Izvestiya Instituta Matematiki i Informatiki Udmurtskogo Gosudarstvennogo Universiteta
2019. Volume 53

MSC2010: 13D45, 14B15

© **L. Lu**

STRUCTURAL THEOREM FOR GR-INJECTIVE MODULES OVER GR-NOETHERIAN G-GRADED COMMUTATIVE RINGS AND LOCAL COHOMOLOGY FUNCTORS

It is well known that the decomposition of injective modules over noetherian rings is one of the most aesthetic and important results in commutative algebra. Our aim is to prove similar results for graded noetherian rings. In this paper, we will study the structure theorem for gr-injective modules over gr-noetherian G-graded commutative rings, give a definition of the gr-Bass numbers, and study their properties. We will show that every gr-injective module has an indecomposable decomposition. Let R be a gr-noetherian graded ring and M be a gr-finitely generated R-module, we will give a formula for expressing the Bass numbers using the functor Ext. We will define the section functor Γ_V with support in a specialization-closed subset V of $Spec^{gr}(R)$ and the abstract local cohomology functor. Finally, we will show that a left exact radical functor F is of the form Γ_V for a specialization-closed subset V.

Keywords: graded commutative rings, gr-Bass numbers, local cohomology functors, derived categories, radical functors.

DOI: 10.20537/2226-3594-2019-53-11

Introduction

Recently, considerable interest has been noted in rings and other algebraic structures equipped with grading. This is explained by the fact that many important classes of rings, for example, polynomial rings, matrix rings, group rings, admit a natural grading.

A natural and important question in the theory of graded rings is to find graded analogues of some classical theorems. For example, C. Park in his work [17] proved Krull's principal ideal theorem, the Krull–Akizuki theorem and the Mori–Nagata theorem in the graded case. S. Paul Smith in his work [21] proved some category equivalences involving the quotient category $QGr(kQ) := Gr(kQ)/Fdim(kQ)$ of graded kQ-modules modulo those that are the sum of their finite-dimensional submodules. J. Bell and J. Zhang in their work [2] proved, for two connected graded algebras A and B that are finitely generated in degree one, that if A is isomorphic to B as ungraded algebras, then they are also isomorphic to each other as graded algebras. J. Chen and Y. Kim in their work [3] show that if a graded submodule of a noetherian module cannot be written as a proper intersection of graded submodules, then it cannot be written as a proper intersection of submodules at all. In [4] I. DellAmbrogio and G. Stevenson establish an inclusion-preserving bijection between, on the one hand, the twist-closed localizing subcategories of the derived category, and, on the other hand, subsets of the homogeneous spectrum of prime ideals of the ring.

It is well known that the decomposition of injective modules over noetherian rings is one of the most aesthetic and important results in commutative algebra. Our aim is to prove similar results for graded noetherian rings. This is important for us to understand the structure of the noetherian modules on the graded rings.

We now give a brief outline of the paper.

In Section 1, we set notation and review some basics of graded commutative rings, torsion theories and t-structures.

In Section 2, we give a definition of gr-Bass numbers and obtained the following important theorem:
Theorem 2.1 *Let R be a gr-noetherian G-graded ring and $M \in gr(R)$. Then*

$$\mu_i^{gr}(\mathfrak{p}, g, M) = dim_{\kappa_{(\mathfrak{p})}} Ext^i_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, M_{(\mathfrak{p})}(g)).$$

In Section 3, we show that a left exact radical functor F is of the form Γ_V for a specialization-closed subset V. This is a generalization of Yoshino and Yoshizawa's theorem in the case of graded rings (see also [23, Theorem 2.6]).

Theorem 3.1 *The following conditions are equivalent for a left exact preradical functor F on $Gr(R)$.*

(1) *F is a radical functor.*

(2) *F preserves injectivity.*

(3) *F is a section functor with support in a specialization-closed subset of $Spec^{gr}(R)$.*

(4) *$\mathbf{R}F$ is an abstract local cohomology functor.*

Symbols

G	totally ordered abelian group
R	G-graded ring
$Mod(R)$	category of R-modules
$Gr(R)$	category of graded R-modules
$gr(R)$	category of gr-noetherian graded R-modules
$\mathcal{A}$	abelian category
$\mathbf{C}(\mathcal{A})$	category of $\mathcal{A}$-complexes
$\mathbf{K}(\mathcal{A})$	homotopy category of chain complexes in $\mathcal{A}$
$\mathbf{D}(\mathcal{A})$	derived category of $\mathcal{A}$

For more information on graded rings, see [8, 15–17, 19, 20].

§ 1. Preliminaries

§ 1.1. Graded commutative rings and graded modules

In this article we always assume that G is a totally ordered abelian group. For the definition of G-graded commutative rings and modules see, for example, [15, 16]. Let R be a G-graded ring and $M = \oplus_{g\in G} M_g$ be a graded R-module, we let $h(M) := \cup_{g\in G} M_g$ be the *set of homogeneous elements*. We denote by $Mod(R)$ the category of R-modules, and by $Gr(R)$ the category of graded R-modules whose objects are graded R-modules, and the morphisms are homomorphisms preserving grading:

$$Hom_{Gr(R)}(M, N) = \{f \in Hom_R(M, N) : f(M_g) \subseteq N_g \; for \; all \; g \in G\}.$$

Obviously, $Gr(R)$ is a *Grothendieck category* [16, Proposition 2.2].

Gr-noetherian module is a graded R-module M in which every sequence of graded submodules $N_1 \subset N_2 \subset \cdots \subset N_n \subset \ldots$ is stabilized. If R as a R-module is gr-noetherian, then we call R *gr-noetherian ring*. If R is a gr-noetherian ring, then the category of gr-noetherian R-modules $gr(R)$ is an *abelian category* [16, Proposition 2.2].

A *gr-multiplicative set* is a subset S of a graded ring R such that the following two conditions hold:

(1) $0 \notin S$, $1 \in S$, $S \subset h(R)$.

(2) For all x and y in S, the product xy is in S.

Let $M = \bigoplus_{g\in G} M_g$ be a graded R-module. The grading of the module M induces a grading on the modulus of quotients $S^{-1}M$ over the gr-multiplicative system S:

$$(S^{-1}M)_g = \left\{\frac{m}{s} \in S^{-1}M \,|\, m \in M_h,\, s \in S \cap R_k,\, g = hk^{-1}\right\}.$$

A graded ring R is a *gr-local ring* if R has a unique graded maximal ideal.

Let R be a gr-noetherian G-graded commutative ring, and M a graded module over R. We say that $\mathfrak{p} \in Spec^{gr}(R)$ is a *gr-associated prime* of M if $\mathfrak{p}$ is the annihilator of some $x \in h(M)$. The set of gr-associated primes of M is denoted by $Ass^{gr}_R(M)$ or $Ass^{gr}(M)$.

§ 1.2. Gr-injective modules over gr-noetherian rings

D e f i n i t i o n 1.1. Let R be a G-graded ring.

(1) A graded R-module M is *gr-injective* if it is injective as an object of the category $Gr(R)$.

(2) Let N be a graded submodule of M. N is said to be an *gr-essential submodule* of M if for every gr-submodule H of M the equality $H \cap N = 0$ implies that $H = 0$.

(3) The *gr-injective hull* $E^{gr}(M)$ of a graded module M is both the smallest gr-injective module containing it and the largest gr-essential extension of it.

(4) A graded module is *gr-indecomposable* if it is nonzero and cannot be written as a direct sum of two nonzero graded submodules.

P r o p o s i t i o n 1.1. *If* $\mathfrak{p} \in Spec^{gr}(R)$, *then* $Ass^{gr}(\frac{R}{\mathfrak{p}}) = \{\mathfrak{p}\}$ *and* $Ass^{gr}(E^{gr}(\frac{R}{\mathfrak{p}})) = \{\mathfrak{p}\}$.

P r o o f. $\mathfrak{p}$ is a gr-associated prime of $R/\mathfrak{p}$ because there is a monomorphism from $R/\mathfrak{p}$ to itself. If $Q \in Ass^{gr}(\frac{R}{\mathfrak{p}})$, we must show that $Q = \mathfrak{p}$. Suppose that Q is the kernel of an epimorphism $R \to Rx$. Then $s \in Q$ iff $sx \in \mathfrak{p}$ iff $s \in \mathfrak{p}$ (because $\mathfrak{p}$ is prime).

Let $\mathfrak{q} \in Ass^{gr}(E^{gr}(\frac{R}{\mathfrak{p}}))$, then $\mathfrak{q} = Ann_R(x)$ and $0 \neq x \in h(E^{gr}(\frac{R}{\mathfrak{p}}))$. By the definition of the gr-injective hull, $\frac{R}{\mathfrak{p}}$ is a gr-essential submodule of $E^{gr}(\frac{R}{\mathfrak{p}})$. There is an element $r \in h(R)$, and $0 \neq rx \in \frac{R}{\mathfrak{p}}$, obviously, $r \notin \mathfrak{q}$. We have $Ann(rx) = \mathfrak{p} \supset Ann(x) = \mathfrak{q}$. For any $p \in \mathfrak{p}$, we have $prx = 0$, so $pr \in \mathfrak{q}$, but $r \notin \mathfrak{q}$, hence $p \in \mathfrak{q}$, that is, $\mathfrak{p} = \mathfrak{q}$. $\square$

T h e o r e m 1.1. *Let R be a gr-noetherian G-graded commutative ring.*

(1) *For every family $\{J_\lambda\}_{\lambda \in \Lambda}$ of gr-injective modules, the direct sum $\bigoplus_{\lambda \in \Lambda} J_\lambda$ is also gr-injective.*

(2) *Every gr-injective module has an indecomposable decomposition.*

(3) *If $0 \neq E$ is a gr-indecomposable gr-injective module, then $E = E^{gr}(\frac{R}{\mathfrak{p}}(g))$, where $\mathfrak{p} \in Spec^{gr}(R)$ and $g \in G$.*

(4) *Let $\mathfrak{p}, \mathfrak{q} \in Spec^{gr}(R)$, $g, h \in G$ and let $\mathfrak{p} \neq \mathfrak{q}$, then $E^{gr}(\frac{R}{\mathfrak{p}}(g)) \neq E^{gr}(\frac{R}{\mathfrak{q}}(h))$.*

P r o o f.
(1) See [10, Theorem 5.9].
(2) See [10, Theorem 5.9].
(3) Let $\mathfrak{p} \in Ass^{gr}(E)$, there exist $x \in h(E)$ and $g \in G$ such that $Rx \cong \frac{R}{\mathfrak{p}}(g)$. Hence $E = E^{gr}(Rx)$.
(4) This is Proposition 1.1. $\square$

§ 1.3. Torsion theories

Our main references for the torsion theories are [5, 12, 18].

A torsion theory in a Grothendieck category $\mathcal{A}$ is a couple $(\mathcal{T}, \mathcal{F})$ of strictly full additive subcategories called the torsion class $\mathcal{T}$ and the torsion free class $\mathcal{F}$ such that the following conditions hold:

(1) $Hom(\mathcal{T}, \mathcal{F}) = 0$.

(2) For all $M \in Obj(\mathcal{A})$, there exists $N \subset M$, $N \in Obj(\mathcal{T})$ and $M/N \in Obj(\mathcal{F})$.

For every object M there exist the largest subobject $t(M) \subset M$ which is in $\mathcal{T}$ and it is called the torsion part of M.

$$t : M \mapsto t(M)$$

is an additive functor. A torsion theory is hereditary if $\mathcal{T}$ is closed under subobjects, or equivalently, t is left exact functor.

A radical functor, or more generally, a preradical functor, has its own long history in the theory of categories and functors. See [7] or [13] for the case of module category. Let $F, G : \mathcal{A} \to \mathcal{A}$ be functors. Recall that F is said to be a subfunctor of G, denoted by $F \subset G$, if $F(M)$ is a subobject of $G(M)$ for all $M \in \mathcal{A}$ and if $F(f)$ is a restriction of $G(f)$ to $F(M)$ for all $f \in Hom_{\mathcal{A}}(M, N)$.

D e f i n i t i o n 1.2. A functor $F : \mathcal{A} \to \mathcal{A}$ is called a preradical functor if F is a subfunctor of $\mathbf{1}$.

L e m m a 1.1 (see [23, Lemma 1.1]). *Let $F : \mathcal{A} \to \mathcal{A}$ be a preradical functor and assume that F is a left exact functor on $\mathcal{A}$. If N is a subobject of M, then the equality $F(N) = N \cap F(M)$ holds.*

D e f i n i t i o n 1.3. A preradical functor F is called a radical functor if $F(M/F(M)) = 0$ for all $M \in Obj(\mathcal{A})$.

If $F : \mathcal{A} \to \mathcal{A}$ is a left exact radical functor, then there is a hereditary torsion theory $(\mathcal{T}_F, \mathcal{F}_F)$ by setting

$$\begin{aligned} \mathcal{T}_F &= \{M \in \mathcal{A} \,|\, F(M) = M\}, \\ \mathcal{F}_F &= \{M \in \mathcal{A} \,|\, F(M) = 0\}. \end{aligned} \tag{1.1}$$

§ 1.4. t-structures

The notion of a t-structure arose in the work [1] of Beilinson, Bernstein, Deligne, and Gabber on perverse sheaves.

Let $\mathcal{D}$ be a triangulated category. t-structure in $\mathcal{D}$ is a pair $\mathbf{t} = (\mathcal{U}, \mathcal{W})$ of full subcategories, closed under taking direct summands in $\mathcal{D}$, which satisfy the following properties:

(t-S.1) $Hom_{\mathcal{D}}(U, W[-1]) = 0$, for all $U \in \mathcal{U}$ and $W \in \mathcal{W}$;

(t-S.2) $\mathcal{U}[1] \subset \mathcal{U}$, $\mathcal{W}[-1] \subset \mathcal{W}$;

(t-S.3) for each $Y \in \mathcal{D}$, there is a triangle $A \to Y \to B \to A[1]$ in $\mathcal{D}$, where $A \in \mathcal{U}$ and $B \in \mathcal{W}[-1]$.

A t-structure $\mathbf{t} = (\mathcal{U}, \mathcal{W})$ in $\mathcal{D}$ is called a stable t-structure on $\mathcal{D}$ if $\mathcal{U}$ and $\mathcal{W}$ are triangulated subcategories.

T h e o r e m 1.2 (see [14, Proposition 2.6]). *Let $\mathcal{D}$ be a triangulated category and $\mathcal{U}$ a triangulated subcategory of $\mathcal{D}$. Then the following conditions are equivalent for $\mathcal{U}$.*

(1) *There is a triangulated subcategory $\mathcal{W}$ of $\mathcal{D}$ such that $(\mathcal{U}, \mathcal{W})$ is a stable t-structure on $\mathcal{D}$.*

(2) *The natural embedding functor $i : \mathcal{U} \to \mathcal{D}$ has a right adjoint $\rho : \mathcal{D} \to \mathcal{U}$.*

If this is the case, setting $\delta = i \circ \rho : \mathcal{D} \to \mathcal{D}$,we have the equalities $\mathcal{U} = Im(\delta)$ and $\mathcal{W} = Ker(\delta)$. There is an natural morphism $\phi : \delta \to \mathbf{1}$, where $\mathbf{1}$ is the identity functor on $\mathcal{D}$. Every $C \in \mathcal{D}$ can be embedded in a triangle of the form

$$\delta(C) \xrightarrow{\phi(C)} C \to D \to \delta(C)[1].$$

§ 2. Gr-Bass numbers

Let R be a gr-noetherian G-graded ring. For each $\mathfrak{p} \in Spec^{gr}(R)$, $S = h(R) - \mathfrak{p}$ is a gr-multiplicative system and $R_{(\mathfrak{p})} := S^{-1}R$ is a gr-local ring. $\mathfrak{p}R_{(\mathfrak{p})}$ is the unique gr-maximal ideal of $R_{(\mathfrak{p})}$. We define $\kappa_{(\mathfrak{p})} = \frac{R_{(\mathfrak{p})}}{\mathfrak{p}R_{(\mathfrak{p})}}$.

P r o p o s i t i o n 2.1. *Suppose that $f : M \to N$ is a monomorphism in $Gr(R)$. Then N is an essential extension of M if and only if for every $\mathfrak{p} \in Spec^{gr}(R)$ the induced morphism*

$$f_{(\mathfrak{p})} : Hom_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, M_{(\mathfrak{p})}) \to Hom_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, N_{(\mathfrak{p})})$$

is an isomorphism.

P r o o f. Since localization is exact and $Hom_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, -)$ is left exact, $f_{(\mathfrak{p})}$ is always a monomorphism. Let $S = h(R) - \mathfrak{p}$. Since $\frac{R}{\mathfrak{p}}$ is finitely presented, we have $S^{-1}Hom_R(\frac{R}{\mathfrak{p}}, M) \cong Hom_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, M_{(\mathfrak{p})})$ and $S^{-1}Hom_R(\frac{R}{\mathfrak{p}}, N) \cong Hom_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, N_{(\mathfrak{p})})$.

First, let N be an essential extension of M and let $0 \neq \frac{f}{s} \in S^{-1}Hom_R(\frac{R}{\mathfrak{p}}, N)$. We have $0 \neq f(1) \in N$, so there is an element $r \in h(R)$ such that $0 \neq rf(1) = f(r) \in M$. It is obvious that $r \notin \mathfrak{p}$, that is, $r \in S$, so $\frac{rf}{rs} \in S^{-1}Hom_R(\frac{R}{\mathfrak{p}}, M)$ and its image in $S^{-1}Hom_R(\frac{R}{\mathfrak{p}}, N)$ is $\frac{f}{s}$, so $f_{(\mathfrak{p})}$ is an isomorphism.

Now let $f_{(\mathfrak{p})}$ be an isomorphism for any $\mathfrak{p} \in Spec^{gr}(R)$. Let $x \in h(N)$. There is a graded prime ideal $\mathfrak{p} \in Ass^{gr}(Rx)$ and an element $r \in h(R)$ such that $y = rx$ and $Ann_R(y) = \mathfrak{p}$. We define a homomorphism $f \in Hom_R(\frac{R}{\mathfrak{p}}, N)$ such that $f(1) = y$. There is a homomorphism $h \in Hom_R(\frac{R}{\mathfrak{p}}, M)$ and an element $s \in S$ such that $\frac{h}{s}(\frac{1}{1}) = \frac{f}{1}(\frac{1}{1}) = \frac{y}{1}$. There is an element $u \in S$ such that $h(1) = usy = usrx \in M$, hence N is an essential extension of M. □

Let M be a graded R-module. A *gr-injective resolution* is an exact sequence

$$E^\bullet = 0 \to M \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} \dots,$$

where E^i is a gr-injective module. If $Ker(d^i) \to E^i$ is an essential extension, then we say that $E^\bullet$ is a *gr-minimal injective resolution*.

P r o p o s i t i o n 2.2. *Let R be a gr-noetherian ring and M be an R-module with a gr-injective resolution $E^\bullet$. Then $E^\bullet$ is a gr-minimal injective resolution if and only if for all $\mathfrak{p} \in Spec^{gr}(R)$ and all i, the induced morphisms $Hom_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, E^i_{(\mathfrak{p})}) \to Hom_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, E^{i+1}_{(\mathfrak{p})})$ are zero.*

P r o o f. Let $Z^i = ker(d^i) = Im(d^{i-1})$. $E^\bullet$ is gr-minimal if and only if E^i is an essential extension of Z^i, if and only if the induced morphism $Hom_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, Z^i_{(\mathfrak{p})}) \to Hom_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, E^i_{(\mathfrak{p})})$ is an isomorphism by Proposition 2.1. Since there are exact sequences $0 \to Z^i \to E^i \to E^{i+1}$, and $Hom_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, -)$ is left exact, the latter condition holds if and only if the induced morphisms $Hom_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, E^i_{(\mathfrak{p})}) \to Hom_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, E^{i+1}_{(\mathfrak{p})})$ are 0. □

P r o p o s i t i o n 2.3. *Let R be a gr-noetherian G-graded ring and S be a gr-multiplicative system. If M is a gr-injective R-module, then $S^{-1}M$ is a gr-injective $S^{-1}R$-module.*

P r o o f. In the nongraded case this is a classical result (see [9, Proposition 2.8]). In the graded case, it is possible to repeat this proof. □

P r o p o s i t i o n 2.4. *Let R be a gr-noetherian G-graded ring and M be a graded R-module with a gr-minimal injective resolution $E^\bullet$. Then for any $\mathfrak{p} \in Spec^{gr}(R)$, $E^\bullet_{(\mathfrak{p})}$ are gr-minimal injective resolutions of $M_{(\mathfrak{p})}$.*

P r o o f. We know that $E^i_{(\mathfrak{p})}$ are gr-injective $R_{(\mathfrak{p})}$-module by Proposition 2.3. By Proposition 2.2 it is only necessary to prove that $Hom_{R_{(\mathfrak{q})}}(\kappa_{(\mathfrak{q})}, (E^i_{(\mathfrak{p})})_{(\mathfrak{q})}) \to Hom_{R_{(\mathfrak{q})}}(\kappa_{(\mathfrak{q})}, (E^{i+1}_{(\mathfrak{p})})_{(\mathfrak{q})})$ are 0 for all $\mathfrak{q} \subset \mathfrak{p}$. We know that $(E^i_{(\mathfrak{p})})_{(\mathfrak{q})} = E^i_{(\mathfrak{q})}$, hence $E^\bullet_{(\mathfrak{p})}$ are gr-minimal injective resolutions of $M_{(\mathfrak{p})}$ by Proposition 2.2. □

D e f i n i t i o n 2.1. (Gr-Bass numbers) Let R be a gr-noetherian G-graded ring and let $M \in gr(R)$ have a gr-minimal injective resolution $E^\bullet$. By Theorem 1.1, we have

$$E^i = \bigoplus_{\mathfrak{p} \in Spec^{gr}(R)} \bigoplus_{g \in G} [E^{gr}_R(\frac{R}{\mathfrak{p}})(g)]^{\mu^{gr}_i(\mathfrak{p}, g, M)}.$$

$\mu^{gr}_i(\mathfrak{p}, g, M)$ are called *gr-Bass numbers*.

P r o p o s i t i o n 2.5. *Let R be a gr-noetherian G-graded ring and $M \in gr(R)$. Then, for any i,*

$$\mu^{gr}_i(\mathfrak{p}, g, M) = \mu^{gr}_i(\mathfrak{p}_{(\mathfrak{p})}, g, M_{(\mathfrak{p})}).$$

P r o o f. Take a gr-minimal injective resolution $E^\bullet$. By Proposition 2.4, $E^\bullet_{(\mathfrak{p})}$ is the gr-minimal injective resolution of $M_{(\mathfrak{p})}$. Since $E^i = \bigoplus_{\mathfrak{p} \in Spec^{gr}(R)} \bigoplus_{g \in G} [E^{gr}_R(\frac{R}{\mathfrak{p}})(g)]^{\mu^{gr}_i(\mathfrak{p}, g, M)}$, we make localization and get the result. □

T h e o r e m 2.1. *Let R be a gr-noetherian G-graded ring and $M \in gr(R)$. Then*

$$\mu_i^{gr}(\mathfrak{p}, g, M) = dim_{\kappa_{(\mathfrak{p})}} Ext^i_{R_{(\mathfrak{p})}}(\kappa_{(\mathfrak{p})}, M_{(\mathfrak{p})}(g)).$$

P r o o f. By Proposition 2.5, it suffices to consider the case when $(R, \mathfrak{m}, \kappa)$ is a gr-noetherian gr-local ring and $\mathfrak{p} = \mathfrak{m}$. Let $E^\bullet$ be a minimal gr-injective resolution of M. Then $Ext^i_{Gr(R)}(\kappa, M)$ is the ith cohomology of the complex $Hom_{Gr(R)}(\kappa, E^\bullet)$. By the minimality of the resolution all maps in this complex are zero, so that $Ext^i_{Gr(R)}(\kappa, M) = Hom_{Gr(R)}(\kappa, E^i)$. To finish the proof it suffices to show that

$$Hom_{Gr(R)}(\kappa, E_R^{gr}(\frac{R}{\mathfrak{p}})) = \begin{cases} 0\ (if\ \mathfrak{p} \neq \mathfrak{m}) \\ \kappa\ (if\ \mathfrak{p} = \mathfrak{m}). \end{cases}$$

If $\mathfrak{p} = \mathfrak{m}$, then, by Proposition 2.1, $Hom_{Gr(R)}(\kappa, E_R^{gr}(\kappa)) = Hom_{Gr(R)}(\kappa, \kappa) = \kappa$.

If $\mathfrak{p} \neq \mathfrak{m}$, then there is a homogeneous element x of $\mathfrak{m}$ not in $\mathfrak{p}$. Let $f \in Hom_{Gr(R)}(\kappa, \frac{R}{\mathfrak{p}})$, then $0 = f(x) = xf(1)$, but x in $\frac{R}{\mathfrak{p}}$ is not a zero divisor, so $f(1) = 0$, that is, $Hom_{Gr(R)}(\kappa, \frac{R}{\mathfrak{p}}) = 0$. By Proposition 2.1, $Hom_{Gr(R)}(\kappa, E_R^{gr}(\frac{R}{\mathfrak{p}})) = Hom_{Gr(R)}(\kappa, \frac{R}{\mathfrak{p}}) = 0$. □

§ 3. Local cohomology

D e f i n i t i o n 3.1. Let R be a gr-noetherian G-graded ring. Let $M \in Gr(R)$.

(1) The *small gr-support* (following [6]) of M is

$$supp^{gr}(M) = \{\mathfrak{p} \in Spec^{gr}(R) \mid Tor_*^{R_{(\mathfrak{p})}}(M_{(\mathfrak{p})}, \kappa_{(\mathfrak{p})}) \neq 0\}.$$

(2) The *(usual) gr-support* of M is

$$Supp^{gr}(M) = \{\mathfrak{p} \in Spec^{gr}(R) \mid M_{(\mathfrak{p})} \neq 0\}.$$

R e m a r k 3.1. Note that $supp^{gr}(M) \subset Supp^{gr}(M)$ and the equality holds if $M \in gr(R)$ (see, e.g., [6, Lemma 2.6]).

D e f i n i t i o n 3.2. Let R be a gr-noetherian G-graded ring. Let $M \in Gr(R)$.

(1) For any subset $V \subset Spec^{gr}(R)$ we say that V is a specialization-closed subset if for any $\mathfrak{p} \in V$ and any $\mathfrak{q} \in Spec^{gr}(R)$ we have $\mathfrak{q} \in V$ whenever $\mathfrak{p} \subset \mathfrak{q}$.

(2) Let V be a specialization-closed subset of $Spec^{gr}(R)$. We can define the section functor Γ_V with support in V as

$$\Gamma_V(M) = \bigcup\{N \subset M \mid Supp^{gr}(N) \subset V\} = \bigcup\{N \subset M \mid supp^{gr}(N) \subset V\}$$

for all $M \in Gr(R)$.

D e f i n i t i o n 3.3. We denote $\mathcal{C} = \mathbf{D}^+(Gr(R))$ in this definition. Let $\delta : \mathcal{C} \to \mathcal{C}$ be a triangle functor. We say that δ is an *abstract local cohomology functor* if the following conditions are satisfied.

(1) The natural embedding functor $i : Im(\delta) \to \mathcal{C}$ has a right adjoint $\rho : \mathcal{C} \to Im(\delta)$ and $\delta \cong i \circ \rho$.

(2) The t-structure $(Im(\delta), Ker(\delta))$ divides indecomposable injective objects, by which we mean that each indecomposable injective object belongs to either $Im(\delta)$ or $Ker(\delta)$.

T h e o r e m 3.1. *The following conditions are equivalent for a left exact preradical functor F on $Gr(R)$.*

(1) *F is a radical functor.*

(2) *F preserves injectivity.*

(3) *F is a section functor with support in a specialization-closed subset of $Spec^{gr}(R)$.*

(4) *$\mathbf{R}F$ is an abstract local cohomology functor.*

The proof of Theorem 3.1 consists of a succession of relatively short lemmas.

L e m m a 3.1. *Let R be a gr-noetherian G-graded ring, and V be a specialization-closed subset of $Spec^{gr}(R)$.*

(1) *If N is a graded submodule of M, then the equality $\Gamma_V(N) = N \cap \Gamma_V(M)$ holds.*

(2) *$\Gamma_V(M/\Gamma_V(M)) = 0$ for every $M \in Gr(R)$.*

(3) *Γ_V is a left exact radical functor.*

P r o o f. (1) If $H \in Gr(R)$, then

$$H \subset \Gamma_V(N) \Leftrightarrow H \subset N \ and\ Supp^{gr}(H) \subset V \Leftrightarrow H \subset \Gamma_V(M) \cap N.$$

(2) If $H \in Gr(R)$, then

$$H \subset \Gamma_V(M/\Gamma_V(M)) \Leftrightarrow H \subset M/\Gamma_V(M) \ and\ Supp^{gr}(H) \subset V \Rightarrow H = 0.$$

(3) Let $0 \to K \xrightarrow{f} M \xrightarrow{g} N$ be an exact sequence in $Gr(R)$. By (1) we have $\Gamma_V(K) = K \cap \Gamma_V(M)$, hence $0 \to \Gamma_V(K) \to \Gamma_V(M)$ is an exact sequence. Let $H \in Gr(R)$, we have

$$H \subset Ker\Gamma_V(g) \Leftrightarrow H \subset Ker g \cap \Gamma_V(M) = Im f \cap \Gamma_V(M) \Leftrightarrow H \subset Im\Gamma_V(f),$$

hence $0 \to \Gamma_V(K) \to \Gamma_V(M) \to \Gamma_V(N)$ is an exact sequence. □

L e m m a 3.2. *Let R be a gr-noetherian G-graded ring. Let $F : Gr(R) \to Gr(R)$ be a left exact radical functor.*

(1) *Let $\mathfrak{p} \in Spec^{gr}(R)$, then $F(E_R^{gr}(\frac{R}{\mathfrak{p}}))$ is identical to either $E_R^{gr}(\frac{R}{\mathfrak{p}})$ or 0.*

(2) *F preserves injectivity.*

P r o o f. (1) Since F is a left exact radical functor, there is a hereditary torsion theory $(\mathcal{T}_F, \mathcal{F}_F)$, which is defined in (1.1). Hence there is an exact sequence

$$0 \to N \to E_R^{gr}(\frac{R}{\mathfrak{p}}) \to H \to 0$$

with $N \in \mathcal{T}_F$ and $H \in \mathcal{F}_F$. If $N = 0$, then $E_R^{gr}(\frac{R}{\mathfrak{p}}) \cong H \in \mathcal{F}_F$, therefore $F(E_R^{gr}(\frac{R}{\mathfrak{p}})) = 0$. If $N \neq 0$, since $Ass^{gr}(E_R^{gr}(\frac{R}{\mathfrak{p}})) = \{\mathfrak{p}\}$, we have $Ass^{gr}(N) = \{\mathfrak{p}\}$, hence $R/\mathfrak{p} \subset N$. Since $\mathcal{T}_F$ is a localizing subcategory and $N \in \mathcal{T}_F$, we have $R/\mathfrak{p} \in \mathcal{T}_F$. By Proposition 1.1, $E_R^{gr}(\frac{R}{\mathfrak{p}}) \in \mathcal{T}_F$. Therefore, $F(E_R^{gr}(\frac{R}{\mathfrak{p}})) = E_R^{gr}(\frac{R}{\mathfrak{p}})$.

(2) For an injective $J \in Gr(R)$, by Theorem 1.1 it has an indecomposable decomposition $J = \bigoplus_{i\in\mathcal{I}} E_R^{gr}(\frac{R}{\mathfrak{p}_i})(g_i)$. We set $J_1 = \bigoplus_{i\in\mathcal{I}_1} E_R^{gr}(\frac{R}{\mathfrak{p}_i})(g_i)$ and $J_2 = \bigoplus_{i\in\mathcal{I}_2} E_R^{gr}(\frac{R}{\mathfrak{p}_i})(g_i)$, where

$$\mathcal{I}_1 = \{i \in \mathcal{I} \,|\, F(E_R^{gr}(\frac{R}{\mathfrak{p}_i})) = E_R^{gr}(\frac{R}{\mathfrak{p}_i})\},$$

$$\mathcal{I}_2 = \{i \in \mathcal{I} \,|\, F(E_R^{gr}(\frac{R}{\mathfrak{p}_i})) = 0\}.$$

By (1) we have $J = J_1 \oplus J_2$. Since $\mathcal{T}_F$ is closed under taking direct sums and $\mathcal{F}_F$ is closed under taking direct products and subsheaves, we have $J_1 \in \mathcal{T}_F$, and $J_2 \in \mathcal{F}_F$. Therefore, we have an equality $F(J) = F(J_1) \oplus F(J_2) = J_1$, which is a gr-injective module. □

P r o p o s i t i o n 3.1. *Let R be a gr-noetherian G-graded ring. Let $F : Gr(R) \to Gr(R)$ be a left exact preradical functor which preserves injectivity. Then*

(1) *$F(E_R^{gr}(\frac{R}{\mathfrak{p}}))$ is identical to either $E_R^{gr}(\frac{R}{\mathfrak{p}})$ or 0.*

(2) *$F(\frac{R}{\mathfrak{p}})$ is identical to either $\frac{R}{\mathfrak{p}}$ or 0.*

P r o o f. (1) Since $F(E_R^{gr}(\frac{R}{\mathfrak{p}}))$ is a gr-injective submodule of a gr-indecomposable injective module $E_R^{gr}(\frac{R}{\mathfrak{p}})$, it is a direct summand of $E_R^{gr}(\frac{R}{\mathfrak{p}})$. Thus the indecomposability of $E_R^{gr}(\frac{R}{\mathfrak{p}})$ forces $F(E_R^{gr}(\frac{R}{\mathfrak{p}}))$ is either $E_R^{gr}(\frac{R}{\mathfrak{p}})$ or 0.
(2) It follows from Lemma 1.1 that $F(\frac{R}{\mathfrak{p}}) = \frac{R}{\mathfrak{p}} \cap F(E_R^{gr}(\frac{R}{\mathfrak{p}}))$, therefore $F(\frac{R}{\mathfrak{p}})$ is either $\frac{R}{\mathfrak{p}}$ or 0 by (1). □

For a left exact preradical functor F which preserves injectivity, we define a subset V_F of $Spec^{gr}(R)$ as follows:

$$V_F = \left\{ \mathfrak{p} \in Spec^{gr}(R) : F(\frac{R}{\mathfrak{p}}) = \frac{R}{\mathfrak{p}} \right\}.$$

Note from the proof of Proposition 3.1 that V_F is the same as the set

$$\left\{ \mathfrak{p} \in Spec^{gr}(R) : F(E_R^{gr}(\frac{R}{\mathfrak{p}})) = E_R^{gr}(\frac{R}{\mathfrak{p}}) \right\}.$$

P r o p o s i t i o n 3.2. *Let F be a left exact preradical functor which preserves injectivity. Then V_F is a specialization-closed subset.*

P r o o f. Let $\mathfrak{p} \in V_F$ and $\mathfrak{q} \supset \mathfrak{p}$. There is an exact sequence

$$0 \to K \to \frac{R}{\mathfrak{p}} \to \frac{R}{\mathfrak{q}}.$$

Since F is a left exact preradical functor, there is an exact sequence

$$0 \to F(K) \to F(\frac{R}{\mathfrak{p}}) \to F(\frac{R}{\mathfrak{q}}).$$

We have $F(\frac{R}{\mathfrak{p}}) = \frac{R}{\mathfrak{p}}$ and $F(K) = K \cap F(\frac{R}{\mathfrak{p}}) = K$ by Lemma 1.1, therefore $F(\frac{R}{\mathfrak{q}}) = \frac{R}{\mathfrak{q}}$, hence $\mathfrak{q} \in V_F$. □

L e m m a 3.3. *Let F be a left exact preradical functor which preserves injectivity. Then the equality $F = \Gamma_{V_F}$ holds as subfunctors of $\mathbf{1}$, where V_F is a specialization-closed subset of $Spec^{gr}(R)$ defined in Proposition 3.2.*

P r o o f. First of all, we consider the case that M is a finite direct sum of indecomposabe injective objects $\bigoplus_{i=1}^{n} E_R^{gr}(\frac{R}{\mathfrak{p}_i})(g_i)$ in $Gr(R)$. Then we have an equality

$$F(M) = \bigoplus_{\mathfrak{p}_i \in V_F} E_R^{gr}(\frac{R}{\mathfrak{p}_i})(g_i) = \Gamma_{V_F}(M)$$

by Proposition 3.1.

Next, we consider the case that $M \in gr(R)$. Since the gr-injective hull $E^{gr}(M)$ of M is a finite direct sum of indecomposable gr-injective modules, we have already shown that $F(E^{gr}(M)) = \Gamma_{V_F}(E^{gr}(M))$. Thus, using Lemma 1.1, we have

$$F(M) = M \cap F(E^{gr}(M)) = M \cap \Gamma_{V_F}(E^{gr}(M)) = \Gamma_{V_F}(M).$$

Finally, we show the claimed equality for an object M in $Gr(R)$ without any assumption. We should notice that a graded submodule $N \subset M$ belongs to $F(M)$ if and only if the equality $F(N) = N$ holds. In fact, this equivalence is easily observed from the equality $F(N) = N \cap F(M)$ by Lemma 1.1. This equivalence is true for the section functor Γ_{V_F} as well. So $N \subset M$ belongs to $\Gamma_{V_F}(M)$ if and only if $\Gamma_{V_F}(N) = N$. Therefore, we see that $N \subset F(M)$ if and only if $N \subset \Gamma_{V_F}(M)$, and the proof is completed. □

L e m m a 3.4. *Let $M^\bullet \in \mathbf{D}(Gr(R))$ and let V be a specialization-closed subset of $Spec^{gr}(R)$. Then*

(1) *$M^\bullet$ belongs to $Im(\mathbf{R}\Gamma_V)$ if and only if $M^\bullet$ is quasi-isomorphic to a gr-injective complex whose components are direct sums of $E_R^{gr}(\frac{R}{\mathfrak{p}})(g)$ with $\mathfrak{p} \in V$.*

(2) *$M^\bullet$ belongs to $Ker(\mathbf{R}\Gamma_V)$ if and only if $M^\bullet$ is quasi-isomorphic to a gr-injective complex whose components are direct sums of $E_R^{gr}(\frac{R}{\mathfrak{p}})(g)$ with $\mathfrak{p} \in Spec^{gr}(R) - V$.*

(3) *$\mathbf{R}F$ is an abstract local cohomology functor.*

P r o o f. By [22, Theorem 5.4] or [11, Proposition B.2], every complex of graded R-module has a K-injective resolution. For any gr-injective complex $J^\bullet \in \mathbf{K}(\mathcal{I})$, $\mathbf{R}\Gamma_V(J^\bullet) = \Gamma_V(J^\bullet)$ is the subcomplex of $J^\bullet$ consisting of gr-injective modules supported in V. Hence every object of $Im(\mathbf{R}\Gamma_V)$ (resp. $Ker(\mathbf{R}\Gamma_V)$) is a gr-injective complex whose components are direct sums of $E_R^{gr}(\frac{R}{\mathfrak{p}})(g)$ with $\mathfrak{p} \in V$ (resp. $\mathfrak{p} \in Spec^{gr}(R) - V$). In particular, if $\mathfrak{p} \in V$ (resp. $\mathfrak{p} \in Spec^{gr}(R) - V$), then $E_R^{gr}(\frac{R}{\mathfrak{p}})(g) \in Im(\mathbf{R}\Gamma_V)$ (resp. $E_R^{gr}(\frac{R}{\mathfrak{p}})(g) \in Ker(\mathbf{R}\Gamma_V)$). Since $Hom_{Gr(R)}(E_R^{gr}(\frac{R}{\mathfrak{p}})(g)), E_R^{gr}(\frac{R}{\mathfrak{q}})(h)) = 0$ for $\mathfrak{p} \in V$ and $\mathfrak{q} \in Spec^{gr}(R) - V$, we can see that $Hom_{\mathbf{K}(\mathcal{I})}(J_1^\bullet, J_2^\bullet) = Hom_{\mathbf{K}(\mathcal{I})}(J_1^\bullet, \Gamma_V(J_2^\bullet))$ for any $J_1^\bullet \in Im(\mathbf{R}\Gamma_V)$ and $J_2^\bullet \in \mathbf{K}(\mathcal{I})$. Hence it follows from the above equivalence that $\mathbf{R}\Gamma_V$ is a right adjoint of the natural embedding $i : Im(\mathbf{R}\Gamma_V) \to \mathbf{D}(Gr(R))$. $\square$

P r o o f o f T h e o r e m 3.1. $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$, $(3) \Rightarrow (1)$, and $(3) \Rightarrow (4)$ have already been proved, respectively, in Lemmas 3.2 (2), 3.3, 3.1 (3), and 3.4 (3).

$(4) \Rightarrow (1)$. Assume that $\mathbf{R}F$ is an abstract local cohomology functor. We have to show that $F(M/F(M)) = 0$ for any graded module M. It is enough to show that $F(E/F(E)) = 0$ for any gr-injective module E. In fact, for any graded module M, taking the gr-injective hull $E^{gr}(M)$ of M, we have $F(M/F(M)) \subset F(E^{gr}(M)/F(E^{gr}(M)))$ by Lemma 1.1.

Note that the natural inclusion $F \subset \mathbf{1}$ of functors on $Gr(R)$ induces a natural morphism $\phi : \mathbf{R}F \to \mathbf{1}$ of functors on $\mathbf{D}^+(Gr(R))$. Since $(Im(\mathbf{R}F), Ker(\mathbf{R}F))$ is a stable t-structure on $\mathbf{D}^+(Gr(R))$, it follows from Theorem 1.2 that every gr-injective module E is embedded in a triangle

$$\mathbf{R}F(E) \xrightarrow{\phi(E)} E \to N \to \mathbf{R}F(E)[1],$$

with $\mathbf{R}F(E) \in Im(\mathbf{R}F)$ and $N \in Ker(\mathbf{R}F)$. Since E is a gr-injective module and since $\mathbf{R}F$ is the right derived functor of a left-exact functor, $\mathbf{R}F(E) = F(E)$ is a submodule of E via the morphism $\phi(E)$. Therefore, we have $N \cong E/F(E)$ in $\mathbf{D}^+(Gr(R))$. In particular, $H^0(\mathbf{R}F(E/F(E))) \cong H^0(\mathbf{R}F(N)) = 0$. Since F is a left exact functor, it is concluded that $F(E/F(E)) = 0$ as desired. This completes the proof of Theorem 3.1. $\square$

Funding. This work was supported by the Chinese Scholarship Council.

REFERENCES

1. Beilinson A.A., Bernstein J., Deligne P. Faisceaux pervers, *Asterisque*, 1982, vol. 1, no. 100, pp. 5–171.
2. Bell J., Zhang J. An isomorphism lemma for graded rings, *Proceedings of the American Mathematical Society*, 2017, vol. 145, pp. 989–994. https://doi.org/10.1090/proc/13276
3. Chen J., Kim Y. Graded-irreducible modules are irreducible, *Communications in Algebra*, 2017, vol. 45, issue 5, pp. 1907–1913. https://doi.org/10.1080/00927872.2016.1226864
4. DellAmbrogio I., Stevenson G. On the derived category of a graded commutative Noetherian ring, *Journal of Algebra*, 2013, vol. 373, pp. 356–376. https://doi.org/10.1016/j.jalgebra.2012.09.038
5. Dickson S.E. A torsion theory for abelian categories, *Transactions of the American Mathematical Society*, 1966, vol. 121, pp. 223–235. https://doi.org/10.2307/1994341
6. Foxby H.B. Bounded complexes of flat modules, *Journal of Pure and Applied Algebra*, 1979, vol. 15, issue 2, pp. 149–172. https://doi.org/10.1016/0022-4049(79)90030-6
7. Goldman O. Rings and modules of quotients, *Journal of Algebra*, 1969, vol. 13, issue 1, pp. 10–47. https://doi.org/10.1016/0021-8693(69)90004-0

8. Heinzer W., Roitman M. The homogeneous spectrum of a graded commutative ring, *Proceedings of the American Mathematical Society*, 2002, vol. 130, pp. 1573–1580. https://doi.org/10.1090/S0002-9939-01-06231-1
9. Hochster M. Local cohomology, *Unpublished notes*, 2011. https://www.math.lsa.umich.edu/∼hochster/615W11/loc.pdf
10. Kanda R. Classifying Serre subcategories via atom spectrum, *Advances in Mathematics*, 2012, vol. 231, pp. 1572–1588. https://doi.org/10.1016/j.aim.2012.07.009
11. Krause H. The stable derived category of a Noetherian scheme, *Compositio Mathematica*, 2005, vol. 141, pp. 1128–1162. https://doi.org/10.1112/S0010437X05001375
12. Lamber J. *Torsion theories, additive semantics, and rings of quotients*, Berlin–Heidelberg: Springer, 2006. https://doi.org/10.1007/BFb0061029
13. Maranda J.-M. Injective structures, *Transactions of the American Mathematical Society*, 1964, vol. 110, pp. 98–135. https://doi.org/10.1090/S0002-9947-1964-0163937-X
14. Miyachi J. Localization of triangulated categories and derived categories, *Journal of Algebra*, 1991, vol. 141, pp. 463–483. https://doi.org/10.1016/0021-8693(91)90243-2
15. Nastasescu C., Oystaeyen F. *Graded ring theory*, Elsevier, 2011.
16. Nastasescu C., Oystaeyen F. *Methods of graded rings*, Springer Science, 2004.
17. Park C., Park M. Integral closure of a graded Noetherian domain, *Journal of the Korean Mathematical Society*, 2011, vol. 48, pp. 449–464. https://doi.org/10.4134/JKMS.2011.48.3.449
18. Popescu N. *Abelian categories with applications to rings and modules*, Academic Press, 1973.
19. Ratliff L., Rush D. Two notes on homogeneous prime ideals in graded Noetherian rings, *Journal of Algebra*, 2003, vol. 264, pp. 211–230. https://doi.org/10.1016/S0021-8693(03)00102-9
20. Rush D. Noetherian properties in monoid rings, *Journal of Pure and Applied Algebra*, 2003, vol. 185, pp. 259–278. https://doi.org/10.1016/S0022-4049(03)00103-8
21. Smith S. Category equivalences involving graded modules over path algebras of quivers, *Advances in Mathematics*, 2012, vol. 230, pp. 1780–1810. https://doi.org/10.1016/j.aim.2012.03.031
22. Tarrio L.A., Lopez A.J., Salorio M.J.S. Localization in categories of complexes and unbounded resolutions, *Canadian Journal of Mathematics*, 2000, vol. 52, pp. 225–247. https://doi.org/10.4153/CJM-2000-010-4
23. Yoshino Y., Yoshizawa T. Abstract local cohomology functors, *Mathematical Journal of Okayama University*, 2011, vol. 53, pp. 129–154.

Received 05.04.2019

Lu Li, Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, GSP-1, Leninskie Gory, Moscow, 119991, Russia.
E-mail: lu.li.bo1990@gmail.com

Л. Лу

Структурная теорема для gr-инъективных модулей над gr-нётеровыми G-градуированными коммутативными кольцами и локальные когомологические функторы

Цитата: *Известия Института математики и информатики Удмуртского государственного университета*. 2019. Т. 53. С. 127–137.

Ключевые слова: градуированные коммутативные кольца, gr-бассовы числа, локальные когомологические функторы, производные категории, радикальные функторы.

УДК: 512.7

DOI: 10.20537/2226-3594-2019-53-11

Хорошо известно, что разложение инъективных модулей над нётеровыми кольцами является одним из наиболее эстетичных и важных результатов в коммутативной алгебре. Наша цель — доказать аналогичные результаты для градуированных нётеровых колец. В этой статье мы изучаем структурную теорему для gr-инъективных модулей над gr-нётеровыми G-градуированными коммутативными кольцами, даем определение gr-бассовых чисел и изучаем их свойства. Мы покажем, что каждый gr-инъективный модуль имеет неразложимое разложение. Пусть R — gr-нётерово градуированное кольцо, а M — gr-конечно порожденный R-модуль. Мы дадим формулу для выражения чисел Басса с помощью функтора Ext. Мы определяем функтор сечения Γ_V с носителем в замкнутом по специализации подмножестве V из $Spec^{gr}(R)$ и абстрактный локальный когомологический функтор. В заключение мы покажем, что левый точный радикальный функтор F имеет вид Γ_V для замкнутого по специализации подмножества V.

Финансирование. Работа выполнена при поддержке Китайского стипендиального совета.

Поступила в редакцию 05.04.2019

Лу Ли, механико-математический факультет, Московский государственный университет им. М. В. Ломоносова, 119991, Россия, г. Москва, ГСП-1, Ленинские горы.
E-mail: lu.li.bo1990@gmail.com