu_1}, \ldots, e^{\nu_s}\bigr)$ for the system (4.2). A coefficient matrix of the multiplicatively perturbed system

$$\psi(n+1) = C(n)H\psi(n), \quad n \in \mathbb{N},\ \psi \in \mathbb{R}^s, \tag{6.3}$$

is diagonal; hence the Lyapunov spectrum of this system consists of the numbers

$$\lambda_j = \limsup_{n\to\infty} n^{-1} \sum_{k=1}^{n} \ln |c_j(k)e^{\nu_j}| = \lambda_j(A) + \nu_j = \mu_j, \quad j = 1, \ldots, s.$$

Applying the Lyapunov transformation $z = L^{-1}(n)\psi$ to the system (6.3) and using the equality (4.3), we obtain the system

$$\begin{aligned} z(n+1) &= L^{-1}(n+1)\psi(n+1) \\ &= L^{-1}(n+1)C(n)H\psi(n) \\ &= L^{-1}(n+1)L(n+1)A(n)L^{-1}(n)HL(n)z(n) \\ &= A(n)L^{-1}(n)HL(n)z(n). \end{aligned}$$

Let us denote

$$R(n) = L^{-1}(n)HL(n), \quad n \in \mathbb{N}. \tag{6.4}$$

From the fact that $L = \bigl(L(n)\bigr)_{n\in\mathbb{N}}$ is a Lyapunov sequence and because H is invertible, it follows that $R = \bigl(R(n)\bigr)_{n\in\mathbb{N}}$ is a Lyapunov sequence, i.e., $R \in \mathcal{R}$. Thus, we have the multiplicatively perturbed system (5.1), wherein $\lambda(AR) = \mu$.

Let $\hat{l} \doteq \max\bigl\{\|L\|_\infty, \|L^{-1}\|_\infty\bigr\}$ and

$$\widehat{\ell} = \widehat{\ell}(\Delta) = \hat{l}^2(e^\Delta - 1)/\Delta.$$

We shall show that

$$\|R - I\|_\infty \le \widehat{\ell} \max_{j=1,\ldots,s} |\mu_j - \lambda_j(A)|.$$

In fact, from (6.4) we obtain that $R(n) - I = L^{-1}(n)\bigl(H - I\bigr)L(n)$; therefore,

$$\|R - I\|_\infty \le \hat{l}^2 \|H - I\|$$

and

$$\|H-I\| = \max_{j=1,\dots,s} |e^{\nu_j}-1| \le \max_{j=1,\dots,n}\bigl(e^{|\nu_j|}-1\bigr).$$

Observe that the function $f(t) = (e^t-1)/t$ is strictly increasing for $t>0$, and therefore, for any $\delta\in(0,\Delta)$ the inequality $\bigl(e^{\delta}-1\bigr)/\delta < \bigl(e^{\Delta}-1\bigr)/\Delta$ holds. It implies that $e^{\delta}-1 < \frac{e^{\Delta}-1}{\Delta}\delta$. Since $|\nu_j|<\Delta$ for all $j\in\{1,\dots,s\}$, it follows that

$$\|H-I\| \le \max_{j=1,\dots,s}\frac{e^{\Delta}-1}{\Delta}|\nu_j| = \max_{j=1,\dots,s}|\nu_j|\widehat{\ell}/\hat{l}^2 = \max_{j=1,\dots,s}|\mu_j-\lambda_j(A)|\widehat{\ell}/\hat{l}^2,$$

and consequently,

$$\|R-I\|_\infty \le \hat{l}^2\|H-I\| \le \widehat{\ell}\max_{j=1,\dots,s}|\mu_j-\lambda_j(A)|. \qquad \square$$

THEOREM 6.7. *If the system* (2.2) *is regular, then the Lyapunov spectrum of the system* (5.1) *is proportionally globally assignable.*

Proof. From the Perron theorem (see [6], [14, p. 70]) it follows that the system (2.2) is dynamically equivalent to the system (4.2) with an upper triangular matrix $C(n) = \{c_{ij}(n)\}_{i,j=1}^n$ and $\lambda(A)=\lambda(C)$. Assume that this dynamic equivalence is provided by the Lyapunov transformation (4.1). Taking into account the equality (4.3), we obtain

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n-1}\ln|\det C(j)| \\
&\quad= \liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n-1}\ln\Bigl|\det\bigl(L(j+1)A(j)L^{-1}(j)\bigr)\Bigr| \\
&\quad= \liminf_{n\to\infty}\frac{1}{n}\left(\ln|\det L(n)| - \ln|\det L(1)| + \sum_{j=1}^{n-1}\ln|\det A(j)|\right) \\
&\quad= \liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n-1}\ln|\det A(j)| = \sum_{i=1}^{s}\lambda_i(A) = \sum_{i=1}^{s}\lambda_i(C),
\end{aligned}$$

and therefore, the system (4.2) is regular. From the Lyapunov theorem on the regularity of the triangular system [14, p. 67] we have that for the diagonal elements $c_{jj}(n)$ of the matrix $C(n)$ there exist the limits

$$\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}\ln|c_{jj}(k)|, \quad j=1,\dots,s,$$

which form the Lyapunov spectrum of the system (4.2). Since the systems (2.2) and (4.2) are dynamically equivalent, it follows that

$$\lambda_j(A) = \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}\ln|c_{jj}(k)|, \quad j=1,\dots,s.$$

For any fixed $\Delta>0$ let us consider any $\mu=(\mu_1,\dots,\mu_s)\in O_\Delta\bigl(\lambda(A)\bigr)$ and denote $\nu_j = \mu_j-\lambda_j(A)$, $j\in\{1,\dots,s\}$. For the system (4.2) consider the multiplicatively perturbed system (6.3) with the matrix H from Theorem 6.6. It is clear that the coefficient matrix of the system (6.3) is upper triangular, its diagonal elements are equal to $c_{jj}(n)e^{\nu_j}$, $j = 1, \dots, s$, and

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \ln |c_{jj}(k)e^{\nu_j}| = \mu_j, \quad j = 1, \dots, s.$$

Using again the Lyapunov theorem on the regularity of the triangular system we get that the system (6.3) is regular and that its spectrum coincides with the sequence $\mu = (\mu_1, \dots, \mu_s)$. The end of the proof is the same as in the proof of Theorem 6.6. □

THEOREM 6.8. *If the Lyapunov spectrum of the system* (2.2) *is stable, then the Lyapunov spectrum of the system* (5.1) *is proportionally globally assignable.*

Proof. Let us assume that the Lyapunov spectrum of the system (2.2) consists of the numbers $\Lambda_1 < \Lambda_2 < \cdots < \Lambda_q$, where the number Λ_j is repeated s_j times; $s_1+s_2+\cdots+s_q = s$. It is known [5, Theorem 1, Remark 3] (see also [8, 28]) that if the Lyapunov spectrum of the system (2.2) is stable, then there exists a Lyapunov transformation (4.1), reducing the system (2.2) to the system (4.2) with block-triangular matrix $C(n) = \operatorname{diag}\bigl(C_1(n), \dots, C_q(n)\bigr)$, with the following properties:
(1) for each $j \in \{1, \dots, q\}$ the matrix $C_j(n)$ is lower-triangular of order $s_j \times s_j$;
(2) the diagonal elements $c_{ii}(n)$ of the matrix $C(n)$ are such that for all $j \in \{1, \dots, q\}$ and $i \in s_j$ we have

$$\limsup_{n\to\infty} n^{-1} \sum_{k=1}^{n-1} \ln |c_{ii}(k)| = \Lambda_j.$$

The notation $i \in s_j$ means that

$$i \in \{s_0 + \cdots + s_{j-1} + 1, \dots, s_0 + \cdots + s_j\},$$

where $s_0 \doteq 0$.

Let us fix $\Delta > 0$. Consider any sequence $\mu = (\mu_1, \dots, \mu_s) \in O_\Delta\bigl(\lambda(A)\bigr)$ and denote $\nu_j = \mu_j - \lambda_j(A)$, $j \in \{1, \dots, s\}$. Let us consider $i \in \{1, \dots, q\}$. For each $j \in s_i$ the equality $\nu_j = \mu_j - \Lambda_i(A)$ holds; therefore, the numbers ν_j, $j \in s_i$, are increasingly ordered. Let us order them in decreasing order and denote the obtained sequence η_j, $j \in s_i$. Let H_i be a diagonal matrix of the order $s_i \times s_i$ with sequential diagonal elements e^{η_j}, $j \in s_i$. Then the matrix $H \doteq \operatorname{diag}\bigl(e^{\eta_1}, \dots, e^{\eta_n}\bigr)$ is equal to the block-diagonal matrix $\operatorname{diag}\bigl(H_1, \dots, H_q\bigr)$. For the system (4.2) consider the multiplicatively perturbed system (6.3) with the constructed matrix H. This system has a block-diagonal coefficient matrix with diagonal blocks in the form of lower-triangular matrices $C_i(n)H_i$, $i = 1, \dots, q$. In the proof of Theorem 3 in [5] it was shown that the Lyapunov spectrum of the system (6.3) coincides with the sequence μ. The end of the proof is the same as in the proof of Theorem 6.6. □

Now we may formulate the main result of this paper which directly follows from Theorems 6.2 and 6.6–6.8.

THEOREM 6.9. *Let the system* (2.1) *be uniformly completely controllable and assume that at least one of the following conditions holds:*

1. *the system* (2.2) *is regular;*
2. *the system* (2.2) *is diagonalizable;*
3. *the Lyapunov spectrum of the system* (2.2) *is stable.*

Then the Lyapunov spectrum of the system (2.4) *is proportionally locally assignable.*

7. Discussion of the results. First, we discuss the question of the values of the constants ℓ and δ from Definition 2.2, existence of which is guaranteed by our results about local proportional assignability of the Lyapunov spectrum. To begin with, note that if the Kalman matrix of system (2.1) satisfies the inequality (3.2) for each $k_0 \in \mathbb{N}$, then the conditions of Definition 3.1 are satisfied for this system, where the value of p_0 may be estimated from above by the inequality [15, p. 34]

$$p_0 \leq a^{K+1} \|W^{-1}(k_0, k_0 + K)\|.$$

Notice [27, p. 94] that

$$\|W^{-1}(k_0, k_0 + K)\| \leq \frac{\|W(k_0, k_0 + K)\|^{s-1}}{\det W(k_0, k_0 + K)} \leq \frac{\left(Ka^{2(K+1)}\right)^{s-1}}{\gamma^s}.$$

Consequently, for the constant

$$p_0 = a^{K+1} \frac{\left(Ka^{2(K+1)}\right)^{s-1}}{\gamma^s} = \frac{K^{s-1}}{\gamma^s} a^{(K+1)(2s-1)}$$

the estimate (3.1) from Definition 3.1 of uniform complete controllability is satisfied.

Further, from the proof of Lemma 3.3 we obtain that the value of the constant p from the formulation of this lemma may be taken as

$$p = p_0 s = \frac{sK^{s-1}}{\gamma^s} a^{(K+1)(2s-1)},$$

and, from the proof of Theorem 3.4, we obtain that the constants α and r from the formulation of this theorem may be taken as

$$\alpha = 2pa^K = \frac{2sK^{s-1}}{\gamma^s} a^{2s(K+1)-1},$$

$$r = \frac{a-1}{2a(a^K - 1)p} = \frac{(a-1)\gamma^s}{2s(a^K - 1)K^{s-1}a^{2sK+2s-K}}. \tag{7.1}$$

Now we consider the statement of Theorem 5.3. Analyzing the proof of this theorem, we have

$$\begin{aligned} l = \alpha K a^{2K} (\widehat{\delta} + 1)^K &= \frac{2sK^s}{\gamma^s} a^{2s(K+1)-1} a^{2K} (\widehat{\delta} + 1)^K \\ &= \frac{2sK^s}{\gamma^s} a^{2K(s+1)+2s-1} (\widehat{\delta} + 1)^K, \end{aligned}$$

where $\widehat{\delta} > 0$ is a sufficiently small constant satisfying the inequality (5.4). Taking into account the equality (7.1), we find that the constant $\widehat{\delta} > 0$ must satisfy the inequality

$$\widehat{\delta}(\widehat{\delta} + 1)^K < \frac{(a-1)\gamma^s}{2s(a^K - 1)K^s a^{2sK+2s+K}}. \tag{7.2}$$

Now we turn to the analysis of Theorem 6.2. The constant ℓ in the inequality

$$\|U\|_\infty \leq l \|R - I\|_\infty \leq l\widehat{\ell} \max_{j=1,\ldots,s} |\lambda_j(A) - \mu_j| \doteq \ell \max_{j=1,\ldots,s} |\lambda_j(A) - \mu_j| \tag{7.3}$$

may be taken as

$$\ell = l\widehat{\ell} = \frac{2\widehat{\ell} s K^s}{\gamma^s} a^{2K(s+1)+2s-1}(\widehat{\delta}+1)^K,$$

and the inequality (7.3) holds for any sequence $\mu \in O_{\delta_1}\big(\lambda(A)\big)$, where $\delta_1 = \min\{1, \widehat{\delta}/\widehat{\ell}\}$ and $\widehat{\ell} = \widehat{\ell}(1)$ is the constant from the Definition 6.1. From the proofs of Theorems 6.6–6.8, it follows that $\widehat{\ell} = \widehat{\ell}(1) = \hat{l}^2(e-1)$.

Now we are ready to give the values of the constants ℓ and δ, which guarantee the fulfillment of Definition 2.2:

$$\ell = 2\hat{l}^2(e-1)\frac{sK^s}{\gamma^s} a^{2(s+1)K+2s-1}(\widehat{\delta}+1)^K,$$
$$\delta = \min\big\{1, \widehat{\delta}/\big(\hat{l}^2(e-1)\big)\big\},$$

where the constant $\widehat{\delta}$ is given by the inequality (7.2).

Since $\hat{l} \geq 1$, $K \geq 1$, $s \geq 1$, $a \geq 2$, it follows that the constant ℓ is sufficiently large, and the constant δ is sufficiently small.

The only nonconstructive step in our proofs is the one where we use the norm of the Lyapunov sequence, which transforms the free system (2.2) to a diagonal form in the case of diagonalizability, and to a special triangular form in the case of regularity or stability of the Lyapunov spectrum. If the given system (2.2) has the required form, then Lyapunov transformation in Theorems 6.6–6.8 is the identity transformation, so $\hat{l} = 1$ and the constants ℓ, δ may be calculated exactly. If the given system does not have the required form, then we have only a lower bound on ℓ and an upper bound on δ, because $\hat{l} \geq 1$. Thus, the following corollary holds.

COROLLARY 7.1. *Let the system* (2.1) *be K-uniformly completely controllable and let the inequality* (3.2) *hold for it. Then the values of the constants ℓ and δ that guarantee the fulfillment of the conditions of Definition* 2.2 *satisfy the inequalities*

$$\ell \geq 2(e-1)\frac{sK^s}{\gamma^s} a^{2(s+1)K+2s-1}(\widehat{\delta}+1)^K,$$
$$\delta \leq \min\big\{1, \widehat{\delta}/(e-1)\big\},$$

where the constant $\widehat{\delta}$ is given by the inequality (7.2).

Now we discuss the question of how our main result is formulated in the special case of the time-invariant system.

COROLLARY 7.2. *Suppose that the matrices A and B are time-invariant. If the condition of complete controllability of the system* (2.1)

$$\operatorname{rank}[B, AB, \dots, A^{s-1}B] = s \tag{7.4}$$

is satisfied, then the Lyapunov spectrum of system (2.4) *is locally proportionally assignable and the constants ℓ and δ can be chosen as*

$$\ell = 2(e-1)\frac{s^{s+1}}{\gamma^s} a^{2s(s+2)-1}(\widehat{\delta}+1)^s, \quad \delta = \min\big\{1, \widehat{\delta}/(e-1)\big\},$$

where $\widehat{\delta} > 0$ is a sufficiently small constant satisfying the inequality

$$\widehat{\delta}(\widehat{\delta}+1)^s < \frac{(a-1)\gamma^s}{2s^{s+1}(a^s-1)a^{s(2s+3)}}.$$

Proof. For a time-invariant system (2.1), uniform complete controllability is equivalent to its complete controllability, which, in turn, is equivalent to the fulfillment of condition (7.4) [13, p. 433], [14, pp. 238–239]. In this case, the value K of the length of the segment of complete controllability can be chosen from the condition

$$\operatorname{rank}[B, AB, \ldots, A^{K-1}B] = s.$$

Obviously, for a completely controllable system, we can always assume that $K = s$. Since the time-invariant system is regular [14, p. 64], we may use Theorem 6.7 and reduce the system to a triangular form by a Perron transformation [14, p. 70], that is, by a transformation (4.1), in which the matrix $L(n)$ is orthogonal for each n. In this case $\|L(n)\| = \|L^{-1}(n)\| = 1$, i.e., $\hat{l} = 1$. From this and by Corollary 7.1 we obtain the required formulas for ℓ and δ. ∎

REMARK 7.3. *From Corollary* 7.2 *we see that our main result applied to time-invariant systems does not coincide with the classical pole placement theorem, formulation of which is as follows* [13, p. 458]*: Let $\Lambda = \{\mu_1, \mu_2, \ldots, \mu_s\}$ be an arbitrary set of s complex numbers such that $\overline{\Lambda} = \{\overline{\mu}_1, \overline{\mu}_2, \ldots, \overline{\mu}_s\} = \Lambda$. Then the pair (A, B) is completely controllable if and only if there exists a constant matrix U such that the eigenvalues of $A + BU$ are the set Λ.*

The main differences are as follows:

(i) *The problem is posed and solved for a system, not for a pair of matrices.*

(ii) *In our problem the Lyapunov spectrum is assigned, which coincides with the logarithms of the absolute values of eigenvalues of the coefficient matrix of the system, and not with the usual spectrum of this matrix as in the classical statement of the problem.*

(iii) *The assigned values of the spectrum lie in some neighborhood of the spectrum of the original system, and not in the whole set of possible values of the spectrum.*

(iv) *There is a Lipschitz-type estimate of the value of the matrix feedback coefficient needed to shift the spectrum by a given value from the original one.*

(v) *The feedback U constructed by us to assign the spectrum depends on time and is periodic with period K. In fact, from the time-invariance of the matrices A, B and from the method of constructing the matrix H, in the proof of Theorem* 6.6*, it follows that all sequences of matrices $V(n)$ in Lemma* 3.3 *can be chosen to be the same on each segment $n = k_0, \ldots, k_0 + K - 1$, $k_0 \in \mathbb{N}$. Finally, the method of constructing the matrices $V(n)$ in the proofs of Theorems* 3.4 *and* 5.3 *ensures their repeatability on each segment $n = (m-1)K + 1, \ldots, mK$, $m \in \mathbb{N}$.*

(vi) *In contrast to the classical theorem, our result provides only sufficient conditions for the assignability of the spectrum of the system* (2.1).

Now we shall construct an example where, using our results, a stabilizing feedback for a linear control system is designed.

EXAMPLE 7.4. *Consider a model of digital positioning system, described in* [23, Example 6.2, pp. 447–448]. *After discretization, we obtain the following discrete-time linear system:*

$$x(n+1) = A(n)x(n) + b(n)u(n), \quad n \in \mathbb{N},\ x \in \mathbb{R}^2,\ u \in \mathbb{R}, \tag{7.5}$$

where

$$A(n) \equiv A = \begin{pmatrix} 1 & 0,08015 \\ 0 & 0,6313 \end{pmatrix}, \quad b(n) \equiv b = \begin{pmatrix} 0,003396 \\ 0,06308 \end{pmatrix}.$$

Let us check for this system the condition (7.4) *of complete controllability. Since*

$$\det[b, Ab] = \det \begin{pmatrix} 0,003396 & 0,008451862 \\ 0,06308 & 0,039822404 \end{pmatrix} \approx -0,0004 \neq 0,$$

the condition (7.4) *is satisfied and the length of the segment of uniform complete controllability is* $K = 2$. *We note that the free system*

$$x(n+1) = Ax(n), \quad n \in \mathbb{N},\ x \in \mathbb{R}^2, \tag{7.6}$$

is regular and the Lyapunov spectrum $\lambda(A)$ *consists of logarithms of the diagonal elements of the matrix* A, *i.e.,*

$$\lambda_1(A) = \ln 0,6313 < 0, \quad \lambda_2(A) = 0.$$

The free system is stable but not asymptotically [13, p. 187]. *Its solution with the initial condition* $x(0) = x_0 e_1$ *is constant for all* $x_0 \in \mathbb{R}$.

Closing the system (7.5) *by the linear feedback* $u(n) = U(n)x(n)$, *where* $U(n) = \big(u_1(n), u_2(n)\big) \in \mathbb{R}^{1\times 2}$, *we obtain the closed-loop system*

$$\begin{aligned} x(n+1) &= \big(A + bU(n)\big)x(n) \\ &= \begin{pmatrix} 1 + 0,003396u_1(n) & 0,08015 + 0,003396u_2(n) \\ 0,06308u_1(n) & 0,6313 + 0,06308u_2(n) \end{pmatrix} x(n), \quad n \in \mathbb{N}. \end{aligned} \tag{7.7}$$

Since for the system (7.5) *the conditions of Corollary* 7.2 *are fulfilled, it follows that for any sequence*

$$\mu = (\mu_1,\ \mu_2) \in O_\delta\big(\lambda(A)\big)$$

there exists 2*-periodic control* $U(n)$, *such that*

$$\|U\|_\infty \le \ell \max\big\{|\mu_1 - \lambda_1(A)|,\ |\mu_2 - \lambda_2(A)|\big\}$$

and in that case

$$\lambda(A + bU) = \mu.$$

Let us take

$$\mu_1 = \lambda_1(A), \quad \mu_2 = \nu,$$

where ν *is any number satisfying* $|\nu| < \delta$. *Then the condition*

$$\mu \in O_\delta\big(\lambda(A)\big)$$

is satisfied. We shall construct the corresponding control $U(n) = U_\nu(n)$ *for which*

$$\|U_\nu\|_\infty \le \ell|\nu| \quad \textit{and} \quad \lambda(A + bU_\nu) = \mu.$$

If $\nu < 0$, *then the closed-loop system* (7.7) *with the control* $U(n) = U_\nu(n)$ *has the Lyapunov exponents*

$$\lambda_1(A + bU_\nu) = \ln 0,6313 < 0, \quad \lambda_2(A + bU_\nu) = \nu < 0,$$

and therefore, the closed loop-system (7.7) *is asymptotically stable for all* $\nu \in (-\delta, 0)$. *We may stabilize the system* (7.5) *by a small* 2*-periodic feedback.*

If $\nu > 0$*, then*

$$\lambda_1(A + bU_\nu) = \ln 0,6313 < 0, \quad \lambda_2(A + bU_\nu) = \nu > 0,$$

and therefore, the closed loop-system (7.7) *is unstable for all* $\nu \in (0, \delta)$*. In that case we may destabilize the system* (7.5) *by a small* 2*-periodic feedback.*

Let us notice that the determinant of the matrix $[b, Ab]$ *is closed to zero. It means that the system* (7.5) *has "bad" controllability characteristics, that is, the* ℓ *is large and* δ *is closed to zero. In fact, let us calculate the values of* ℓ *and* δ*, using Corollary* 7.2*. For the Kalman matrix* $W(k_0, k_0 + K)$ *we have*

$$\begin{aligned} W(k_0, k_0 + K) &= W(k_0, k_0 + 2) = W(0, 2) \\ &= \Phi_A(0, 1)bb^T\Phi_A^T(0, 1) + \Phi_A(0, 2)bb^T\Phi_A^T(0, 2) \\ &= A^{-1}bb^T(A^{-1})^T + A^{-2}bb^T(A^{-2})^T \\ &= A^{-2}\big(bb^T + Abb^TA^T\big)(A^{-2})^T \end{aligned}$$

for each positive integer k_0*. From the nonsingularity of the matrix* A *we obtain that*

$$\min_{\xi \in \mathbb{R}^2} \xi^T W(k_0, k_0 + 2)\xi = \min_{\xi \in \mathbb{R}^2} \xi^T\big(bb^T + Abb^TA^T\big)\xi.$$

Direct calculations show that the inequality (3.2) *holds for* $\gamma = 10^{-7}$*. As a number* a *we may take* $a = 6$*. Then*

$$\ell = 16(e - 1) \cdot 10^{14}6^{15}(\widehat{\delta} + 1)^2, \quad \delta = \widehat{\delta}/(e - 1),$$

where $\widehat{\delta} > 0$ *is sufficiently small constant satisfying the inequality*

$$\widehat{\delta}(\widehat{\delta} + 1)^2 < \frac{5 \cdot 10^{-14}}{16 \cdot 35 \cdot 6^{14}}.$$

The construction of the control $U(n)$ *is carried out on the basis of Lemma* 3.3 *and Theorems* 3.4*,* 5.3*, and* 6.7*. Taking in Lemma* 3.3 *as the control* $u_l(j)$ *the Kalman control*

$$u_l(j) = -b^T\Phi_A^T(1, j + 1)W^{-1}(1, 3)z_l, \quad l, j = 1, 2,$$

and generating the control $\big(U(n)\big)_{n=1,2}$ *according to the proofs of Theorems* 3.4*,* 5.3*, and* 6.7*, we get*

$$\begin{aligned} U(1) &= -b^T(A^{-1})^TW^{-1}(1, 3)(H - I), \\ U(2) &= -b^T(A^{-2})^TW^{-1}(1, 3)\big(I + A^{-1}bU(1)\big)^{-1}A^{-2}, \end{aligned}$$

where $H = A^{-1}RAR$*,* $R = \operatorname{diag}(e^\nu, 1)$*.*

Note that the method of local assignment of the Lyapunov spectrum that we proposed allows us to construct the feedback $U(n)$ not only in the time-invariant case, but also in the time-varying case. The only nonconstructive element is the Lyapunov transformation $L(n)$. In addition, in the time-varying case, the construction of the feedback $U(n)$ must be carried out for all $n \in \mathbb{N}$.

In conclusion, we discuss the necessity and robustness of the sufficient conditions for the local proportional assignability of the Lyapunov spectrum.

One can prove that the set of uniformly completely controllable systems is open in the set of all systems of the form (2.1) with the topology of uniform convergence on $\mathbb{N}$, while the set of systems with stable Lyapunov spectrum and the sets of diagonalizable and regular systems are not open in the set of all systems of the form (2.2) with the topology of uniform convergence on $\mathbb{N}$. Therefore, the properties of diagonalizability, regularity, and stability of the Lyapunov spectrum cannot be a robust property for the local proportional assignability of the Lyapunov spectrum.

The proofs of the main results of our paper do not give reasons to suppose that the properties of diagonalizability, regularity, and stability of the Lyapunov exponents are necessary for the proportional local assignability or even close to those. The instability of the Lyapunov exponents of the original system means that the Lyapunov spectrum, considered as a function defined on the space of systems with the topology of the uniform convergence on $\mathbb{N}$, has a discontinuity at the point corresponding to the system under consideration; i.e., for arbitrarily small perturbations some of exponents may vary considerably having the so-called jumps. In this case, if the free system is neither diagonalizable nor regular, some of the corresponding controlled systems may not have the property of local proportional assignability of the exponents. However, even the construction of examples of such systems, not to mention the study of the assignability of their exponents, is a difficult task that must be further investigated. The problem of the necessity of the condition of uniform complete controllability for local proportional assignability of the Lyapunov spectrum is also unsolved.

8. Conclusions. In this paper we consider a refined version of the pole assignment problem for discrete time-varying linear systems, namely the proportional local pole assignment problem. We show that diagonalizability, as well as Lyapunov regularity or stability of Lyapunov spectrum, ensures the solvability of this problem for any uniformly completely controllable open-loop system. Now at least two questions remain open. First, are there some other and more general sufficient conditions for the local assignability of the Lyapunov spectrum? Second, what are the necessary conditions for the local proportional assignment of the Lyapunov spectrum?

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