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Matrix eigenvalue spectrum assignment for linear control systems by static output feedback

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ABSTRACT

For a linear time-invariant control system defined by a linear differential equation of the n-th order with a multidimensional state, input and output, we set and study the problem of arbitrary matrix eigenvalue spectrum assignment by linear static output feedback. We obtain controllability-like rank conditions that are necessary and sufficient for the problem of arbitrary matrix eigenvalue spectrum assignment and are sufficient for the problem of arbitrary eigenvalue spectrum assignment by linear static output feedback. It is proved that, in particular cases, when the system has block scalar matrix coefficients, these conditions can be relaxed. The obtained results generalize the known results for the corresponding problem by static state feedback and by static output feedback in the case of the one-dimensional equation. Examples are presented to illustrate the results.

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1. Introduction

The problem of eigenvalue spectrum assignment relates to the classical problems of mathematical control theory. Consider a linear control system

$$x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n x = b_1 u. \tag{1}$$

Here $x \in \mathbb{K}$ is a state, $u \in \mathbb{K}$ is a control, $a_i \in \mathbb{K}$, $i = \overline{1,n}$, $b_1 \in \mathbb{K}$ are constants, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. System (1) is called *arbitrary eigenvalue spectrum assignable (AESA) by linear static state feedback (LSSF)* if for any $\gamma_i \in \mathbb{K}$, $i = \overline{1,n}$, there exists a linear state feedback control

$$u = k_1 x^{(n-1)} + \ldots + k_n x \tag{2}$$

such that the closed-loop system (1), (2) has the form

$$x^{(n)} + \gamma_1 x^{(n-1)} + \ldots + \gamma_n x = 0. \tag{3}$$

It is clear that if $b_1 \neq 0$, then the feedback control (2), where $k_i = b_1^{-1}(a_i - \gamma_i)$, $i = \overline{1,n}$, reduces system (1) to the form (3), i.e., the problem is resolvable. In particular, one can ensure required asymptotics to solutions of (3) (e.g., stability or instability etc.). If $b_1 = 0$, then system (1) is not controllable, and the problem is not resolvable.

Consider a linear control system in the vector form

$$\dot{z} = Fz + Gv. \tag{4}$$

Here $z \in \mathbb{K}^r$ is a state, $v \in \mathbb{K}^q$ is a control, $F \in M_r(\mathbb{K})$, $G \in M_{r,q}(\mathbb{K})$, $M_{r,q}(\mathbb{K})$ is a space of $r \times q$-matrices with elements of $\mathbb{K}$, $M_r(\mathbb{K}) := M_{r,r}(\mathbb{K})$ (we will denote $M_{r,q} := M_{r,q}(\mathbb{K})$, $M_r := M_r(\mathbb{K})$). System (4) is called *AESA by LSSF* if for any $\delta_i \in \mathbb{K}$, $i = \overline{1,r}$, there exists a linear state feedback control

$$v = Lz \tag{5}$$

with $L \in M_{q,r}(\mathbb{K})$ such that the characteristic polynomial $\chi(F + GL; \lambda)$ of the matrix of the closed-loop system

$$\dot{z} = (F + GL)z$$

satisfies the condition

$$\chi(F + GL; \lambda) = \lambda^r + \delta_1 \lambda^{r-1} + \ldots + \delta_r.$$

It was proved (in [21] for $\mathbb{K} = \mathbb{C}$ and in [35] for $\mathbb{K} = \mathbb{R}$) that system (4) is AESA by LSSF (5) iff system (4) is completely controllable, i.e.,

$$\operatorname{rank}[G, FG, \ldots, F^{r-1}G] = r. \tag{6}$$

Consider the corresponding problem for the differential equation (1) when the state vector and the control vector are multidimensional. Let $s \in \mathbb{N}$ be given. Consider a linear control system

$$x^{(n)} + A_1 x^{(n-1)} + \ldots + A_n x = B_1 u. \tag{7}$$

Here $x \in \mathbb{K}^s$ is a state vector, $u \in \mathbb{K}^s$ is a control vector, $A_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, $B_1 \in M_s(\mathbb{K})$ are constant matrices.

Definition 1. We say that system (7) is *arbitrary matrix eigenvalue spectrum assignable (AMESA) by LSSF* if for any $\Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, there exists a linear static state feedback control

$$u = K_1 x^{(n-1)} + \ldots + K_n x, \tag{8}$$

where $K_i \in M_s(\mathbb{K})$, such that the closed-loop system has the form

$$x^{(n)} + \Gamma_1 x^{(n-1)} + \ldots + \Gamma_n x = 0. \tag{9}$$

The following proposition is evident.

Proposition 1. *System* (7) *is AMESA by LSSF iff* $\det B_1 \neq 0$.

In fact, if $\det B_1 \neq 0$, then the feedback control (8), where

$$K_i = B_1^{-1}(A_i - \Gamma_i), \quad i = \overline{1,n},$$

reduces system (7) to system (9). Vice versa, if $\det B_1 = 0$, then it is clear that not for any $\Gamma_i \in M_n(\mathbb{K})$, $i = \overline{1,n}$, there exists a feedback control (8) that reduces system (7) to system (9).

By using the standard replacement $z_1 = x$, $z_2 = x'$, $\ldots$, $z_n = x^{(n-1)}$, one can rewrite the control system (7), (8) in the form (4), (5) where $z = \operatorname{col}[z_1, \ldots, z_n] \in \mathbb{K}^{ns}$, $v = u \in \mathbb{K}^s$,

$$F = \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I \\ -A_n & -A_{n-1} & -A_{n-2} & \ldots & -A_1 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ B_1 \end{bmatrix}, \tag{10}$$
$$L = [K_n, K_{n-1}, \ldots, K_1], \qquad r = ns, \quad q = s.$$

Here $0 \in M_s$, $I \in M_s$ is the identity matrix. We say that system (4) with matrices (10) is *the big system corresponding to system* (7). System (7) is called *AESA by LSSF* if the corresponding big system (4), (10) is AESA by LSSF. If $\det B_1 \neq 0$, then it is easy to see that $\operatorname{rank}[G, FG, \dots, F^{n-1}G] = ns$. It follows that condition (6) is satisfied. Thus, the following proposition holds.

Proposition 2. *If* $\det B_1 \neq 0$ *then system* (7) *is AESA by LSSF* (8).

Remark 1. In contrast to Proposition 1, the condition $\det B_1 \neq 0$ in Proposition 2 is only sufficient but not necessary. The following example shows this. Let $n = 2$, $s = 2$,

$$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}. \tag{11}$$

One can check that, for system (4), (10), (11), condition (6) holds, i.e., system (4), (10), (11) (and, hence, system (7)) is AESA by LSSF but $\det B_1 = 0$. This example shows, in particular, that, for system (7), the properties of AMESA and AESA by LSSF are different.

Consider a control system defined by a linear differential equation of n-th order where the input is a linear combination of m variables and their derivatives of order $\leq n - p$ and the output is a k-dimensional vector of linear combinations of the state x and its derivatives of order $\leq p - 1$ $(1 \leq p \leq n)$:

$$\begin{aligned} x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n x = \\ = b_{p1} u_1^{(n-p)} + b_{p+1,1} u_1^{(n-p-1)} + \ldots + b_{n1} u_1 + \ldots \\ + b_{pm} u_m^{(n-p)} + b_{p+1,m} u_m^{(n-p-1)} + \ldots + b_{nm} u_m, \end{aligned} \tag{12}$$

$$\begin{aligned} y_1 &= c_{11} x + c_{21} x' + \ldots + c_{p1} x^{(p-1)}, \quad \ldots, \\ y_k &= c_{1k} x + c_{2k} x' + \ldots + c_{pk} x^{(p-1)}. \end{aligned} \tag{13}$$

Here $x \in \mathbb{K}$ is a state variable, $u_\alpha \in \mathbb{K}$ are control variables, $y_\beta \in \mathbb{K}$ are output variables, a_i, $b_{l\alpha}$, $c_{\nu\beta} \in \mathbb{K}$, $i = \overline{1, n}$, $l = \overline{p, n}$, $\nu = \overline{1, p}$, $\alpha = \overline{1, m}$, $\beta = \overline{1, k}$. Let us construct the vectors $u = \operatorname{col}(u_1, \dots, u_m) \in \mathbb{K}^m$, $y = \operatorname{col}(y_1, \dots, y_k) \in \mathbb{K}^k$.

Suppose that the control in system (12), (13) has the form of linear static output feedback (LSOF):

$$u = Qy. \tag{14}$$

Here $Q = \{q_{\alpha\beta}\} \in M_{m,k}(\mathbb{K})$, $q_{\alpha\beta} \in \mathbb{K}$, $\alpha = \overline{1, m}$, $\beta = \overline{1, k}$. System (12), (13) is called *AESA by LSOF* if for any $\gamma_i \in \mathbb{K}$, $i = \overline{1, n}$, there exists a linear static output feedback control (14) such that the closed-loop system (12), (13), (14) has the form (3). System (12), (13) with (14) is a generalization of system (1) with (2): in the case if $m = 1$, $p = n$,

$k = n$, and $\{c_{ij}\}_{i,j=1}^{n} = I \in M_n$, system (12), (13), (14) is coinciding with (1), (2). The conditions imposed on the orders of derivatives in (12) and (13) are natural because one needs the orders of the derivatives on the right-hand side of the closed-loop system to be less than n (see also [44]).

In the vector form, the problem of eigenvalue spectrum assignment by LSOF is as follows. For time-invariant plant described by

$$\dot{z} = Fz + Gv, \qquad \xi = Hz,$$

with $z \in \mathbb{K}^r$, $v \in \mathbb{K}^q$, $\xi \in \mathbb{K}^d$, $F \in M_r(\mathbb{K})$, $G \in M_{r,q}(\mathbb{K})$, $H \in M_{d,r}(\mathbb{K})$, one needs to construct a linear static output feedback control

$$v = L\xi$$

with $L \in M_{q,d}(\mathbb{K})$ ensuring for the characteristic polynomial $\chi(F + GLH; \lambda)$ of the matrix of the closed-loop system

$$\dot{z} = (F + GLH)z \tag{15}$$

the equality

$$\chi(F + GLH; \lambda) = \lambda^r + \delta_1 \lambda^{r-1} + \ldots + \delta_r \tag{16}$$

with an arbitrary pregiven $\delta_i \in \mathbb{K}$, $i = \overline{1, r}$.

The static output feedback problem of eigenvalue assignment is one of the most important open questions in control theory [24,3], see also reviews [29,25,28]. This problem has been studied for over 40 years by many authors. The most essential results at different times have been obtained by Davison and Wang [7] ($\mathbb{K} = \mathbb{R}$), Kimura [16] ($\mathbb{K} = \mathbb{R}$), Hermann and Martin [14] ($\mathbb{K} = \mathbb{C}$), Willems and Hesselink [34] ($\mathbb{K} = \mathbb{R}$), Brockett and Byrnes [4] ($\mathbb{K} = \mathbb{C}$), Wang [31,32] ($\mathbb{K} = \mathbb{R}$), Rosenthal, Schumacher, and Willems [23] ($\mathbb{K} = \mathbb{R}$). Some new results on eigenvalue assignment by static output feedback were obtained in works [40–43,33,2,17,20,22,19,11,30,1,5].

Although there is a huge amount of papers on static output feedback, however, as noted in [25], “so far, there has been no exact solution to this prominent problem which can guarantee the design of static output feedback or determine that such a feedback does not exist”.

For the scalar system (12), (13), (14), this problem has been solved in [39]. Let us construct the matrices $B = \{b_{l\alpha}\}$, $l = \overline{1, n}$, $\alpha = \overline{1, m}$, and $C = \{c_{\nu\beta}\}$, $\nu = \overline{1, n}$, $\beta = \overline{1, k}$, where $b_{l\alpha} := 0$ for $l < p$ and $c_{\nu\beta} := 0$ for $\nu > p$. Let $J := \{\vartheta_{ij}\} \in M_n(\mathbb{R})$ where $\vartheta_{ij} = 1$ for $j = i + 1$ and $\vartheta_{ij} = 0$ for $j \neq i + 1$. Let T denote the transposition of a matrix. The following theorem holds [39].

Theorem 1. *System (12), (13) is AESA by LSOF (14) iff the matrices*

$$C^T B, \quad C^T JB, \quad \dots, \quad C^T J^{n-1} B$$

are linearly independent.

Consider the corresponding problem for system (12), (13), (14) when the state, input and output variables are multidimensional. Suppose that numbers $s, n, m, k \in \mathbb{N}$, and $p \in \{\overline{1,n}\}$ are given. Consider the following system:

$$\begin{gathered} x^{(n)} + A_1 x^{(n-1)} + \ldots + A_n x = \\ = B_{p1} u_1^{(n-p)} + B_{p+1,1} u_1^{(n-p-1)} + \ldots + B_{n1} u_1 + \ldots \\ + B_{pm} u_m^{(n-p)} + B_{p+1,m} u_m^{(n-p-1)} + \ldots + B_{nm} u_m, \end{gathered} \tag{17}$$

$$\begin{gathered} y_1 = C_{11} x + C_{21} x' + \ldots + C_{p1} x^{(p-1)}, \quad \ldots, \\ y_k = C_{1k} x + C_{2k} x' + \ldots + C_{pk} x^{(p-1)}. \end{gathered} \tag{18}$$

Here $x \in \mathbb{K}^s$ is a state variable, $u_\alpha \in \mathbb{K}^s$ are control variables, $y_\beta \in \mathbb{K}^s$ are output variables, A_i, $B_{l\alpha}$, $C_{\nu\beta} \in M_s(\mathbb{K})$, $i = \overline{1,n}$, $l = \overline{p,n}$, $\nu = \overline{1,p}$, $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$. Let us construct the vectors $u = \operatorname{col}(u_1, \dots, u_m) \in \mathbb{K}^{ms}$, $y = \operatorname{col}(y_1, \dots, y_k) \in \mathbb{K}^{ks}$.

Suppose that the control in system (17), (18) has the form of linear static output feedback:

$$u = Qy. \tag{19}$$

Here $Q = \{Q_{\alpha\beta}\} \in M_{ms,ks}(\mathbb{K})$, $Q_{\alpha\beta} \in M_s(\mathbb{K})$, $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$. By (18), we have

$$y_\beta = \sum_{\nu=1}^{p} C_{\nu\beta} x^{(\nu-1)}, \quad \beta = \overline{1,k}.$$

Hence,

$$u_\alpha = \sum_{\beta=1}^{k} Q_{\alpha\beta} \Big(\sum_{\nu=1}^{p} C_{\nu\beta} x^{(\nu-1)} \Big), \quad \alpha = \overline{1,m}.$$

The closed-loop system (17), (18), (19) take the form

$$x^{(n)} + \sum_{i=1}^{n} A_i x^{(n-i)} - \sum_{\alpha=1}^{m} \sum_{l=p}^{n} B_{l\alpha} \left(\sum_{\beta=1}^{k} Q_{\alpha\beta} \Big(\sum_{\nu=1}^{p} C_{\nu\beta} x^{(n-l+\nu-1)} \Big) \right) = 0. \tag{20}$$

Definition 2. We say that system (17), (18) is *AMESA by LSOF* (19) if for any $\Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, there exists a gain matrix $Q \in M_{ms,ks}(\mathbb{K})$ such that the closed-loop system (20) has the form (9).

By using the replacement $z_1 = x$, $z_2 = x'$, ..., $z_n = x^{(n-1)}$, the closed-loop system (20) can be rewritten in the form of the big system (15) where $r = ns$, $q = ms$, $d = ks$.

Definition 3. System (17), (18) is called *AESA by LSOF* (19) if for any $\delta_i \in \mathbb{K}$, $i = \overline{1, ns}$, there exists a gain matrix $Q \in M_{ms,ks}(\mathbb{K})$ such that the characteristic polynomial of the matrix of the closed-loop big system (15) has the form (16).

For system (4), by reducing to the upper block Hessenberg form, AESA problem by state feedback was provided in [26]. For system (7), AESA problem by state feedback was studied in [27]. The eigenstructure assignment problem for system (7) was studied in [8]. For second order systems (7), AESA problems by LSSF were studied in [10,15,6], robust eigenvalue problems were studied in [13]. AESA problem by LSOF for system (17), (18), (19) with $n = 2$, $s = 2$, $m = 1$, $k = 1$, $p = 1$ was considered in [20]. AESA problem for system (17), (18) by LSOF (19) for any $n \in \mathbb{N}$ with $p = n$, $m = 1$, $k = n$ was studied in [37,38,36]. For system (17) with $m = p = 1$, the problem of a state-space realization in the descriptor system canonical form was studied in [9]. For system (17), (18), AESA problem by state feedback was studied in [44], and necessary and sufficient conditions for AESA have been obtained there in the terms of the so-called Sylvester mapping.

In this paper, we study the problem of matrix eigenvalue spectrum assignment for system (17), (18) by LSOF (19). In general case, the problem of arbitrary matrix eigenvalue spectrum assignment for system (17), (18) by static output feedback (19), to the best of our knowledge, has not been considered. The main aim of the work is to obtain necessary and sufficient conditions of AMESA and sufficient conditions of AESA for system (17), (18) by LSOF (19) similarly to Propositions 1 and 2, and to extend Theorem 1 for the case $s > 1$.

2. Notations, definitions, and preliminary statements

Here and throughout, we suppose that the numbers $s, n, m, k \in \mathbb{N}$, and $p \in \{\overline{1, n}\}$ are fixed. Let us give necessary denotations and definitions. For any matrix $H \in M_\omega$, we suppose, by definition, $H^0 = I \in M_\omega$. Denote by $\otimes$ the right Kronecker product of matrices $G = \{g_{ij}\} \in M_{\omega,\rho}$, $i = \overline{1, \omega}$, $j = \overline{1, \rho}$, and $H \in M_{\sigma,\tau}$ [18, Ch. 12] defined by the formula

$$G \otimes H := \begin{bmatrix} g_{11}H & g_{12}H & \dots & g_{1\rho}H \\ g_{21}H & g_{22}H & \dots & g_{2\rho}H \\ \vdots & \vdots & & \vdots \\ g_{\omega 1}H & g_{\omega 2}H & \dots & g_{\omega\rho}H \end{bmatrix} \in M_{\omega\sigma,\rho\tau}.$$

Denote $\mathcal{J} := J \otimes I \in M_{ns}$ where $I \in M_s$ and $J := \{\vartheta_{ij}\} \in M_n$, $\vartheta_{ij} = 1$ for $j = i + 1$ and $\vartheta_{ij} = 0$ for $j \neq i + 1$. We will use the mappings vecc, vecr that unroll a matrix

$H = \{h_{ij}\} \in M_{\omega,\rho}(\mathbb{K})$, $i = \overline{1,\omega}$, $j = \overline{1,\rho}$, column-by-column and row-by-row respectively into the column vector and the row vector respectively:

$$\operatorname{vecc} H = \operatorname{col}(h_{11}, \dots, h_{\omega 1}, \dots, h_{1\rho}, \dots, h_{\omega\rho}) \in M_{\omega\rho,1}(\mathbb{K}),$$
$$\operatorname{vecr} H = [h_{11}, \dots, h_{1\rho}, \dots, h_{\omega 1}, \dots, h_{\omega\rho}] \in M_{1,\omega\rho}(\mathbb{K}).$$

Lemma 1. *If $X \in M_{\omega,\rho}$, $Y \in M_{\rho,\sigma}$, $Z \in M_{\sigma,\tau}$, then*

$$\operatorname{vecc}(XYZ) = (Z^T \otimes X) \operatorname{vecc} Y.$$

The proof is given in [18, Sect. 12.1, Proposition 4] for square matrices X, Z. The proof remains the same for arbitrary rectangular matrices X, Z.

Definition 4. For the fixed $s \in \mathbb{N}$, let us introduce the operation of the block trace $\mathrm{SP}_s \colon M_{qs} \to M_s$ by the following rule: if $H = \{H_{ij}\} \in M_{qs}$, $H_{ij} \in M_s$, $i, j = \overline{1,q}$, then $\mathrm{SP}_s H = \sum\limits_{i=1}^{q} H_{ii}$.

Lemma 2. *Suppose that X and Y are block matrices with $s \times s$-blocks, there exist XY and YX, and the blocks of the matrix Y are scalar matrices, i.e.,*

$$X = \{X_{ij}\} \in M_{qs,rs}, \quad X_{ij} \in M_s, \quad i = \overline{1,q}, \quad j = \overline{1,r};$$
$$Y = \{Y_{ji}\} \in M_{rs,qs}, \quad Y_{ji} = y_{ji} I, \quad y_{ji} \in \mathbb{K}, \quad I \in M_s, \quad j = \overline{1,r}, \quad i = \overline{1,q}.$$

Then $\mathrm{SP}_s(XY) = \mathrm{SP}_s(YX)$.

Proof.

$$\mathrm{SP}_s(XY) = \sum_{i=1}^{q} \sum_{j=1}^{r} X_{ij} Y_{ji} = \sum_{i=1}^{q} \sum_{j=1}^{r} X_{ij} y_{ji} I = \sum_{i=1}^{q} \sum_{j=1}^{r} X_{ij} y_{ji}, \tag{21}$$

$$\mathrm{SP}_s(YX) = \sum_{j=1}^{r} \sum_{i=1}^{q} Y_{ji} X_{ij} = \sum_{j=1}^{r} \sum_{i=1}^{q} y_{ji} I X_{ij} = \sum_{j=1}^{r} \sum_{i=1}^{q} y_{ji} X_{ij}. \tag{22}$$

From (21) and (22) it follows the required. Q.E.D.

Lemma 3. *Let $D = \{D_{\omega\rho}\} \in M_{ns}$, $D_{\omega\rho} \in M_s$, $\omega, \rho = \overline{1,n}$. Then*

$$\mathrm{SP}_s(D\mathcal{J}^{i-1}) = \mathrm{SP}_s(\mathcal{J}^{i-1} D) = \sum_{\eta=1}^{n-i+1} D_{\eta+i-1,\eta}, \quad i = \overline{1,n}. \tag{23}$$

Proof. The equality $\mathrm{SP}_s(D\mathcal{J}^{i-1}) = \mathrm{SP}_s(\mathcal{J}^{i-1} D)$ follows from Lemma 2 because, for any $i = \overline{1,n}$, the blocks of the matrix $\mathcal{J}^{i-1}$ are scalar. Let us prove the second equality in (23).

For $i = 1$, we have $\mathcal{J}^{i-1} = I \in M_{ns}$. Therefore, (23) follows from the definition of $\mathrm{SP_s}$.

Next, we have $\mathcal{J} = \{G_{\rho\omega}\}_{\rho,\omega=1}^{n} \in M_{ns}$, where $G_{\rho\omega} = I \in M_s$ if $\omega = \rho+1$, $\rho = \overline{1, n-1}$, and $G_{\rho\omega} = 0 \in M_s$ otherwise. Let us calculate the degrees of the matrix $\mathcal{J}$. For any $j = \overline{1, n-1}$, we have:

$$\mathcal{J}^j = \{G^{(j)}_{\rho\omega}\}_{\rho,\omega=1}^{n} \in M_{ns},\ G^{(j)}_{\rho\omega} = \begin{cases} I \in M_s, \text{ if } \omega = \rho + j,\ \rho = \overline{1, n-j}, \\ 0 \in M_s, \text{ otherwise.} \end{cases} \tag{24}$$

Therefore,

$$\mathrm{SP}_s(\mathcal{J}^j D) = \sum_{\rho=1}^{n}\sum_{\omega=1}^{n} G^{(j)}_{\rho\omega} D_{\omega\rho} = \sum_{\rho=1}^{n-j} D_{\rho+j,\rho}.$$

Replacing j by $i-1$ and ρ by η in the last equality, we obtain (23). Q.E.D.

Definition 5. Suppose that X and Y are block matrices with $s \times s$-blocks such that the number of the (block) columns of X is equal to the number of the (block) rows of Y:

$$\begin{aligned} X &= \{X_{ij}\} \in M_{qs,rs}, \quad X_{ij} \in M_s, \quad i = \overline{1, q}, \quad j = \overline{1, r}; \\ Y &= \{Y_{j\nu}\} \in M_{rs,ts}, \quad Y_{j\nu} \in M_s, \quad j = \overline{1, r}, \quad \nu = \overline{1, t}. \end{aligned}$$

For the matrices X and Y, let us introduce the operation of the block multiplication by the following rule:

$$Z = X \star Y := \{Z_{i\nu}\}, \quad Z_{i\nu} := \sum_{j=1}^{r} X_{ij} \otimes Y_{j\nu}, \quad i = \overline{1, q}, \quad \nu = \overline{1, t}.$$

We have $Z_{i\nu} \in M_{s^2}$ for all $i = \overline{1, q}$, $\nu = \overline{1, t}$, therefore, $Z := X \star Y \in M_{qs^2, ts^2}$.

For convenience, so as not to write brackets, we assume

$$P \star RS := P \star (R \cdot S), \qquad PR \star S := (P \cdot R) \star S$$

where matrices P, R, S have the corresponding dimensions.

Lemma 4. *Let*

$$\begin{aligned} X &= \{X_{i\rho}\} \in M_{qs,ns}, \quad X_{i\rho} \in M_s, \quad i = \overline{1, q}, \quad \rho = \overline{1, n}; \\ Y &= \{Y_{\rho\nu}\} \in M_{ns,rs}, \quad Y_{\rho\nu} \in M_s, \quad \rho = \overline{1, n}, \quad \nu = \overline{1, r}. \end{aligned}$$

Then, for any $j = \overline{0, n-1}$,

$$X\mathcal{J}^j \star Y = X \star \mathcal{J}^j Y. \tag{25}$$

Proof. For $j = 0$, we have $\mathcal{J}^j = I \in M_{ns}$ and, therefore, (25) is true.

Let $j \in \{\overline{1, n-1}\}$. Denote

$$X\mathcal{J}^j =: V^{(j)} = \{V^{(j)}_{i\rho}\} \in M_{qs,ns}, \quad V^{(j)}_{i\rho} \in M_s, \quad i = \overline{1,q}, \quad \rho = \overline{1,n};$$
$$\mathcal{J}^j Y =: W^{(j)} = \{W^{(j)}_{\rho\nu}\} \in M_{ns,rs}, \quad W^{(j)}_{\rho\nu} \in M_s, \quad \rho = \overline{1,n}, \quad \nu = \overline{1,r}.$$

Then, by (24), we have

$$V^{(j)}_{i\rho} = \sum_{\eta=1}^{n} X_{i\eta} G^{(j)}_{\eta\rho} = \begin{cases} X_{i,\rho-j} \in M_s, & j+1 \le \rho \le n, \\ 0 \in M_s, & 1 \le \rho \le j. \end{cases} \tag{26}$$

$$W^{(j)}_{\rho\nu} = \sum_{\omega=1}^{n} G^{(j)}_{\rho\omega} Y_{\omega\nu} = \begin{cases} Y_{\rho+j,\nu} \in M_s, & 1 \le \rho \le n-j, \\ 0 \in M_s, & n-j+1 \le \rho \le n. \end{cases} \tag{27}$$

Thus, by Definition 5, taking into account (26) and (27), we have

$$(X\mathcal{J}^j \star Y)_{i\nu} = (V^{(j)} \star Y)_{i\nu} = \sum_{\rho=1}^{n} V^{(j)}_{i\rho} \otimes Y_{\rho\nu} = \sum_{\rho=j+1}^{n} X_{i,\rho-j} \otimes Y_{\rho\nu}, \tag{28}$$

$$(X \star \mathcal{J}^j Y)_{i\nu} = (X \star W^{(j)})_{i\nu} = \sum_{\eta=1}^{n} X_{i\eta} \otimes W^{(j)}_{\eta\nu} = \sum_{\eta=1}^{n-j} X_{i\eta} \otimes Y_{\eta+j,\nu}. \tag{29}$$

By replacing ρ by $\eta + j$ in (28), we obtain that (28) and (29) are coincident. Therefore, (25) is true. Q.E.D.

Definition 6. For the fixed $s \in \mathbb{N}$, let us introduce the operation of the block transposition $\mathcal{T}$ by the following rule: if $H = \{H_{ij}\} \in M_{qs,rs}$, $H_{ij} \in M_s$, $i = \overline{1,q}$, $j = \overline{1,r}$, then

$$H^{\mathcal{T}} := G = \{G_{ji}\} \in M_{rs,qs}, \quad G_{ji} := H_{ij}, \quad j = \overline{1,r}, \quad i = \overline{1,q}.$$

Lemma 5. *The following properties hold.*

1. $(H^{\mathcal{T}})^{\mathcal{T}} = H$.

2. If X and Y are block matrices with $s \times s$-blocks, there exists XY, and the blocks of the matrix Y are scalar matrices, i.e.,

$$X = \{X_{ij}\} \in M_{qs,rs}, \quad X_{ij} \in M_s, \quad i = \overline{1,q}, \quad j = \overline{1,r};$$
$$Y = \{Y_{j\sigma}\} \in M_{rs,ts}, \quad Y_{j\sigma} = y_{j\sigma} I, \quad y_{j\sigma} \in \mathbb{K}, \quad I \in M_s, \quad j = \overline{1,r}, \quad \sigma = \overline{1,t},$$

then

$$(XY)^{\mathcal{T}} = Y^{\mathcal{T}} X^{\mathcal{T}}. \tag{30}$$

These properties can be checked directly.

Definition 7. Let X be a block matrix with $s \times s$-blocks:

$$X = \{X_{ij}\} \in M_{qs,rs}, \quad X_{ij} \in M_s, \quad i = \overline{1,q}, \quad j = \overline{1,r}.$$

Let us construct the mappings VECCR_s, $\mathrm{VECRR}_s : M_{qs,rs} \to M_{s,qrs}$ that unroll the matrix $X = \{X_{ij}\} \in M_{qs,rs}$ by block columns and by block rows respectively into the block row with $s \times s$-blocks:

$$\mathrm{VECCR}_s\, X = [X_{11}, \ldots, X_{q1}, \ldots, X_{1r}, \ldots, X_{qr}],$$
$$\mathrm{VECRR}_s\, X = [X_{11}, \ldots, X_{1r}, \ldots, X_{q1}, \ldots, X_{qr}],$$

and the mappings VECRC_s, $\mathrm{VECCC}_s : M_{qs,rs} \to M_{qrs,s}$ that unroll the matrix $X = \{X_{ij}\} \in M_{qs,rs}$ by block rows and by block columns respectively into the block column with $s \times s$-blocks:

$$\mathrm{VECRC}_s\, X = \begin{bmatrix} X_{11} \\ \vdots \\ X_{1r} \\ \vdots \\ X_{q1} \\ \vdots \\ X_{qr} \end{bmatrix}, \quad \mathrm{VECCC}_s\, X = \begin{bmatrix} X_{11} \\ \vdots \\ X_{q1} \\ \vdots \\ X_{1r} \\ \vdots \\ X_{qr} \end{bmatrix}.$$

The following equalities are clear:

$$\mathrm{VECCR}_s\, X = \mathrm{VECRR}_s\,(X^{\mathcal{T}}), \tag{31}$$
$$(\mathrm{VECRC}_s\, X)^T = \mathrm{VECCR}_s\,(X^T). \tag{32}$$

Lemma 6. *If*

$$X = \{X_{ij}\} \in M_{qs,rs}, \quad X_{ij} \in M_s, \quad i = \overline{1,q}, \quad j = \overline{1,r},$$
$$Y = \{Y_{ji}\} \in M_{rs,qs}, \quad Y_{ji} \in M_s, \quad j = \overline{1,r}, \quad i = \overline{1,q},$$

then

$$\mathrm{SP}_s(XY) = \mathrm{VECRR}_s X \cdot \mathrm{VECCC}_s Y = \tag{33}$$
$$= \mathrm{VECCR}_s X \cdot \mathrm{VECRC}_s Y. \tag{34}$$

The proof follows directly from Definitions 4 and 7.

Lemma 7. *Let*

$$F = \{F_{l\alpha}\} \in M_{ns,ms}, \quad F_{l\alpha} \in M_s, \quad l = \overline{1,n}, \quad \alpha = \overline{1,m};$$
$$Q = \{Q_{\alpha\beta}\} \in M_{ms,ks}, \quad Q_{\alpha\beta} \in M_s, \quad \alpha = \overline{1,m}, \quad \beta = \overline{1,k};$$
$$H = \{H_{l\beta}\} \in M_{ns,ks}, \quad H_{l\beta} \in M_s, \quad l = \overline{1,n}, \quad \beta = \overline{1,k}.$$

Suppose that $R = \mathrm{SP}_s(FQH^{\mathcal{T}})$. *Then*

$$\mathrm{vecc}\, R = \mathrm{VECRR}_{s^2}(H^T \star F) \cdot \mathrm{vecc}\,(\mathrm{VECCR}_s\, Q). \tag{35}$$

Proof. Denote

$$FQ =: W = \{W_{l\beta}\} \in M_{ns,ks}, \quad W_{l\beta} \in M_s, \quad l = \overline{1,n}, \quad \beta = \overline{1,k}.$$

Then $W_{l\beta} = \sum\limits_{\alpha=1}^{m} F_{l\alpha} Q_{\alpha\beta}$. Therefore,

$$R = \mathrm{SP}_s(WH^{\mathcal{T}}) = \sum_{l=1}^{n}\sum_{\beta=1}^{k} W_{l\beta}H_{l\beta} = \sum_{l=1}^{n}\sum_{\beta=1}^{k}\sum_{\alpha=1}^{m} F_{l\alpha}Q_{\alpha\beta}H_{l\beta}. \tag{36}$$

By applying Lemma 1 to matrices $F_{l\alpha}Q_{\alpha\beta}H_{l\beta}$ for every $l = \overline{1,n}$, $\beta = \overline{1,k}$, $\alpha = \overline{1,m}$, we obtain from (36) that

$$\mathrm{vecc}\, R = \sum_{l=1}^{n}\sum_{\beta=1}^{k}\sum_{\alpha=1}^{m} \Big(\left(H_{l\beta}\right)^T \otimes F_{l\alpha} \Big) \cdot \mathrm{vecc}\, Q_{\alpha\beta}. \tag{37}$$

Denote $Z := H^T \star F$. Since $H^T \in M_{ks,ns}$, $F \in M_{ns,ms}$, hence, $Z \in M_{ks^2,ms^2}$. We have $Z = \{Z_{\beta\alpha}\}$, $Z_{\beta\alpha} \in M_{s^2}$, $\beta = \overline{1,k}$, $\alpha = \overline{1,m}$, and the matrix $Z_{\beta\alpha}$, by Definition 5, has the form

$$Z_{\beta\alpha} = \sum_{l=1}^{n} (H_{l\beta})^T \otimes F_{l\alpha}. \tag{38}$$

By definition,

$$\mathrm{VECRR}_{s^2} Z = [Z_{11}, \ldots, Z_{1m}, \ldots, Z_{k1}, \ldots, Z_{km}] \in M_{s^2,kms^2}. \tag{39}$$

Next,

$$\begin{gathered} \mathrm{vecc}(\mathrm{VECCR}_s Q) = \mathrm{vecc}\,[Q_{11}, \ldots, Q_{m1}, \ldots, Q_{1k}, \ldots, Q_{mk}] = \\ = \mathrm{col}\,(\mathrm{vecc}\, Q_{11}, \ldots, \mathrm{vecc}\, Q_{m1}, \ldots, \mathrm{vecc}\, Q_{1k}, \ldots, \mathrm{vecc}\, Q_{mk}) \in \mathbb{K}^{kms^2}. \end{gathered} \tag{40}$$

By multiplying (39) by (40) and taking into account (38), we obtain that the right-hand side of (35) has the form

$$\mathrm{VECRR}_{s^2}(H^T \star F) \cdot \mathrm{vecc}\,(\mathrm{VECCR}_s\, Q) =$$
$$= Z_{11} \cdot \mathrm{vecc}\, Q_{11} + \ldots + Z_{1m} \cdot \mathrm{vecc}\, Q_{m1} + \ldots + Z_{k1} \cdot \mathrm{vecc}\, Q_{1k} + \ldots + Z_{km} \cdot \mathrm{vecc}\, Q_{mk} =$$
$$= \sum_{\beta=1}^{k} \sum_{\alpha=1}^{m} Z_{\beta\alpha} \cdot \mathrm{vecc}\, Q_{\alpha\beta} = \sum_{\beta=1}^{k} \sum_{\alpha=1}^{m} \sum_{l=1}^{n} \Big((H_{l\beta})^T \otimes F_{l\alpha} \Big) \cdot \mathrm{vecc}\, Q_{\alpha\beta}. \tag{41}$$

From (37) and (41) it follows (35). Q.E.D.

3. Necessary and sufficient conditions for AMESA by LSOF

On the basis of system (17), (18), let us construct the block matrices $B \in M_{ns,ms}$, $C \in M_{ns,ks}$ (where $0 \in M_s$):

$$B = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ B_{p1} & \ldots & B_{pm} \\ \vdots & & \vdots \\ B_{n1} & \ldots & B_{nm} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & \ldots & C_{1k} \\ \vdots & & \vdots \\ C_{p1} & \ldots & C_{pk} \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{bmatrix}. \tag{42}$$

Theorem 2. *System (17), (18) is AMESA by LSOF (19) if and only if for any $\Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, there exists a matrix $Q \in M_{ms,ks}(\mathbb{K})$ such that the following equalities hold:*

$$\Gamma_i = A_i - \mathrm{SP}_s(\mathcal{J}^{i-1} B Q C^{\mathcal{T}}), \quad i = \overline{1,n}. \tag{43}$$

Proof. Let matrices $\Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, be given. One needs to construct a matrix $Q = \{Q_{\alpha\beta}\} \in M_{ms,ks}$, $Q_{\alpha\beta} \in M_s$, $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$, such that the closed-loop system (20) has the form (9).

System (20) can be written in the form

$$x^{(n)} + A_1 x^{(n-1)} + \ldots + A_n x - \Delta = 0, \tag{44}$$

where

$$\Delta = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{k} \sum_{l=p}^{n} \sum_{\nu=1}^{p} B_{l\alpha} Q_{\alpha\beta} C_{\nu\beta} x^{(n-l+\nu-1)}. \tag{45}$$

Let us replace the summation index ν by $i = l - \nu + 1$ in (45). Since ν ranges from 1 to p, hence, i ranges from $l - p + 1$ to l. So,

$$\Delta=\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}\sum_{l=p}^{n}\sum_{i=l-p+1}^{l}B_{l\alpha}Q_{\alpha\beta}C_{l+1-i,\beta}x^{(n-i)}.$$

If $i\in\{\overline{1,l-p}\}$, then $l+1-i\geq p+1$, hence, $C_{l+1-i,\beta}=0$. Thus,

$$\Delta=\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}\sum_{l=p}^{n}\sum_{i=1}^{l}B_{l\alpha}Q_{\alpha\beta}C_{l+1-i,\beta}x^{(n-i)}.$$

If $l\in\{\overline{1,p-1}\}$, then $B_{l\alpha}=0$, hence,

$$\Delta=\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}\sum_{l=1}^{n}\sum_{i=1}^{l}B_{l\alpha}Q_{\alpha\beta}C_{l+1-i,\beta}x^{(n-i)}.$$

Let us change the summation order: we replace $\sum\limits_{l=1}^{n}\sum\limits_{i=1}^{l}$ by $\sum\limits_{i=1}^{n}\sum\limits_{l=i}^{n}$; then we obtain

$$\Delta=\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}\sum_{i=1}^{n}\sum_{l=i}^{n}B_{l\alpha}Q_{\alpha\beta}C_{l+1-i,\beta}x^{(n-i)}. \tag{46}$$

Let us construct $D=BQC^{\mathcal{T}}$. Then $D=\{D_{\omega\rho}\}\in M_{ns}$, $D_{\omega\rho}\in M_s$, $\omega,\rho=\overline{1,n}$, and, by construction,

$$D_{\omega\rho}=\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}B_{\omega\alpha}Q_{\alpha\beta}C_{\rho\beta}. \tag{47}$$

By using Lemma 3, equality (47), and replacing the summation variable $l=\eta+i-1$, we obtain

$$\begin{aligned}\mathrm{SP}_s(\mathcal{J}^{i-1}D)&=\sum_{\eta=1}^{n-i+1}D_{\eta+i-1,\eta}=\sum_{\eta=1}^{n-i+1}\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}B_{\eta+i-1,\alpha}Q_{\alpha\beta}C_{\eta\beta}\\&=\sum_{l=i}^{n}\sum_{\alpha=1}^{m}\sum_{\beta=1}^{k}B_{l\alpha}Q_{\alpha\beta}C_{l+1-i,\beta}.\end{aligned} \tag{48}$$

It follows from (46) and (48) that

$$\Delta=\sum_{i=1}^{n}\mathrm{SP}_s(\mathcal{J}^{i-1}D)x^{(n-i)}. \tag{49}$$

Substituting (49) in (44), we obtain that the closed-loop system has the form

$$x^{(n)} + \sum_{i=1}^{n} \Big(A_i - \mathrm{SP}_s(\mathcal{J}^{i-1}BQC^{\mathcal{T}})\Big)x^{(n-i)} = 0. \tag{50}$$

System (50) is coinciding with (9) iff equalities (43) hold. Q.E.D.

Remark 2. By Lemma 3, equalities (43) are equivalent to

$$\Gamma_i = A_i - \mathrm{SP}_s(BQC^{\mathcal{T}}\mathcal{J}^{i-1}), \quad i = \overline{1,n}.$$

Equalities (43) represent a system of linear equations with respect to the coefficients of the matrix Q. Let us find conditions for solvability of this system.

Consider the matrices

$$C^T \star B, \quad C^T \star \mathcal{J}B, \quad \dots, \quad C^T \star \mathcal{J}^{n-1}B.$$

We have $C^T \in M_{ks,ns}$, $B \in M_{ns,ms}$, hence, $C^T \star \mathcal{J}^{i-1}B \in M_{ks^2,ms^2}$ for all $i = \overline{1,n}$. Let us construct the matrices $\mathrm{VECRR}_{s^2}(C^T \star \mathcal{J}^{i-1}B) \in M_{s^2,kms^2}$, $i = \overline{1,n}$, and the matrix

$$\Theta = \begin{bmatrix} \mathrm{VECRR}_{s^2}(C^T \star B) \\ \mathrm{VECRR}_{s^2}(C^T \star \mathcal{J}B) \\ \dots\dots\dots\dots\dots\dots\dots \\ \mathrm{VECRR}_{s^2}(C^T \star \mathcal{J}^{n-1}B) \end{bmatrix} \in M_{ns^2,kms^2}. \tag{51}$$

Theorem 3. *System (17), (18) is AMESA by LSOF (19) if and only if*

$$\operatorname{rank}\Theta = ns^2. \tag{52}$$

Proof. Let us apply the mapping vecc to equalities (43) and apply Lemma 7 to the matrices $F = \mathcal{J}^{i-1}B$, $H = C$. Then (43) takes the form

$$\mathrm{vecc}\,(A_i - \Gamma_i) = \mathrm{VECRR}_{s^2}(C^T \star \mathcal{J}^{i-1}B)\cdot \mathrm{vecc}\,(\mathrm{VECCR}_s Q),\ i = \overline{1,n}. \tag{53}$$

Denote

$$\begin{gathered} v := \mathrm{vecc}\,(\mathrm{VECCR}_s Q) \in \mathbb{K}^{kms^2}, \\ w := \mathrm{col}\,\big(\mathrm{vecc}\,(A_1 - \Gamma_1), \dots, \mathrm{vecc}\,(A_n - \Gamma_n)\big) \in \mathbb{K}^{ns^2}. \end{gathered} \tag{54}$$

Then systems (53) can be rewritten in the form

$$\Theta v = w. \tag{55}$$

The condition (52) is equivalent to solvability of system (55) with respect to v for any $\Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, and hence, to arbitrary matrix eigenvalue spectrum assignability for system (17), (18) by feedback (19). In particular, system (55) has the solution

$$v = \Theta^T(\Theta\Theta^T)^{-1}w. \tag{56}$$

The required matrix Q can be found from the equality

$$Q = \mathrm{VECCR}_s^{-1}(\mathrm{vecc}^{-1}v). \tag{57}$$

Theorem 3 is proved. Q.E.D.

Remark 3. By Lemma 4, the matrices $C^T \star \mathcal{J}^{i-1}B$ in (51) can be replaced by $C^T\mathcal{J}^{i-1}\star B$, $i=\overline{1,n}$.

Remark 4. Theorem 3 is an analogue of Proposition 1 for AMESA problem by LSOF.

Remark 5. Note that the condition $mk \geq n$ is necessary for (52).

4. Sufficient conditions for AESA by LSOF

Consider system (9). Denote $\Gamma = [\Gamma_1, \ldots, \Gamma_n] \in M_{s,ns}$. Construct the matrix characteristic polynomial for system (9):

$$\Upsilon(\Gamma;\lambda) = I\lambda^n + \Gamma_1\lambda^{n-1} + \ldots + \Gamma_n.$$

Here $\lambda \in \mathbb{K}$, $\Upsilon(\Gamma;\lambda) \in M_s(\mathbb{K})$. From system (9), construct the big system with the block companion matrix:

$$\dot z = \Phi z, \quad z \in \mathbb{K}^{ns},$$
$$\Phi = \Phi(\Gamma) = \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I \\ -\Gamma_n & -\Gamma_{n-1} & -\Gamma_{n-2} & \ldots & -\Gamma_1 \end{bmatrix}. \tag{58}$$

Denote by $\chi(\Phi;\lambda)$ the characteristic polynomial of the matrix of system (58), i.e., $\chi(\Phi;\lambda) = \det(\lambda I - \Phi)$. The following theorem holds [12, Theorem 1.1].

Theorem 4. $\det\Upsilon(\Gamma;\lambda) = \chi(\Phi;\lambda)$.

Let us prove the following theorem.

Theorem 5. *For any numbers $\delta_i \in \mathbb{K}$, $i=\overline{1,ns}$, there exist matrices $\Gamma_j \in M_s(\mathbb{K})$, $j=\overline{1,n}$, such that*

$$\chi(\Phi(\Gamma);\lambda) = \lambda^{ns} + \delta_1\lambda^{ns-1} + \delta_2\lambda^{ns-2} + \ldots + \delta_{ns}. \tag{59}$$

Proof. Suppose that $\delta_i \in \mathbb{K}$, $i = \overline{1, ns}$, are given. Let us construct the matrices $\Gamma_j \in M_s(\mathbb{K})$, $j = \overline{1, n}$, as follows: for $j = \overline{1, n-1}$, we set

$$\Gamma_j = \begin{bmatrix} \delta_j & 0 & \dots & 0 \\ \delta_{n+j} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \delta_{(s-1)n+j} & 0 & \dots & 0 \end{bmatrix};$$

for $j = n$, we set

$$\Gamma_n = \begin{bmatrix} \delta_n & -1 & 0 & \dots & 0 \\ \delta_{2n} & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_{(s-1)n} & 0 & 0 & \dots & -1 \\ \delta_{sn} & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then

$$\Upsilon(\Gamma; \lambda) = \begin{bmatrix} \lambda^n + \delta_1\lambda^{n-1} + \delta_2\lambda^{n-2} + \ldots + \delta_n & -1 & 0 & \dots & 0 \\ \delta_{n+1}\lambda^{n-1} + \delta_{n+2}\lambda^{n-2} + \ldots + \delta_{2n} & \lambda^n & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_{(s-2)n+1}\lambda^{n-1} + \delta_{(s-2)n+2}\lambda^{n-2} + \ldots + \delta_{(s-1)n} & 0 & 0 & \dots & -1 \\ \delta_{(s-1)n+1}\lambda^{n-1} + \delta_{(s-1)n+2}\lambda^{n-2} + \ldots + \delta_{sn} & 0 & 0 & \dots & \lambda^n \end{bmatrix}.$$

Calculating the determinant of $\Upsilon(\Gamma; \lambda)$, by using the expansion by the first column, we obtain

$$\det \Upsilon(\Gamma; \lambda) = (\lambda^n + \delta_1\lambda^{n-1} + \delta_2\lambda^{n-2} + \ldots + \delta_n)\lambda^{n(s-1)} + \\ + (\delta_{n+1}\lambda^{n-1} + \delta_{n+2}\lambda^{n-2} + \ldots + \delta_{2n})\lambda^{n(s-2)} + \ldots + (\delta_{(s-1)n+1}\lambda^{n-1} + \ldots + \delta_{sn}). \tag{60}$$

From (60) and Theorem 4, it follows (59). Q.E.D.

Theorem 6. *If system (17), (18) is AMESA by LSOF (19) then system (17), (18) is AESA by LSOF (19).*

Theorem 6 follows from Theorem 5.

Theorem 7. *If* $\operatorname{rank}\Theta = ns^2$ *then system (17), (18) is AESA by LSOF (19).*

Theorem 7 follows from Theorem 3 and Theorem 6.

Remark 6. Theorem 3 is an analogue of Proposition 2 for AESA problem by LSOF.

Remark 7. Condition $\operatorname{rank}\Theta = ns^2$ in Theorem 7 is only sufficient but not necessary (as in Proposition 2). Let us show it. Suppose that $s = 2$, $n = 3$, $m = k = p = 2$, $A_i = 0 \in M_2$, $i = 1, 2, 3$, $B_{21} = B_{32} = C_{11} = I \in M_2$, $B_{22} = B_{31} = C_{12} = C_{21} = 0 \in M_2$, $C_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Constructing the matrix (51), we obtain

$$\Theta = \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix} \in M_{12,16},$$

where $0 \in M_2$, $I \in M_2$. Hence, $\operatorname{rank}\Theta = 10 < 12 = ns^2$, i.e., condition (52) does not hold. Let us prove that this system is AESA by LSOF. Suppose that arbitrary $\delta_i \in \mathbb{K}$, $i = \overline{1, 6}$, are given. Let us construct the gain matrix

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in M_4, \quad Q_{11} = \begin{bmatrix} -\delta_2 & 0 \\ -\delta_5 & 0 \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} -\delta_1 & 0 \\ -\delta_4 & 0 \end{bmatrix},$$
$$Q_{21} = \begin{bmatrix} -\delta_3 & 1 \\ -\delta_6 & 0 \end{bmatrix}, \quad Q_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The closed-loop system (20) has the form

$$x''' + \Gamma_1 x'' + \Gamma_2 x' + \Gamma_3 x = 0,$$

where the matrices Γ_j (according to the proof of Theorem 2) have the form

$$\Gamma_1 = -\mathrm{SP}_s(BQC^{\mathcal{T}}), \quad \Gamma_2 = -\mathrm{SP}_s(\mathcal{J}BQC^{\mathcal{T}}), \quad \Gamma_3 = -\mathrm{SP}_s(\mathcal{J}^2BQC^{\mathcal{T}}). \tag{61}$$

Calculating (61), we obtain that

$$\Gamma_1 = \begin{bmatrix} \delta_1 & 0 \\ \delta_4 & 0 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} \delta_2 & 0 \\ \delta_5 & 0 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} \delta_3 & -1 \\ \delta_6 & 0 \end{bmatrix}.$$

Hence,

$$\Upsilon(\Gamma; \lambda) = \begin{bmatrix} \lambda^3 + \delta_1\lambda^2 + \delta_2\lambda + \delta_3 & -1 \\ \delta_4\lambda^2 + \delta_5\lambda + \delta_6 & \lambda^3 \end{bmatrix}.$$

Calculating $\det\Upsilon(\Gamma; \lambda)$ and using Theorem 4, we obtain that $\chi(\Phi(\Gamma); \lambda) = \lambda^6 + \sum_{i=1}^{6} \delta_i\lambda^{6-i}$, what is required.

5. Special cases

5.1. Blocks of C are scalar matrices

Suppose that the blocks of the matrix C are scalar matrices, i.e.,

$$\begin{gathered} C = \{C_{\nu\beta}\} \in M_{ns,ks}, \\ C_{\nu\beta} = c_{\nu\beta} I, \quad c_{\nu\beta} \in \mathbb{K}, \quad I \in M_s, \quad \nu = \overline{1,n}, \quad \beta = \overline{1,k}, \\ c_{\nu\beta} = 0, \quad \nu = \overline{p+1,n}, \quad \beta = \overline{1,k}. \end{gathered} \tag{62}$$

Then

$$C^{\mathcal{T}} = C^T. \tag{63}$$

By applying Lemma 2 to $X = \mathcal{J}^{i-1}BQ$, $Y = C^{\mathcal{T}}$, we obtain

$$\mathrm{SP}_s(\mathcal{J}^{i-1}BQC^{\mathcal{T}}) = \mathrm{SP}\,(C^{\mathcal{T}}\mathcal{J}^{i-1}BQ). \tag{64}$$

By (33),

$$\mathrm{SP}_s(C^{\mathcal{T}}\mathcal{J}^{i-1}BQ) = \mathrm{VECRR}_s(C^{\mathcal{T}}\mathcal{J}^{i-1}B) \cdot \mathrm{VECCC}_s Q. \tag{65}$$

Taking into account (63), (64), (65), one can rewrite system (43) with respect to coefficients of Q in the following form:

$$\Omega \cdot V = W. \tag{66}$$

Here

$$\Omega := \begin{bmatrix} \mathrm{VECRR}_s(C^T B) \\ \mathrm{VECRR}_s(C^T \mathcal{J} B) \\ \dots\dots\dots\dots\dots\dots \\ \mathrm{VECRR}_s(C^T \mathcal{J}^{n-1} B) \end{bmatrix} \in M_{ns,kms}, \tag{67}$$

$$\begin{gathered} W := \mathrm{col}\,(A_1 - \Gamma_1, \dots, A_n - \Gamma_n) \in M_{ns,s}(\mathbb{K}), \\ V := \mathrm{VECCC}_s\, Q \in M_{kms,s}. \end{gathered} \tag{68}$$

System (66) is solvable with respect to V for any $\Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, iff

$$\mathrm{rank}\,\Omega = ns. \tag{69}$$

In particular, system (66) has the solution

$$V = \Omega^T(\Omega\Omega^T)^{-1}W. \tag{70}$$

The required matrix Q can be found from the equality

$$Q = \mathrm{VECCC}_s^{-1} V. \tag{71}$$

Thus, the following theorem holds.

Theorem 8. *Suppose that the blocks of the matrix C are scalar matrices. Then system (17), (18) is AMESA by LSOF (19) if and only if (69) holds.*

Corollary 1. *Suppose that the blocks of the matrix C are scalar matrices. Then system (17), (18) is AESA by LSOF (19) if (69) holds.*

Corollary 1 follows from Theorem 8 and Theorem 6.

Remark 8. Note that the condition $mk \geq n$ is necessary for (69).

Remark 9. Condition (69) in Corollary 1 is only sufficient but not necessary. Let us show it. Suppose that $s = 2$, $n = 3$, $m = k = p = 2$, $A_i = 0 \in M_2$, $i = 1, 2, 3$, $B_{32} = C_{11} = C_{22} = I \in M_2$, $B_{22} = B_{31} = C_{12} = C_{21} = 0 \in M_2$, $B_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Constructing the matrix (67), we obtain

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \in M_{6,8}.$$

Hence, $\operatorname{rank} \Theta = 5 < 6 = ns$, i.e., condition (69) does not hold. Let us prove that this system is AESA by LSOF. Suppose that arbitrary $\delta_i \in \mathbb{K}$, $i = \overline{1,6}$, are given. Let us construct the gain matrix

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in M_4, \quad Q_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} 0 & 0 \\ -\delta_4 & -\delta_1 \end{bmatrix},$$
$$Q_{21} = \begin{bmatrix} 0 & 1 \\ -\delta_6 & -\delta_3 \end{bmatrix}, \quad Q_{22} = \begin{bmatrix} 0 & 0 \\ -\delta_5 & -\delta_2 \end{bmatrix}.$$

The closed-loop system (20) has the form

$$x''' + \Gamma_1 x'' + \Gamma_2 x' + \Gamma_3 x = 0,$$

where the matrices Γ_j (according to the proof of Theorem 2) have the form

$$\Gamma_1 = -\mathrm{SP}_s(BQC^{\mathcal{T}}), \quad \Gamma_2 = -\mathrm{SP}_s(\mathcal{J}BQC^{\mathcal{T}}), \quad \Gamma_3 = -\mathrm{SP}_s(\mathcal{J}^2 BQC^{\mathcal{T}}). \tag{72}$$

Calculating (72), we obtain that

$$\Gamma_1 = \begin{bmatrix} 0 & 0 \\ \delta_4 & \delta_1 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 & 0 \\ \delta_5 & \delta_2 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} 0 & -1 \\ \delta_6 & \delta_3 \end{bmatrix}.$$

Hence,

$$\Upsilon(\Gamma;\lambda) = \begin{bmatrix} \lambda^3 & -1 \\ \delta_4\lambda^2 + \delta_5\lambda + \delta_6 & \lambda^3 + \delta_1\lambda^2 + \delta_2\lambda + \delta_3 \end{bmatrix}.$$

Calculating $\det \Upsilon(\Gamma;\lambda)$ and using Theorem 4, we obtain that $\chi(\Phi(\Gamma);\lambda) = \lambda^6 + \sum_{i=1}^{6} \delta_i \lambda^{6-i}$, what is required.

Remark 10. Suppose that $m = 1$, $p = n$, $k = n$, and $C = I \in M_{ns}$. Hence, $B = \operatorname{col}(0,\dots,0,\widehat{B}) \in M_{ns,s}$ $(0 \in M_s)$, and system (17), (18), (19) is coinciding with (7), (9) where $B_1 = \widehat{B}$. Constructing (67), we obtain that

$$\Omega = \begin{bmatrix} 0 & 0 & \dots & 0 & \widehat{B} \\ 0 & 0 & \dots & \widehat{B} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \widehat{B} & \dots & 0 & 0 \\ \widehat{B} & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Condition (69) is equivalent to $\operatorname{rank} \widehat{B} = s$, i.e., $\det \widehat{B} \neq 0$. Thus, Theorem 8 coincides with Proposition 1, and Corollary 1 coincides with Proposition 2. So, Theorem 8 and Corollary 1 are generalization of Propositions 1 and 2 to systems with static output feedback.

5.2. Blocks of B are scalar matrices

Suppose that the blocks of the matrix B are scalar matrices, i.e.,

$$\begin{gathered} B = \{B_{l\alpha}\} \in M_{ns,ms}, \\ B_{l\alpha} = b_{l\alpha} I, \quad b_{l\alpha} \in \mathbb{K}, \quad I \in M_s, \quad l = \overline{1,n}, \quad \alpha = \overline{1,m}, \\ b_{l\alpha} = 0, \quad l = \overline{1,p-1}, \quad \alpha = \overline{1,m}. \end{gathered} \tag{73}$$

Then, for any $i = \overline{1,n}$, the blocks of the matrices $\mathcal{J}^{i-1}B$ are scalar also. Hence,

$$(\mathcal{J}^{i-1}B)^{\mathcal{T}} = (\mathcal{J}^{i-1}B)^{T}. \tag{74}$$

By applying Lemma 2 to $X = QC^{\mathcal{T}}$, $Y = \mathcal{J}^{i-1}B$, we obtain

$$\mathrm{SP}_s(\mathcal{J}^{i-1}BQC^{\mathcal{T}}) = \mathrm{SP}_s(QC^{\mathcal{T}}\mathcal{J}^{i-1}B). \tag{75}$$

By (34),

$$\mathrm{SP}_s(QC^{\mathcal{T}}\mathcal{J}^{i-1}B) = \mathrm{VECCR}_sQ \cdot \mathrm{VECRC}_s(C^{\mathcal{T}}\mathcal{J}^{i-1}B). \tag{76}$$

Taking into account (75), (76), one can rewrite system (43) with respect to coefficients of Q in the following form:

$$X \cdot \Xi = Y. \tag{77}$$

Here

$$\Xi := \left[\mathrm{VECRC}_s(C^{\mathcal{T}}B), \ldots, \mathrm{VECRC}_s(C^{\mathcal{T}}\mathcal{J}^{n-1}B)\right] \in M_{mks,ns}, \tag{78}$$

$$Y := [A_1 - \Gamma_1, \ldots, A_n - \Gamma_n] \in M_{s,ns}, \tag{79}$$

$$X := \mathrm{VECCR}_s\, Q \in M_{s,mks}.$$

System (77) is solvable with respect to X for any $\Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, iff $\operatorname{rank}\Xi = ns$. In particular, system (77) has the solution

$$X = Y(\Xi^T\Xi)^{-1}\Xi^T. \tag{80}$$

The required matrix Q can be found from the equality

$$Q = \mathrm{VECCR}_s^{-1}X. \tag{81}$$

Let us rewrite system (77) in the form

$$\Xi^T \cdot X^T = Y^T.$$

Consider the matrix

$$\Xi^T = \begin{bmatrix} \left[\mathrm{VECRC}_s(C^{\mathcal{T}}B)\right]^T \\ \left[\mathrm{VECRC}_s(C^{\mathcal{T}}\mathcal{J}B)\right]^T \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \left[\mathrm{VECRC}_s(C^{\mathcal{T}}\mathcal{J}^{n-1}B)\right]^T \end{bmatrix} \in M_{ns,mks}. \tag{82}$$

For any $i = \overline{1,n}$, by (32), we have

$$\left[\mathrm{VECRC}_s(C^{\mathcal{T}}\mathcal{J}^{i-1}B)\right]^T = \mathrm{VECCR}_s\big((C^{\mathcal{T}}\mathcal{J}^{i-1}B)^T\big). \tag{83}$$

By (31),

$$\mathrm{VECCR}_s\big((C^{\mathcal{T}}\mathcal{J}^{i-1}B)^T\big) = \mathrm{VECRR}_s\Big(\big((C^{\mathcal{T}}\mathcal{J}^{i-1}B)^T\big)^{\mathcal{T}}\Big). \tag{84}$$

By (74),

$$(C^{\mathcal{T}}\mathcal{J}^{i-1}B)^{T} = (\mathcal{J}^{i-1}B)^{T}(C^{\mathcal{T}})^{T} = (\mathcal{J}^{i-1}B)^{\mathcal{T}}(C^{T})^{\mathcal{T}}. \tag{85}$$

By (30),

$$(\mathcal{J}^{i-1}B)^{\mathcal{T}}(C^{T})^{\mathcal{T}} = (C^{T}\mathcal{J}^{i-1}B)^{\mathcal{T}}. \tag{86}$$

It follows from (85), (86), and assertion 1 of Lemma 5 that

$$\left((C^{\mathcal{T}}\mathcal{J}^{i-1}B)^{T}\right)^{\mathcal{T}} = \left((C^{T}\mathcal{J}^{i-1}B)^{\mathcal{T}}\right)^{\mathcal{T}} = C^{T}\mathcal{J}^{i-1}B. \tag{87}$$

It follows from (83), (84), and (87) that the matrix (82) has the form

$$\Xi^{T} = \begin{bmatrix} \mathrm{VECRR}_s(C^{T}B) \\ \mathrm{VECRR}_s(C^{T}\mathcal{J}B) \\ \dots\dots\dots\dots\dots\dots\dots \\ \mathrm{VECRR}_s(C^{T}\mathcal{J}^{n-1}B) \end{bmatrix} \in M_{ns,mks},$$

that is the matrix Ξ^{T} is coinciding with (67). Thus, the following theorem holds.

Theorem 9. *Suppose that the blocks of the matrix B are scalar matrices. Then system (17), (18) is AMESA by LSOF (19) if and only if (69) holds.*

Corollary 2. *Suppose that the blocks of the matrix B are scalar matrices. Then system (17), (18) is AESA by LSOF if (69) holds.*

Corollary 2 follows from Theorem 9 and Theorem 6.

Remark 11. Condition (69) in Corollary 2 is only sufficient but not necessary. Let us show it. Consider the example in Remark 7. In this example, the blocks of the matrix B are scalar matrices. Constructing the matrix (67), we obtain

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \in M_{6,8}.$$

Hence, $\operatorname{rank}\Omega = 5 < 6 = ns$, i.e., condition (69) does not hold. However, as it is shown in Remark 7, this system is AESA by LSOF.

5.3. Blocks of B and C are scalar matrices

Suppose that the blocks of the matrices B and C are scalar matrices, i.e., the matrices (42) have the form (62), (73). Denote

$$B_0 = \{b_{i\alpha}\} \in M_{n,m}, \quad C_0 = \{c_{i\beta}\} \in M_{n,k}, \quad i = \overline{1,n}, \quad \alpha = \overline{1,m}, \quad \beta = \overline{1,k}.$$

Then

$$B = B_0 \otimes I, \quad C = C_0 \otimes I, \quad I \in M_s. \tag{88}$$

Due to the properties of the Kronecker product, for all $i = \overline{1,n}$, we have

$$\mathrm{VECRR}_s(C^T \mathcal{J}^{i-1} B) = \mathrm{VECRR}_s\big((C_0^T \otimes I)(J^{i-1} \otimes I)(B_0 \otimes I)\big) =$$
$$\mathrm{VECRR}_s\big((C_0^T J^{i-1} B_0) \otimes I\big) = \Big(\mathrm{vecr}\,(C_0^T J^{i-1} B_0)\Big) \otimes I.$$

Thus, the matrix (67) has the form

$$\Omega = \begin{bmatrix} \Big(\mathrm{vecr}\,(C_0^T B_0)\Big) \otimes I \\ \Big(\mathrm{vecr}\,(C_0^T J B_0)\Big) \otimes I \\ \dots\dots\dots\dots\dots\dots\dots \\ \Big(\mathrm{vecr}\,(C_0^T J^{n-1} B_0)\Big) \otimes I \end{bmatrix} = P \otimes I,$$

where

$$P = \begin{bmatrix} \mathrm{vecr}\,(C_0^T B_0) \\ \mathrm{vecr}\,(C_0^T J B_0) \\ \dots\dots\dots\dots\dots \\ \mathrm{vecr}\,(C_0^T J^{n-1} B_0) \end{bmatrix} \in M_{n,km}.$$

Since $\mathrm{rank}\,(X \otimes Y) = \mathrm{rank}\, X \cdot \mathrm{rank}\, Y$, we obtain that $\mathrm{rank}\,\Omega = ns$ iff $\mathrm{rank}\, P = n$. This condition is equivalent to linear independence of matrices

$$C_0^T B_0, \quad C_0^T J B_0, \quad \dots, \quad C_0^T J^{n-1} B_0. \tag{89}$$

Thus, from Theorem 8, it follows the theorem.

Theorem 10. *Suppose that the blocks of the matrices B and C are scalar matrices, i.e., B and C have the form (88). Then system (17), (18) is AMESA by LSOF (19) if and only if the matrices (89) are linearly independent.*

Corollary 3. *Suppose that the blocks of the matrices B and C are scalar matrices, i.e., B and C have the form (88). Then system (17), (18) is AESA by LSOF (19) if the matrices (89) are linearly independent.*

Corollary 3 follows from Theorem 10 and Theorem 6.

Remark 12. The converse assertion to Corollary 3 is not true, in general case. We give the proof of this below in Example 1.

Example 1. Here we prove that the converse assertion to Corollary 3 is not true, in general case. Suppose that $\mathbb{K} = \mathbb{C}$, $s = 2$, $n = 4$, $m = 1$, $k = 4$, $p = 4$, $A_i = 0 \in M_2$, $i = \overline{1,4}$,

$$B_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have $C_0^T J B_0 = 0 \in M_{4,1}$, hence, matrices (89) are linearly dependent. Let us show that this system is AESA by LSOF. Suppose that arbitrary $\delta_i \in \mathbb{C}$, $i = \overline{1,8}$, are given. Let $Q = \{Q_{\alpha\beta}\} \in M_{ms,ks}$ have the form

$$Q = [Q_{11} \quad Q_{12} \quad Q_{13} \quad Q_{14}] \in M_{2,8}, \quad Q_{1\beta} = \begin{bmatrix} \rho_\beta & \eta_\beta \\ \xi_\beta & \omega_\beta \end{bmatrix}, \quad \beta = \overline{1,4}.$$

The closed-loop system (20) has the form

$$x'''' + \Gamma_1 x''' + \Gamma_2 x'' + \Gamma_3 x' + \Gamma_4 x = 0,$$

where the matrices Γ_i (according to the proof of Theorem 2) have the form (43). Calculating (43), we obtain

$$\Gamma_1 = -Q_{14}, \quad \Gamma_2 = 0 \in M_2, \quad \Gamma_3 = -Q_{12}, \quad \Gamma_4 = -Q_{11}.$$

Therefore, we have

$$\Upsilon(\Gamma;\lambda) = \begin{bmatrix} \lambda^4 - \rho_4\lambda^3 - \rho_2\lambda - \rho_1 & -\eta_4\lambda^3 - \eta_2\lambda - \eta_1 \\ -\xi_4\lambda^3 - \xi_2\lambda - \xi_1 & \lambda^4 - \omega_4\lambda^3 - \omega_2\lambda - \omega_1 \end{bmatrix}.$$

One needs to construct $\rho_i, \eta_i, \xi_i, \omega_i$, $i = 1, 2, 4$, such that

$$\det \Upsilon(\Gamma;\lambda) = \lambda^8 + \sum_{i=1}^{8} \delta_i \lambda^{n-i}. \tag{90}$$

Set $\xi_1 := 1$, $\xi_2 := 0$, $\xi_4 := 0$, $\rho_1 := 0$. Calculating $\det \Upsilon(\Gamma;\lambda)$, we obtain

$$\det \Upsilon(\Gamma;\lambda) = \lambda^8 - (\rho_4 + \omega_4)\lambda^7 + \rho_4\omega_4\lambda^6 - (\rho_2 + \omega_2)\lambda^5 +$$
$$+ (\rho_2\omega_4 + \rho_4\omega_2 - \omega_1)\lambda^4 + (\rho_4\omega_1 - \eta_4)\lambda^3 + \rho_2\omega_2\lambda^2 + (\rho_2\omega_1 - \eta_2)\lambda - \eta_1.$$

Equality (90) holds iff the following equalities are fulfilled:

$$\begin{aligned} -(\rho_4+\omega_4) &= \delta_1, & \rho_4\omega_1-\eta_4 &= \delta_5, \\ \rho_4\omega_4 &= \delta_2, & \rho_2\omega_2 &= \delta_6, \\ -(\rho_2+\omega_2) &= \delta_3, & \rho_2\omega_1-\eta_2 &= \delta_7, \\ \rho_2\omega_4+\rho_4\omega_2-\omega_1 &= \delta_4, & -\eta_1 &= \delta_8. \end{aligned} \tag{91}$$

The problem of AESA is resolvable if the system of nonlinear equations (91) is resolvable for arbitrary $\delta_1,\ldots,\delta_8$. It is clear that system (91) is resolvable. In fact, from the first and second equations of (91), we find ρ_4 and ω_4, namely,

$$\rho_4,\omega_4 = \left(-\delta_1 \pm \sqrt{\delta_1^2-4\delta_2}\right)\Big/2. \tag{92}$$

From the third and sixth equations of (91), we find ρ_2 and ω_2, namely,

$$\rho_2,\omega_2 = \left(-\delta_3 \pm \sqrt{\delta_3^2-4\delta_6}\right)\Big/2. \tag{93}$$

Substituting (92) and (93) into the fourth equation of (91), we find

$$\omega_1 = \rho_2\omega_4+\rho_4\omega_2-\delta_4. \tag{94}$$

Substituting (92), (93), and (94) into the fifth and seventh equation of (91), we find

$$\eta_4 = \rho_4\omega_1-\delta_5, \qquad \eta_2 = \rho_2\omega_1-\delta_7.$$

Finally, $\eta_1 = -\delta_8$. So, the system is AESA.

Remark 13. Similar examples can be constructed for any $n \geq 4$. This can be proved by induction. If $s \geq 3$ then the construction of the corresponding examples becomes very complicated. For $n \leq 3$ (for any s), such examples cannot be constructed, both for $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$. This can be proven by sequentially parsing cases. Thus, for $n \leq 3$, the converse assertion to Corollary 3 is true. We omit the rigorous proofs of this.

Remark 14. The presented example is not valid for the case $\mathbb{K} = \mathbb{R}$ because, due to (92) and (93), the coefficients of the matrices $Q_{\alpha\beta}$ are complex, in general case, even if $\delta_i \in \mathbb{R}$. Is it possible to construct similar examples for the case $\mathbb{K} = \mathbb{R}$ and $n \geq 4$? The answer to this question is open.

Remark 15. Suppose that $s = 1$. Then $B = B_0$, $C = C_0$, and the properties AMESA and AESA are coincident. In this case, the converse to Corollary 3 is true (by Theorem 10), and Theorem 10 is coinciding with Theorem 1. So, Theorem 10 and more general Theorems 9, 8, and 3 are generalizations of Theorem 1 for the case $s > 1$.

6. On properties of systems with AMESA

The property of AMESA (by LSSF or LSOF) is sufficient for AESA (Theorem 5) but is not necessary (see Remarks 1 and 7). The AMESA property is quite conservative and is far from the AESA property. Indeed, the AMESA property allows us to assign ns^2 coefficients of the matrices Γ_j, $j = \overline{1,n}$, and the AESA property allows us to assign ns coefficients of the characteristic polynomial (59). These properties are coinciding only if $s = 1$ in general.

The condition $mk \geq n$ is necessary for AMESA by LSOF (Remark 5). This is clear because if $mk < n$, then the number mks^2 of coefficients of the gain matrix Q in (19) is less than the number ns^2 of coefficients of matrices Γ_j, $j = \overline{1,n}$, that we have to assign, and, in this case, the arbitrary assignability is impossible. This can be compared with the result of Wang [31] for the problem of AESA by LSOF for system (15) but there is no direct relationship between the results presented here and Wang's results.

The AMESA property is much stronger than the AESA property. In particular, it allows us to assign not only eigenvalues for the closed-loop system but also eigenvectors with a high degree of freedom. The problem of the simultaneous assignment of the eigenvalue spectrum together with the eigenvectors is one of the important problems of the theory of eigenvalue placement. As an example, we present the following property that systems with AMESA have.

Theorem 11. *For any different $\lambda_\xi \in \mathbb{R}$, $\xi = \overline{1,ns}$, and for any linear independent vectors $h_1, \ldots, h_s \in \mathbb{R}^s$ there exist matrices $\Gamma_j \in M_s(\mathbb{R})$, $j = \overline{1,n}$, such that the general solution of system (9) has the form*

$$\begin{aligned} x = {} & C_1 h_1 \exp(\lambda_1 t) + C_2 h_2 \exp(\lambda_2 t) + \ldots + C_s h_s \exp(\lambda_s t) \\ & + C_{s+1} h_1 \exp(\lambda_{s+1} t) + \ldots + C_{2s} h_s \exp(\lambda_{2s} t) + \ldots \\ & + C_{(n-1)s+1} h_1 \exp(\lambda_{(n-1)s+1} t) + \ldots + C_{ns} h_s \exp(\lambda_{ns} t). \end{aligned} \tag{95}$$

Proof. Let different $\lambda_\xi \in \mathbb{R}$, $\xi = \overline{1,ns}$, and linear independent vectors $h_1, \ldots, h_s \in \mathbb{R}^s$ be given. Construct $S := [h_1, \ldots, h_s] \in M_s(\mathbb{R})$. Then $\det S \neq 0$. Set

$$\begin{gathered} N_1 := \operatorname{diag}\{\lambda_1, \ldots, \lambda_s\}, \quad N_2 := \operatorname{diag}\{\lambda_{s+1}, \ldots, \lambda_{2s}\}, \quad \ldots, \\ N_n := \operatorname{diag}\{\lambda_{(n-1)s+1}, \ldots, \lambda_{ns}\}. \end{gathered} \tag{96}$$

Then the matrices N_j, $j = \overline{1,n}$, are commuting. Construct

$$L_1 := SN_1S^{-1}, \quad \ldots, \quad L_n := SN_nS^{-1}. \tag{97}$$

Then L_j, $j = \overline{1,n}$, are commuting as well. Let

$$(\lambda I - L_1)(\lambda I - L_2)\cdots(\lambda I - L_n) =: I\lambda^n + \Gamma_1\lambda^{n-1} + \ldots + \Gamma_n, \tag{98}$$

i.e., $\Gamma_1 := -(L_1+\ldots+L_n)$, $\ldots$, $\Gamma_n := (-1)^n L_1 \cdot \ldots \cdot L_n$. Then the matrices $\Gamma_j \in M_s(\mathbb{R})$, $j = \overline{1,n}$, are required.

In fact, let us show that the vector-function $x_{j,i}(t) = h_i \exp(\lambda_{(j-1)s+i}t)$ is a solution of system (9) for every $j = \overline{1,n}$ and $i = \overline{1,s}$. It follows from (97) that $L_jS = SN_j$, $j = \overline{1,n}$. Then, it follows from (96) that

$$L_jh_i = \lambda_{(j-1)s+i}h_i \quad \forall j = \overline{1,n} \quad \forall i = \overline{1,s}. \tag{99}$$

It follows from (98) that

$$x^{(n)} + \Gamma_1x^{(n-1)} + \ldots + \Gamma_nx = \left(\frac{d}{dt} - L_1\right)\left(\frac{d}{dt} - L_2\right)\ldots\left(\frac{d}{dt} - L_n\right)x. \tag{100}$$

Moreover, the operators in the right-hand side of (100) are commuting. Let the operator $\left(\frac{d}{dt} - L_j\right)$ in the right-hand side of (100) be repositioned to the most right position. We have

$$\begin{aligned}\left(\frac{d}{dt} - L_j\right)x_{j,i}(t) &= \left(\frac{d}{dt} - L_j\right)h_i\exp(\lambda_{(j-1)s+i}t)\\ &= (\lambda_{(j-1)s+i}h_i - L_jh_i)\exp(\lambda_{(j-1)s+i}t) = 0\end{aligned}$$

due to (99). Thus, by (100), $x_{j,i}(t)$ is a solution of (9) for every $j = \overline{1,n}$ and $i = \overline{1,s}$.

Since all λ_ξ, $\xi = \overline{1,ns}$, are different, the vector-functions $x_{j,i}(t)$, $j = \overline{1,n}$, $i = \overline{1,s}$, are linearly independent. Therefore, formula (95) gives a general solution of system (9). Q.E.D.

Remark 16. The condition of Theorem 11 that all λ_ξ are different can be weakened to the condition that all the vector-functions $x_{j,i}(t)$ are linearly independent.

Remark 17. One can derive other properties for a closed-loop system from the AMESA property. The AMESA property is poorly studied. The study of this property can be the subject of further research.

7. Examples

Example 2. Let $\mathbb{K} = \mathbb{R}$, $n = 3$, $m = k = p = s = 2$, and the matrices of system (17), (18) have the form

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{101}$$

$$B_{21} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B_{22} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, B_{31} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B_{32} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{102}$$

$$C_{11} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, C_{12} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, C_{21} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, C_{22} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}. \tag{103}$$

Let us construct matrices (42):

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Construct

$$C^T \star B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$C^T \star \mathcal{J} B = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & -2 \\ 1 & 1 & 0 & -1 & -1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -1 & -1 & 1 & 1 & 0 \end{bmatrix},$$

$$C^T \star \mathcal{J}^2 B = \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

Construct (51):

$$\Theta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 2 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & -1 & 0 & 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & -2 & 0 & 1 & 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

We have $\operatorname{rank}\Theta = 12$, hence, by Theorem 3 system (17), (18) is AMESA by LSOF (19). Let us construct this feedback control. Suppose, for example, that

$$\Gamma_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}. \tag{104}$$

Constructing w by formula (54), we obtain

$$w = \operatorname{col}(-4, 0, 0, 0, 0, 1, 0, 2, -2, 0, 0, -4).$$

Calculating v by formula (56), we obtain

$$v = \operatorname{col}(-3, 1, -3, -1, -1, -1, 1, -1, 0, 0, 1, 1, 0, -4, -1, -5).$$

From (57), we obtain

$$Q = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & -4 & -5 \end{bmatrix}. \tag{105}$$

Feedback control (19) with (105) reduces the system (17), (18) with (101), (102), (103) to the system

$$x''' + \Gamma_1 x'' + \Gamma_2 x' + \Gamma_3 x = 0$$

with (104).

Example 3. Let $\mathbb{K} = \mathbb{R}$, $n = 3$, $m = k = p = s = 2$, and the matrices of system (17), (18) have the form

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{106}$$

$$B_{21} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B_{22} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, B_{31} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B_{32} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{107}$$

$$C_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C_{21} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C_{22} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}. \tag{108}$$

Let us construct matrices (42):

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Blocks of C are scalar matrices. Construct

$$C^T B = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ -2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 2 \end{bmatrix}, \quad C^T \mathcal{J} B = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -2 \\ 1 & -2 & 2 & 1 \\ -2 & -1 & 1 & -3 \end{bmatrix},$$

$$C^T \mathcal{J}^2 B = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Construct (67):

$$\Omega = \begin{bmatrix} -1 & 0 & 0 & -1 & -2 & 0 & 0 & -2 \\ 0 & 1 & -1 & 1 & 0 & 2 & -2 & 2 \\ 1 & -1 & 1 & 1 & 1 & -2 & 2 & 1 \\ -1 & -1 & 1 & -2 & -2 & -1 & 1 & -3 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

We have $\operatorname{rank} \Omega = 6$, hence, by Theorem 8, system (17), (18) is AMESA by LSOF (19). Let us construct this feedback control. Suppose, for example, that

$$\Gamma_1 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}. \tag{109}$$

Constructing W by formula (68), we obtain

$$W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -2 \\ 0 & -2 \end{bmatrix}.$$

Calculating V by formula (70), we obtain

$$V = \begin{bmatrix} -1 & -3 \\ 0 & -1 \\ 0 & 1 \\ 1 & -2 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}.$$

From (71), we get

$$Q = \begin{bmatrix} -1 & -3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & 2 \end{bmatrix}. \tag{110}$$

Feedback control (19) with (110) reduces the system (17), (18) with (106), (107), (108) to the system

$$x''' + \Gamma_1 x'' + \Gamma_2 x' + \Gamma_3 x = 0$$

with (109).

Example 4. Let $\mathbb{K} = \mathbb{R}$, $n = 3$, $m = k = p = s = 2$, and the matrices of system (17), (18) have the form

$$A_1 = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{111}$$

$$B_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_{22} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, B_{31} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_{32} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{112}$$

$$C_{11} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, C_{12} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, C_{21} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, C_{22} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{113}$$

Let us construct matrices (42):

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Blocks of B are scalar matrices. Construct

$$C^{\mathcal{T}}B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C^{\mathcal{T}}\mathcal{J}B = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & -1 & 0 & -1 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C^{\mathcal{T}}\mathcal{J}^2B = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Construct (78):

$$\Xi = \begin{bmatrix} 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have $\operatorname{rank}\Omega = \operatorname{rank}\Xi^T = 6$, hence, by Theorem 9, system (17), (18) is AMESA by LSOF (19). Let us construct this feedback control. Suppose, for example, that

$$\Gamma_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}. \tag{114}$$

Constructing Y by formula (79) we obtain

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -2 \\ 2 & 0 & 0 & -2 & 0 & -2 \end{bmatrix}.$$

Calculating X by formula (80), we obtain

$$X = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 3 & 0 \end{bmatrix}.$$

From (81), we get

$$Q = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix}. \tag{115}$$

Feedback control (19) with (115) reduces the system (17), (18) with (111), (112), (113) to the system

$$x''' + \Gamma_1 x'' + \Gamma_2 x' + \Gamma_3 x = 0$$

with (114).

8. Conclusion

In this paper we have studied the problem of arbitrary matrix eigenvalue spectrum assignment by linear static output feedback for a control system defined by a linear time-invariant differential equation of the n-th order with the s-dimensional state, input and output. We have obtained necessary and sufficient conditions for AMESA by LSOF. These conditions are expressed in terms of system coefficients and do not depend on coefficients A_i of the free system. Particular cases have been studied when the system has block scalar matrix coefficients. The results extend the known results on AESA by LSOF obtained for $s = 1$ and on AMESA by LSSF. It is proved that AMESA by LSOF implies AESA by LSOF. As corollaries, sufficient conditions for AESA by LSOF have been obtained. It has been shown that these sufficient conditions are not necessary, in general case, if $s > 1$. Illustrative examples are presented.

A further development of these results may be their extension to block systems of more general form and more detailed study of the AMESA property.

Declaration of competing interest

There is no competing interest.

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References

[1] O. Bachelier, J. Bosche, D. Mehdi, On pole placement via eigenstructure assignment approach, IEEE Trans. Autom. Control 51 (2006) 1554–1558, https://doi.org/10.1109/tac.2006.880809.

[2] V. Belozyorov, New solution method of linear static output feedback design problem for linear control systems, Linear Algebra Appl. 504 (2016) 204–227, https://doi.org/10.1016/j.laa.2016.04.001.

[3] R. Brockett, A stabilization problem, in: Open Problems in Mathematical Systems and Control Theory, Springer, London, 1999, pp. 75–78.

[4] R. Brockett, C. Byrnes, Multivariable Nyquist criteria, root loci, and pole placement: a geometric viewpoint, IEEE Trans. Autom. Control 26 (1981) 271–284, https://doi.org/10.1109/tac.1981.1102571.

[5] C. Champetier, J.F. Magni, On eigenstructure assignment by gain output feedback, SIAM J. Control Optim. 29 (1991) 848–865, https://doi.org/10.1137/0329046.

[6] E. Chu, Pole assignment for second-order systems, Mech. Syst. Signal Process. 16 (2002) 39–59, https://doi.org/10.1006/mssp.2001.1439.

[7] E. Davison, S. Wang, On pole assignment in linear multivariable systems using output feedback, IEEE Trans. Autom. Control 20 (1975) 516–518, https://doi.org/10.1109/tac.1975.1101023.

[8] G.R. Duan, Parametric approaches for eigenstructure assignment in high-order linear systems, Int. J. Control. Autom. Syst. 3 (2005) 419–429.

[9] G.R. Duan, Y.J. Gao, State-space realization and generalized Popov Belevitch Hautus criterion for high-order linear systems—the singular case, Int. J. Control. Autom. Syst. 18 (2020) 2038–2047, https://doi.org/10.1007/s12555-019-0212-4.

[10] G.R. Duan, B. Zhou, Solution to the second-order Sylvester matrix equation $MVF^2+DVF+KV=BW$, IEEE Trans. Autom. Control 51 (2006) 805–809, https://doi.org/10.1109/tac.2006.874989.

[11] M. Fu, Pole placement via static output feedback is NP-hard, IEEE Trans. Autom. Control 49 (2004) 855–857, https://doi.org/10.1109/tac.2004.828311.

[12] I. Gohberg, P. Lancaster, L. Rodman, Matrix Polynomials, Society for Industrial and Applied Mathematics, 2009.

[13] D. Henrion, D. Henrion, M. Šebek, V. Kučera, Robust pole placement for second-order systems: an LMI approach, IFAC Proc. Vol. 36 (2003) 419–424, https://doi.org/10.1016/s1474-6670(17)35700-2.

[14] R. Hermann, C. Martin, Applications of algebraic geometry to systems theory–part I, IEEE Trans. Autom. Control 22 (1977) 19–25, https://doi.org/10.1109/tac.1977.1101395.

[15] Y. Kim, H.S. Kim, J.L. Junkins, Eigenstructure assignment algorithm for mechanical second-order systems, J. Guid. Control Dyn. 22 (1999) 729–731, https://doi.org/10.2514/2.4444.

[16] H. Kimura, Pole assignment by gain output feedback, IEEE Trans. Autom. Control 20 (1975) 509–516, https://doi.org/10.1109/tac.1975.1101028.

[17] K.H. Kiritsis, Arbitrary pole placement by constant output feedback for linear time invariant systems, Asian J. Control 19 (2016) 832–839, https://doi.org/10.1002/asjc.1439.

[18] P. Lancaster, M. Tismenetsky, The Theory of Matrices, second edition with applications, Academic Press, 1985.

[19] H. Narayanan, H. Narayanan, On the linear static output feedback problem: the annihilating polynomial approach, Linear Algebra Appl. 579 (2019) 336–364, https://doi.org/10.1016/j.laa.2019.06.005.

[20] E.A. Perepelkin, Pole assignment problem for a second-order system, Differ. Equ. 53 (2017) 1524–1527, https://doi.org/10.1134/s0012266117110167.

[21] V. Popov, Hyperstability and optimality of automatic systems with several control functions, Rev. Roum. Sci. Tech., Sér. Électrotech. Énerg. 9 (1964) 629–690.

[22] K. Robenack, R. Vobwinkell, M. Franke, On the eigenvalue placement by static output feedback via quantifier elimination, in: 2018 26th Mediterranean Conference on Control and Automation (MED), IEEE, 2018, pp. 133–138.

[23] J. Rosenthal, J. Schumacher, J. Willems, Generic eigenvalue assignment by memoryless real output feedback, Syst. Control Lett. 26 (1995) 253–260, https://doi.org/10.1016/0167-6911(95)00019-6.

[24] J. Rosenthal, J. Willems, Open problems in the area of pole placement, in: Open Problems in Mathematical Systems and Control Theory, Springer, London, 1999, pp. 181–191.

[25] M. Sadabadi, D. Peaucelle, From static output feedback to structured robust static output feedback: a survey, Annu. Rev. Control 42 (2016) 11–26, https://doi.org/10.1016/j.arcontrol.2016.09.014.

[26] B. Shafai, S. Bhattacharyya, An algorithm for pole assignment in high order multivariable systems, IEEE Trans. Autom. Control 33 (1988) 870–876, https://doi.org/10.1109/9.1320.

[27] B. Shafai, L. Keel, New pole placement algorithm: polynomial matrix approach, in: 1990 American Control Conference, IEEE, 1990, pp. 1684–1686.

[28] M. Shumafov, Stabilization of linear control systems and pole assignment problem: a survey, Vestn. St. Petersbg. Univ., Math. 52 (2019) 349–367, https://doi.org/10.1134/S1063454119040095.

[29] V. Syrmos, C. Abdallah, P. Dorato, K. Grigoriadis, Static output feedback—a survey, Automatica 33 (1997) 125–137, https://doi.org/10.1016/s0005-1098(96)00141-0.

[30] V. Syrmos, F. Lewis, Output feedback eigenstructure assignment using two Sylvester equations, IEEE Trans. Autom. Control 38 (1993) 495–499, https://doi.org/10.1109/9.210155.

[31] X. Wang, Pole placement by static output feedback, J. Math. Syst. Estim. Control 2 (1992) 205–218.

[32] X. Wang, Grassmannian, central projection, and output feedback pole assignment of linear systems, IEEE Trans. Autom. Control 41 (1996) 786–794, https://doi.org/10.1109/9.506231.

[33] X. Wang, U. Konigorski, On linear solutions of the output feedback pole assignment problem, IEEE Trans. Autom. Control 58 (2013) 2354–2359, https://doi.org/10.1109/tac.2013.2250077.

[34] J. Willems, W. Hesselink, Generic properties of the pole placement problem, IFAC Proc. Vol. 11 (1978) 1725–1729, https://doi.org/10.1016/s1474-6670(17)66142-1.

[35] W. Wonham, On pole assignment in multi-input controllable linear systems, IEEE Trans. Autom. Control 12 (1967) 660–665, https://doi.org/10.1109/tac.1967.1098739.

[36] H. Yu, D. Bi, Parametric approaches for eigenstructure assignment in high-order linear systems via output feedback, in: 2012 24th Chinese Control and Decision Conference (CCDC), IEEE, 2012, pp. 459–464.

[37] H.H. Yu, G.R. Duan, ESA in high-order linear systems via output feedback, Asian J. Control 11 (2009) 336–343, https://doi.org/10.1002/asjc.111.
[38] H.H. Yu, G.R. Duan, ESA in high-order descriptor linear systems via output feedback, Int. J. Control. Autom. Syst. 8 (2010) 408–417, https://doi.org/10.1007/s12555-010-0228-2.
[39] V. Zaitsev, Modal control of a linear differential equation with incomplete feedback, Differ. Equ. 39 (2003) 145–148, https://doi.org/10.1023/A:1025188512610.
[40] V.A. Zaitsev, Spectrum control in linear systems with incomplete feedback, Differ. Equ. 45 (2009) 1348–1357, https://doi.org/10.1134/s0012266109090109.
[41] V.A. Zaitsev, Necessary and sufficient conditions in a spectrum control problem, Differ. Equ. 46 (2010) 1789–1793, https://doi.org/10.1134/s0012266110120128.
[42] V.A. Zaitsev, Consistent systems and pole assignment: I, Differ. Equ. 48 (2012) 120–135, https://doi.org/10.1134/s001226611110120.
[43] V.A. Zaitsev, Consistent systems and pole assignment: II, Differ. Equ. 48 (2012) 857–866, https://doi.org/10.1134/s0012266112060092.
[44] B. Zhou, G.R. Duan, Pole assignment of high-order linear systems with high-order time-derivatives in the input, J. Franklin Inst. 357 (2020) 1437–1456, https://doi.org/10.1016/j.jfranklin.2019.10.030.