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MIXED TYPE INTEGRO-DIFFERENTIAL EQUATION WITH FRACTIONAL ORDER CAPUTO OPERATORS AND SPECTRAL PARAMETERS

The issues of unique solvability of a boundary value problem for a mixed type integro-differential equation with two Caputo time-fractional operators and spectral parameters are considered. A mixed type integro-differential equation is a partial integro-differential equation of fractional order in both positive and negative parts of multidimensional rectangular domain under consideration. The fractional Caputo operator's order is less in the positive part of the domain, than the order of Caputo operator in the negative part of the domain. Using the method of Fourier series, two systems of countable systems of ordinary fractional integro-differential equations with degenerate kernels are obtained. Further, a method of degenerate kernels is used. To determine arbitrary integration constants, a system of algebraic equations is obtained. From this system, regular and irregular values of spectral parameters are calculated. The solution of the problem under consideration is obtained in the form of Fourier series. The unique solvability of the problem for regular values of spectral parameters is proved. To prove the convergence of Fourier series, the properties of the Mittag–Leffler function, Cauchy–Schwarz inequality and Bessel inequality are used. The continuous dependence of the problem solution on a small parameter for regular values of spectral parameters is also studied. The results are formulated as a theorem.

Keywords: integro-differential equation, mixed type equation, small parameter, spectral parameters, fractional Caputo operators, unique solvability.

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Introduction

Fractional calculus plays an important role for the mathematical modeling in many scientific and engineering disciplines (see more detailed information in [1]). In [2], some basic problems in continuum and statistical mechanics are considered. In [3], the mathematical problems of Ebola epidemic model are studied. In [4, 5], the fractional model for the dynamics of tuberculosis infection and novel coronavirus (nCov-2019), respectively, is studied. The construction of various models of theoretical physics problems by the aid of fractional calculus is described in [6, vol. 4, 5], [7, 8]. A specific physical interpretation of the Hilfer and Caputo fractional derivatives, describing the random motion of a particle moving on a real line at Poisson paced times with finite velocity is given in [9]. A detailed review on the application of fractional calculus in solving applied problems is given in [6, vol. 6–8], [10]. More detailed information as well as a bibliography related to the theory of fractional integro-differentiation, including the Hilfer and Caputo fractional derivatives can also be found in [11].

Applications for the equations of mixed type were studied in [12–14]. In particular, in the work [12] I. M. Gel'fand considered an example of gas motion in a channel surrounded by a porous medium and at that, gas motion in the channel was described by a wave equation, while outside the channel a diffusion equation was posed. Ya. S. Uflyand in [13] considered a problem on propagation of electric oscillations in compound lines, when the losses on a semi-infinite line were neglected and the rest of the line was treated as a cable with no leaks. He reduced this problem to a mixed parabolic-hyperbolic type equation. In [14], a hyperbolic-parabolic system arising in pulse combustion is investigated. Mixed type fractional differential equations are studied in many works of scientists, in particular in [15–24].

One of the important sections of the theory of integral and differential equations is the theory of integro-differential equations. The presence of an integral term in differential equations of the first and second orders has an important role in the theory of dynamical systems with automatic control [25, 26]. Mixed type integer order integro-differential equations with degenerate kernels and spectral parameters are studied in [27, 28].

In the present paper we study the issues of unique solvability of a boundary value problem for a mixed type integro-differential equation with two Caputo time-fractional operators and spectral parameters in multidimensional rectangular domain. We note that boundary value problems for integro-differential equations with spectral parameters have singularities in studying the questions of one value solvability [29, 30].

§ 1. Statement of the problem

In multidimensional domain $\Omega = \{-T < t < T,\ 0 < x_1, \ldots, x_m < l\}$ a mixed integro-differential equation of the following form is considered

$$A_\varepsilon(U) - B_\omega(U) = \begin{cases} \nu \displaystyle\int_0^T K_1(t,s)U(s,x)\,ds, & t > 0, \\ \nu \displaystyle\int_{-T}^0 K_2(t,s)U(s,x)\,ds, & t < 0, \end{cases} \tag{1.1}$$

where

$$A_\varepsilon(U) = \frac{1+\operatorname{sgn}(t)}{2}\left[{}_C D_{0t}^{\alpha_1} - \varepsilon \sum_{i=1}^m \frac{\partial^2}{\partial x_i \partial x_i}\, {}_C D_{0t}^{\beta_1}\right] U(t,x) +$$
$$+ \frac{1-\operatorname{sgn}(t)}{2}\left[{}_C D_{0t}^{\alpha_2} - \varepsilon \sum_{i=1}^m \frac{\partial^2}{\partial x_i \partial x_i}\, {}_C D_{0t}^{\beta_2}\right] U(t,x),$$

$$B_\omega(U) = \begin{cases} \displaystyle\sum_{i=1}^m \frac{\partial^2 U}{\partial x_i \partial x_i}, & t > 0, \\ \omega^2 \displaystyle\sum_{i=1}^m \frac{\partial^2 U}{\partial x_i \partial x_i}, & t < 0, \end{cases}$$

T and l are given positive real numbers, ω is positive spectral parameter, ε is positive small parameter, $x \in \Omega_l^m$, ν is real non-zero spectral parameter, $0 \neq K_j(t,s) = a_j(t) b_j(s)$, $a_j(t) \in C^2[-T;T]$, $b_j(s) \in C[-T;T]$, $\Omega_l^m \equiv [0;l]^m$, $0 < \beta_1 < \alpha_1 \leqslant 1$, $1 < \beta_2 < \alpha_2 \leqslant 2$.

Problem. Find in the domain Ω an unknown function

$$\begin{aligned} &U(t,x) \in C(\overline{\Omega}) \cap C^{\alpha_1,2}(\Omega_+) \cap C^{\alpha_2,2}(\Omega_-) \cap C_{t,x}^{\alpha_1+2}(\Omega_+) \cap C_{t,x}^{\alpha_2+2}(\Omega_-) \cap \\ &\cap C_{t,x_1,x_2,\ldots,x_m}^{\alpha_1+2+0+\ldots+0}(\Omega_+) \cap C_{t,x_1,x_2,\ldots,x_m}^{\alpha_2+2+0+\ldots+0}(\Omega_-) \cap C_{t,x_1,x_2,x_3,\ldots,x_m}^{\alpha_1+0+2+0+\ldots+0}(\Omega_+) \cap \\ &\cap C_{t,x_1,x_2,x_3,\ldots,x_m}^{\alpha_2+0+2+0+\ldots+0}(\Omega_-) \cap \ldots \cap C_{t,x_1,\ldots,x_{m-1},x_m}^{\alpha_1+0+\ldots+0+2}(\Omega_+) \cap C_{t,x_1,\ldots,x_{m-1},x_m}^{\alpha_2+0+\ldots+0+2}(\Omega_-), \end{aligned} \tag{1.2}$$

satisfying the mixed integro-differential equation (1.1) and the following boundary conditions

$$U(-T,x) = \varphi_1(x), \quad {}_C D_{0t}^{\theta} U(-T,x) = \varphi_2(x), \quad x \in \Omega_l^m, \tag{1.3}$$

$$
\begin{gathered}
U(t,0,x_2,x_3,\ldots,x_m)=U(t,l,x_2,x_3,\ldots,x_m)=\\
=U(t,x_1,0,x_3,\ldots,x)=U(t,x_1,l,x_3,\ldots,x_m)=\ldots=\\
=U(t,x_1,\ldots,x_{m-1},0)=U(t,x_1,\ldots,x_{m-1},l)=\\
=U_{x_1x_1}(t,0,x_2,x_3,\ldots,x_m)=U_{x_1x_1}(t,l,x_2,x_3,\ldots,x_m)=\\
=U_{x_1x_1}(t,x_1,0,x_3,\ldots,x_m)=U_{x_1x_1}(t,x_1,l,x_3,\ldots,x_m)=\ldots=\\
=U_{x_1x_1}(t,x_1,\ldots,x_{m-1},0)=U_{x_1x_1}(t,x_1,\ldots,x_{m-1},l)=\ldots=\\
=U_{x_mx_m}(t,0,x_2,x_3,\ldots,x_m)=U_{x_mx_m}(t,l,x_2,x_3,\ldots,x_m)=\\
=U_{x_mx_m}(t,x_1,0,x_3,\ldots,x_m)=U_{x_mx_m}(t,x_1,l,x_3,\ldots,x_m)=\ldots=\\
=U_{x_mx_m}(t,x_1,\ldots,x_{m-1},0)=U_{x_mx_m}(t,x_1,\ldots,x_{m-1},l)=0,\quad -T<t<T,
\end{gathered}
\tag{1.4}
$$

where $0<\theta<1$, $\varphi_i(x)$ are given smooth functions, $\varphi_i(0)=\varphi_i(l)=0$, $i=1,2$, $C^r(\Omega)$ is a class of functions $U(t,x_1,\ldots,x_m)$ with continuous derivatives $\frac{\partial^r U}{\partial t^r}$, $\frac{\partial^r U}{\partial x_1^r},\ldots,\frac{\partial^r U}{\partial x_m^r}$ in Ω, $C^{r,s}_{t,x}(\Omega)$ is a class of functions $U(t,x_1,\ldots,x_m)$ with continuous derivatives $\frac{\partial^r U}{\partial t^r}$, $\frac{\partial^s U}{\partial x_1^s},\ldots,\frac{\partial^s U}{\partial x_m^s}$ in Ω, $C^{r+r+0+\ldots+0}_{t,x_1,x_2,\ldots,x_m}(\Omega)$ is a class of functions $U(t,x_1,\ldots,x_m)$ with continuous derivative $\frac{\partial^{2r} U}{\partial t^r\partial x_1^r}$ in Ω, $\ldots$, $C^{r+0+\ldots+0+r}_{t,x_1,\ldots,x_{m-1},x_m}(\Omega)$ is a class of functions $U(t,x_1,\ldots,x_m)$ with continuous derivative $\frac{\partial^{2r} U}{\partial t^r\partial x_m^r}$ in Ω, r, s are positive real numbers, $\overline{\Omega}=\{-T\leqslant t\leqslant T,\ x\in\Omega_l^m\}$, $\Omega_-=\{-T<t<0,\ 0<x_1,\ldots,x_m<l\}$, $\Omega_+=\{0<t<T,\ 0<x_1,\ldots,x_m<l\}$.

§ 2. Expansion of the solution of the problem (1.1)–(1.4) into Fourier series

The solution of the mixed integro-differential equation (1.1) in domain Ω is sought in the form of a Fourier series

$$
U(t,x)=\sum_{n_1,\ldots,n_m=1}^{\infty}u^{\pm}_{n_1,\ldots,n_m}(t)\,\vartheta_{n_1,\ldots,n_m}(x), \tag{2.1}
$$

where

$$
u^{\pm}_{n_1,\ldots,n_m}(t)=\begin{cases} u^{+}_{n_1,\ldots,n_m}(t)=\displaystyle\int_{\Omega_l^m}U(t,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx, & t>0,\\ u^{+}_{n_1,\ldots,n_m}(t)=\displaystyle\int_{\Omega_l^m}U(t,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx, & t<0,\end{cases} \tag{2.2}
$$

$$
\int_{\Omega_l^m}U(t,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx=\int_0^l\ldots\int_0^l U(t,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx_1\cdot\ldots\cdot dx_m,
$$

$$
\vartheta_{n_1,\ldots,n_m}(x)=\left(\sqrt{\frac{2}{l}}\right)^m\sin\frac{\pi n_1}{l}x_1\cdot\ldots\cdot\sin\frac{\pi n_m}{l}x_m,\quad n_1,\ldots,n_m=1,\,2,\ldots.
$$

Substituting series (2.1) into equation (1.1), we obtain two fractional countable systems of ordinary integro-differential equations

$$
\begin{gathered}
{}_CD^{\alpha_1}_{0t}u^{+}_{n_1,\ldots,n_m}(t)+\varepsilon\mu^2_{n_1,\ldots,n_m}\,{}_CD^{\beta_1}_{0t}u^{+}_{n_1,\ldots,n_m}(t)+\mu^2_{n_1,\ldots,n_m}\,u^{+}_{n_1,\ldots,n_m}(t)=\\
=\nu\int_0^T a_1(t)b_1(s)u^{+}_{n_1,\ldots,n_m}(s)\,ds,\quad t>0,
\end{gathered}
\tag{2.3}
$$

$$
\begin{gathered}
{}_CD^{\alpha_2}_{0t}u^{-}_{n_1,\ldots,n_m}(t)+\varepsilon\mu^2_{n_1,\ldots,n_m}\,{}_CD^{\beta_2}_{0t}u^{-}_{n_1,\ldots,n_m}(t)+\mu^2_{n_1,\ldots,n_m}\omega^2u^{-}_{n_1,\ldots,n_m}(t)=\\
=\nu\int_{-T}^0 a_2(t)b_2(s)u^{-}_{n_1,\ldots,n_m}(s)\,ds,\quad t<0,
\end{gathered}
\tag{2.4}
$$

where $\mu_{n_1,\ldots,n_m}=\frac{\pi}{l}\sqrt{n_1^2+\ldots+n_m^2}$.

By the aid of notations

$$\tau^{+}_{n_1,\ldots,n_m} = \int_0^T b_1(s) u^{+}_{n_1,\ldots,n_m}(s)\, ds, \tag{2.5}$$

$$\tau^{-}_{n_1,\ldots,n_m} = \int_{-T}^0 b_2(s) u^{-}_{n_1,\ldots,n_m}(s)\, ds \tag{2.6}$$

we present the countable systems of equations (2.3) and (2.4) as follows

$$\begin{gathered} {}_CD^{\alpha_1}_{0t} u^{+}_{n_1,\ldots,n_m}(t) + \varepsilon\mu^2_{n_1,\ldots,n_m} {}_CD^{\beta_1}_{0t} u^{+}_{n_1,\ldots,n_m}(t) + \mu^2_{n_1,\ldots,n_m}\, u^{+}_{n_1,\ldots,n_m}(t) = \\ = \nu a_1(t)\tau^{+}_{n_1,\ldots,n_m}, \quad t > 0, \end{gathered} \tag{2.7}$$

$$\begin{gathered} {}_CD^{\alpha_2}_{0t} u^{-}_{n_1,\ldots,n_m}(t) + \varepsilon\mu^2_{n_1,\ldots,n_m} {}_CD^{\beta_2}_{0t} u^{-}_{n_1,\ldots,n_m}(t) + \mu^2_{n_1,\ldots,n_m}\, \omega^2 u^{-}_{n_1,\ldots,n_m}(t) = \\ = \nu a_2(t)\tau^{-}_{n_1,\ldots,n_m}, \quad t < 0. \end{gathered} \tag{2.8}$$

The solutions of the countable systems of differential equations (2.7) and (2.8), satisfying conditions

$$u^{+}_{n_1,\ldots,n_m}(0) = C^{+}_{1n_1,\ldots,n_m}, \quad u^{-}_{n_1,\ldots,n_m}(0) = C^{-}_{1n_1,\ldots,n_m}, \quad \frac{d}{dt} u^{-}_{n_1,\ldots,n_m}(0) = C^{-}_{2n_1,\ldots,n_m}$$

have the form:

$$\begin{gathered} u^{+}_{n_1,\ldots,n_m}(t) = \nu\tau^{+}_{n_1,\ldots,n_m} \times \\ \times \int_0^t a_1(t-s) s^{\alpha_1 - 1} E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} s^{\alpha_1-\beta_1}, -\mu^2_{n_1,\ldots,n_m} s^{\alpha_1}\right) ds + \\ + C^{+}_{1n_1,\ldots,n_m} E_{(\alpha_1-\beta_1,\alpha_1),1}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} t^{\alpha_1-\beta_1}, -\mu^2_{n_1,\ldots,n_m} t^{\alpha_1}\right), \quad t > 0, \end{gathered} \tag{2.9}$$

$$\begin{gathered} u^{-}_{n_1,\ldots,n_m}(t) = \nu\tau^{-}_{n_1,\ldots,n_m} \times \\ \times \int_t^0 a_2(s-t)(-s)^{\alpha_2-1} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2} \cdot \\ \cdot \left(-\varepsilon\mu^2_{n_1,\ldots,n_m}(-s)^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2(-s)^{\alpha_2}\right) ds + \\ + C^{-}_{1n_1,\ldots,n_m} E_{(\alpha_2-\beta_2,\alpha_2),1}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m}(-t)^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\,\omega^2(-t)^{\alpha_2}\right) - \\ - C^{-}_{2n_1,\ldots,n_m} t E_{(\alpha_2-\beta_2,\alpha_2),2}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m}(-t)^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2(-t)^{\alpha_2}\right), \quad t < 0. \end{gathered} \tag{2.10}$$

where $C^{+}_{1n_1,\ldots,n_m},\ C^{-}_{in_1,\ldots,n_m},\ (i = 1, 2)$ are for us unknown constants to be uniquely determined and

$$E_{(\alpha,\beta),\gamma}(z_1, z_2) = \sum_{m_1,m_2=0}^{\infty} \frac{z_1^{m_1} z_2^{m_2}}{\Gamma(\gamma + \alpha m_1 + \beta m_2)}, \ z_i, \alpha, \beta, \gamma \in \mathbb{C}, \ \operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\beta) > 0$$

is Mittag–Leffler function of two variables.

From the nature of the statement of the problem (properties in (1.2)) it follows that the continuous conjugation condition is fulfilled: $U(0+0, x) = U(0-0, x)$. So, taking the formula (2.2) into account, we have

$$\begin{gathered} u^{+}_{n_1,\ldots,n_m}(0+0) = \int_{\Omega^m_l} U(0+0, x)\vartheta_{n_1,\ldots,n_m}(x)\, dx = \\ = \int_{\Omega^m_l} U(0-0, x)\vartheta_{n_1,\ldots,n_m}(x)\, dx = u^{-}_{n_1,\ldots\, n_m}(0-0). \end{gathered} \tag{2.11}$$

Analogously, taking (2.2) into account from the conditions (1.3) we obtain

$$u^-_{n_1,\dots,n_m}(-T) = \int_{\Omega_l^m} U(-T,x)\vartheta_{n_1,\dots,n_m}(x)\,dx = \\ = \int_{\Omega_l^m} \varphi_1(x)\vartheta_{n_1,\dots,n_m}(x)\,dx = \varphi_{1n_1,\dots,n_m}, \tag{2.12}$$

$${}_C D^\theta_{0\,t} u^-_{n_1,\dots,n_m}(-T) = \int_{\Omega_l^m} {}_C D^\theta_{0\,t} U(-T,x)\vartheta_{n_1,\dots,n_m}(x)\,dx = \\ = \int_{\Omega_l^m} \varphi_2(x)\vartheta_{n_1,\dots,n_m}(x)\,dx = \varphi_{2n_1,\dots,n_m}, \tag{2.13}$$

where

$$\varphi_{in_1,\dots,n_m} = \int_{\Omega_l^m} \varphi_i(x)\,\vartheta_{n_1,\dots,n_m}(x)\,dx, \quad i=1,2.$$

By the aid of continuous conjugation condition (2.11) from (2.9) and (2.10) we have that $C^+_{1n_1,\dots,n_m} = C^-_{1n_1,\dots,n_m}$. To find unknown coefficients $C^-_{1\,n_1,\dots,n_m}$ and $C^-_{2n_1,\dots,n_m}$ in (2.10) we use the conditions (2.12) and (2.13) and come to the system of linear algebraic equations:

$$\begin{cases} C^-_{1n_1,\dots,n_m} E_{(\alpha_2-\beta_2,\alpha_2),1}\left(-\varepsilon\mu^2_{n_1,\dots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\dots,n_m}\omega^2 T^{\alpha_2}\right) + \\ + C^-_{2n_1,\dots,n_m} T E_{(\alpha_2-\beta_2,\alpha_2),2}\left(-\varepsilon\mu^2_{n_1,\dots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\dots,n_m}\omega^2 T^{\alpha_2}\right) = \\ = \psi_{1n_1,\dots,n_m}, \\ C^-_{1n_1,\dots,n_m} T^{-\theta}\left[E_{(\alpha_2-\beta_2,\alpha_2),1-\theta}\left(-\varepsilon\mu^2_{n_1,\dots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\dots,n_m}\omega^2 T^{\alpha_2}\right) - \dfrac{1}{\Gamma(1-\theta)}\right] + \\ + C^-_{2n_1,\dots,n_m} T^{1-\theta} E_{(\alpha_2-\beta_2,\,\alpha_2),2-\theta}\left(-\varepsilon\,\mu^2_{n_1,\dots,n_m} T^{\alpha_2-\beta_2},\, -\mu^2_{n_1,\dots,n_m}\omega^2 T^{\alpha_2}\right) = \\ = \psi_{2n_1,\dots,n_m}, \end{cases} \tag{2.14}$$

where

$$\psi_{in_1,\dots,n_m} = \varphi_{in_1,\dots,n_m} - \nu\tau^-_{n_1,\dots,n_m}\overline{\varphi}_{in_1,\dots,n_m}, \quad i=1,2,$$

$$\overline{\varphi}_{1n_1,\dots,n_m} = \int_{-T}^0 a_2(s+T)\,(-s)^{\alpha_2-1} \times \\ \times E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2}\left(-\varepsilon\mu^2_{n_1,\dots,n_m}(-s)^{\alpha_2-\beta_2}, -\mu^2_{n_1,\dots,n_m}\,\omega^2(-s)^{\alpha_2}\right)\,ds, \tag{2.15}$$

$$\overline{\varphi}_{2n_1,\dots,n_m} = a_2(0)\,T^{\alpha_2-2} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2-1}\left(-\varepsilon\mu^2_{n_1,\dots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\dots,n_m}\,\omega^2 T^{\alpha_2}\right) + \\ + \left.\frac{da_2(\xi-s)}{ds}\right|_{\xi=s} T^{\alpha_2-1} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2-1}\left(-\varepsilon\mu^2_{n_1,\dots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\dots,n_m}\omega^2 T^{\alpha_2}\right) + \\ + \int_{-T}^0 (-\xi)^{\alpha_2-1} E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2}\left(-\varepsilon\mu^2_{n_1,\dots,n_m}(-\xi)^{\alpha_2-\beta_2}, -\mu^2_{n_1,\dots,n_m}\omega^2(-\xi)^{\alpha_2}\right)d\xi \times \\ \times \int_\xi^{-p} \frac{1}{(s-\xi)^{\alpha_2-1}}\,\frac{d^2a_2(\xi-s)}{ds^2}\,ds. \tag{2.16}$$

If we assume that

$$\sigma_{n_1,\dots,n_m}(\omega) = T^{1-\theta} E_{(\alpha_2-\beta_2,\alpha_2),1}\left(-\varepsilon\mu^2_{n_1,\dots n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\dots,n_m}\omega^2 T^{\alpha_2}\right) \times \\ \times E_{(\alpha_2-\beta_2,\alpha_2),2-\theta}\left(-\varepsilon\mu^2_{n_1,\dots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\dots n_m}\omega^2 T^{\alpha_2}\right) - \\ - T^{1-\theta} E_{(\alpha_2-\beta_2,\alpha_2),2}\left(-\varepsilon\,\mu^2_{n_1,\dots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\dots,n_m}\omega^2 T^{\alpha_2}\right) \times \\ \times \left[E_{(\alpha_2-\beta_2,\alpha_2),1-\theta}\left(-\varepsilon\mu^2_{n_1,\dots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\dots,n_m}\omega^2 T^{\alpha_2}\right) - \frac{1}{\Gamma(1-\theta)}\right] \neq 0, \tag{2.17}$$

then the system with respect to $C^-_{1n_1,\ldots,n_m}$ and $C^-_{2n_1,\ldots n_m}$ is uniquely solvable. Solving this system (2.14) we arrive at the following presentations for these unknown coefficients

$$C^-_{1n_1,\ldots,n_m} = \frac{1}{\sigma_{n_1,\ldots,n_m}(\omega)}\left[\psi_{1n_1,\ldots,n_m} T^{1-\theta} E_{(\alpha_2-\beta_2,\alpha_2),2-\theta}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2 T^{\alpha_2}\right) - \right.$$

$$\left. - \psi_{2n_1,\ldots,n_m} T \cdot E_{(\alpha_2-\beta_2,\alpha_2),2}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2 T^{\alpha_2}\right)\right],$$

$$C^-_{2n_1,\ldots,n_m} = \frac{1}{\sigma_{n_1,\ldots,n_m}(\omega)}\left[\psi_{2n_1,\ldots,n_m} E_{(\alpha_2-\beta_2,\alpha_2),1}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2 T^{\alpha_2}\right) - \right.$$

$$\left. - \psi_{1n_1,\ldots,n_m} T^{-\theta}\left[E_{(\alpha_2-\beta_2,\alpha_2),1-\theta}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2 T^{\alpha_2}\right) - \frac{1}{\Gamma(1-\theta)}\right]\right].$$

Substituting these results into (2.10) and taking into account $C^+_{1n_1,\ldots,n_m} = C^-_{1n_1,\ldots,n_m}$ in (2.9), we obtain the following representations:

$$\begin{aligned} u^+_{n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) = \varphi_{1n_1,\ldots,n_m} N_{11n_1,\ldots,n_m}(t,\varepsilon,\omega) - \varphi_{2n_1,\ldots,n_m} N_{12n_1,\ldots,n_m}(t,\varepsilon,\omega) + \\ + \nu\tau^+_{n_1,\ldots,n_m} N_{13n_1,\ldots n_m}(t,\varepsilon,\omega), \quad t>0, \end{aligned} \tag{2.18}$$

$$\begin{aligned} u^-_{n_1,\ldots,n_m}(t) = \varphi_{1n_1,\ldots,n_m} N_{21n_1,\ldots,n_m}(t,\varepsilon,\omega) - \varphi_{2n_1,\ldots,n_m} N_{22n_1,\ldots,n_m}(t,\varepsilon,\omega) + \\ + \nu\tau^-_{n_1,\ldots,n_m} N_{23n_1,\ldots,n_m}(t,\varepsilon,\omega), \quad t<0, \end{aligned} \tag{2.19}$$

where

$$N_{11n_1,\ldots,n_m}(t,\varepsilon,\omega) = \frac{T^{1-\theta}}{\sigma_{n_1,\ldots,n_m}(\omega)} E_{(\alpha_2-\beta_2,\alpha_2),2-\theta}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2 T^{\alpha_2}\right) \times$$
$$\times E_{(\alpha_1-\beta_1,\alpha_1),1}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} t^{\alpha_1-\beta_1}, -\mu^2_{n_1,\ldots,n_m} t^{\alpha_1}\right),$$
$$N_{12n_1,\ldots,n_m}(t,\varepsilon,\omega) = \frac{T}{\sigma_{n_1,\ldots,n_m}(\omega)} E_{(\alpha_2-\beta_2,\alpha_2),2}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2 T^{\alpha_2}\right) \times$$
$$\times E_{(\alpha_1-\beta_1,\alpha_1),1}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} t^{\alpha_1-\beta_1}, -\mu^2_{n_1,\ldots,n_m} t^{\alpha_1}\right),$$
$$N_{13n_1,\ldots,n_m}(t,\varepsilon,\omega) = M_{11n_1,\ldots,n_m}(t,\varepsilon) - M_{12n_1,\ldots,n_m}(t,\varepsilon,\omega) + M_{13n_1,\ldots,n_m}(t,\varepsilon,\omega),$$
$$M_{11n_1,\ldots,n_m}(t,\varepsilon) = \int_0^t a_1(t-s)s^{\alpha_1-1} E_{(\alpha_1-\beta_1,\alpha_1),\alpha_1}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} s^{\alpha_1-\beta_1}, -\mu^2_{n_1,\ldots,n_m} s^{\alpha_1}\right) ds,$$
$$M_{12n_1,\ldots,n_m}(t,\varepsilon,\omega) = \overline{\varphi}_{1n_1,\ldots,n_m} N_{11n_1,\ldots,n_m}(t,\varepsilon,\omega),$$
$$M_{13n_1,\ldots,n_m}(t,\varepsilon,\omega) = \overline{\varphi}_{2n_1,\ldots,n_m} N_{12n_1,\ldots,n_m}(t,\varepsilon,\omega),$$
$$N_{21n_1,\ldots,n_m}(t,\varepsilon,\omega) = M_{21n_1,\ldots,n_m}(t,\varepsilon,\omega) + M_{22n_1,\ldots,n_m}(t,\varepsilon,\omega),$$
$$N_{22n_1,\ldots,n_m}(t,\varepsilon,\omega) = M_{23n_1,\ldots,n_m}(t,\varepsilon,\omega) + M_{24n_1,\ldots,n_m}(t,\varepsilon,\omega),$$
$$N_{23n_1,\ldots,n_m}(t,\varepsilon,\omega) = M_{25n_1,\ldots,n_m}(t,\varepsilon,\omega) - \overline{\varphi}_{1n_1,\ldots,n_m}\left[M_{21n_1,\ldots,n_m}(t,\varepsilon,\omega) - M_{22n_1,\ldots,n_m}(t,\varepsilon,\omega)\right] +$$
$$+ \overline{\varphi}_{2n_1,\ldots,n_m}\left[M_{23n_1,\ldots,n_m}(t,\varepsilon,\omega) - M_{24n_1,\ldots,n_m}(t,\varepsilon,\omega)\right],$$
$$M_{21n_1,\ldots,n_m}(t,\varepsilon,\omega) = \frac{T^{1-\theta}}{\sigma_{n_1,\ldots,n_m}(\omega)} E_{(\alpha_2-\beta_2,\alpha_2),2-\theta}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2 T^{\alpha_2}\right) \times$$
$$\times E_{(\alpha_2-\beta_2,\alpha_2),1}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m}(-t)^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2(-t)^{\alpha_2}\right),$$
$$M_{22n_1,\ldots,n_m}(t,\varepsilon,\omega) = \frac{T}{\sigma_{n_1,\ldots,n_m}(\omega)} \times$$
$$\times \left[E_{(\alpha_2-\beta_2,\alpha_2),1-\theta}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m} T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2 T^{\alpha_2}\right) - \frac{1}{\Gamma(1-\theta)}\right] \times$$

$$\times t \cdot E_{(\alpha_2-\beta_2,\alpha_2),2}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m}(-t)^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2(-t)^{\alpha_2}\right),$$

$$M_{23n_1,\ldots,n_m}(t,\varepsilon,\omega) = \frac{T}{\sigma_{n_1,\ldots,n_m}(\omega)} E_{(\alpha_2-\beta_2,\alpha_2),2}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m}T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2 T^{\alpha_2}\right) \times$$

$$\times E_{(\alpha_2-\beta_2,\alpha_2),1}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m}(-t)^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2(-t)^{\alpha_2}\right),$$

$$M_{24n_1,\ldots,n_m}(t,\varepsilon,\omega) = \frac{1}{\sigma_{n_1,\ldots,n_m}(\omega)} E_{(\alpha_2-\beta_2,\alpha_2),1}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m}T^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2 T^{\alpha_2}\right) \times$$

$$\times t \cdot E_{(\alpha_2-\beta_2,\alpha_2),2}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m}(-t)^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2(-t)^{\alpha_2}\right),$$

$$M_{25n_1,\ldots,n_m}(t,\varepsilon,\omega) = \int_t^0 a_2(s-t)(-s)^{\alpha_2-1} \times$$

$$\times E_{(\alpha_2-\beta_2,\alpha_2),\alpha_2}\left(-\varepsilon\mu^2_{n_1,\ldots,n_m}(-s)^{\alpha_2-\beta_2}, -\mu^2_{n_1,\ldots,n_m}\omega^2(-s)^{\alpha_2}\right) ds,$$

the quantity $\overline{\varphi}_{1n_1,\ldots,n_m}$ and $\overline{\varphi}_{1n_1,\ldots,n_m}$ are defined from (2.15) and (2.16), respectively.

Substituting these presentations (2.18) and (2.19) into (2.5) and (2.6), we obtain

$$\tau^+_{n_1,\ldots,n_m}\left[1 - \nu\int_0^T b_1(s) N_{13n_1,\ldots,n_m}(s,\varepsilon,\omega)\, ds\right] =$$
$$= \varphi_{1n_1,\ldots,n_m}\int_0^T b_1(s)\, N_{11n_1,\ldots,n_m}(s,\varepsilon,\omega)\, ds - \varphi_{2n_1,\ldots,n_m}\int_0^T b_1(s) N_{12n_1,\ldots,n_m}(s,\varepsilon,\omega)\, ds, \tag{2.20}$$

$$\tau^-_{n_1,\ldots,n_m}\left[1 - \nu\int_{-T}^0 b_2(s) N_{23n_1,\ldots,n_m}(s,\varepsilon,\omega)\, ds\right] =$$
$$= \varphi_{1n_1,\ldots,n_m}\int_{-T}^0 b_2(s) N_{21n_1,\ldots,n_m}(s,\varepsilon,\omega)\, ds - \tag{2.21}$$
$$- \varphi_{2n_1,\ldots,n_m}\int_{-T}^0 b_2(s) N_{22n_1,\ldots,n_m}(s,\varepsilon,\omega)\, ds.$$

If the following conditions are fulfilled

$$\nu\int_0^T b_1(s)\, N_{13n_1,\ldots,n_m}(s,\varepsilon,\omega)\, ds \neq 1, \quad \nu\int_{-T}^0 b_2(s)\, N_{23n_1,\ldots,n_m}(s,\varepsilon,\omega)\, ds \neq 1, \tag{2.22}$$

then, by virtue of (2.20) and (2.21), from the presentations (2.18) and (2.19) we derive

$$u^+_{n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) =$$
$$= \varphi_{1n_1,\ldots,n_m} Q_{11n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) - \varphi_{2n_1,\ldots,n_m} Q_{12n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu), \quad t > 0, \tag{2.23}$$

$$u^-_{n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) =$$
$$= \varphi_{1n_1,\ldots,n_m} Q_{21n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) - \varphi_{2n_1,\ldots,n_m} Q_{22n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu), \quad t < 0, \tag{2.24}$$

where

$$Q_{11n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) = N_{11n_1,\ldots,n_m}(t,\varepsilon,\omega) +$$
$$+ \nu\frac{N_{13n_1,\ldots,n_m}(t,\varepsilon,\omega)}{1 - \nu\int_0^T b_1(s)\, N_{13n_1,\ldots,n_m}(s,\varepsilon,\omega)\, ds}\int_0^T b_1(s) N_{11n_1,\ldots,n_m}(s,\varepsilon,\omega)\, ds,$$

$$Q_{12n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) = N_{12n_1,\ldots,n_m}(t,\varepsilon,\omega) +$$
$$+ \nu\frac{N_{13n_1,\ldots,n_m}(t,\varepsilon,\omega)}{1 - \nu\int_0^T b_1(s) N_{13n_1,\ldots,n_m}(s,\varepsilon,\omega)\, ds}\int_0^T b_1(s) N_{12n_1,\ldots,n_m}(s,\varepsilon,\omega)\, ds,$$

$$Q_{21n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu) = N_{21n_1,\ldots,n_m}(t,\varepsilon,\omega) +$$

$$+\nu\frac{N_{23n_1,\dots,n_m}(t,\varepsilon,\omega)}{1-\nu\int_{-T}^{0}b_2(s)N_{23n_1,\dots,n_m}(s,\varepsilon,\omega)\,ds}\int_{-T}^{0}b_2(s)N_{21n_1,\dots,n_m}(s,\varepsilon,\omega)\,ds,$$

$$Q_{22n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)=N_{22n_1,\dots,n_m}(t,\varepsilon,\omega)+$$

$$+\nu\frac{N_{23n_1,\dots,n_m}(t,\varepsilon,\omega)}{1-\nu\int_{-T}^{0}b_2(s)N_{23n_1,\dots,n_m}(s,\varepsilon,\omega)\,ds}\int_{-T}^{0}b_2(s)N_{22n_1,\dots,n_m}(s,\varepsilon,\omega)\,ds.$$

Now we substitute presentations (2.23) and (2.24) into the Fourier series (2.1) and obtain the following formal solution of the problem (1.1)–(1.4)

$$U(t,x,\varepsilon,\omega,\nu)=\sum_{n_1,\dots,n_m=1}^{\infty}\vartheta_{n_1,\dots,n_m}(x)\times$$
$$\times\left[\varphi_{1n_1,\dots,n_m}Q_{11n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)-\varphi_{2n_1,\dots,n_m}Q_{12n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\right],\quad t>0, \tag{2.25}$$

$$U(t,x,\varepsilon,\omega,\nu)=\sum_{n_1,\dots,n_m=1}^{\infty}\vartheta_{n_1,\dots,n_m}(x)\times$$
$$\times\left[\varphi_{1n_1,\dots,n_m}Q_{21n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)+\varphi_{2n_1,\dots,n_m}Q_{22n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\right],\quad t<0. \tag{2.26}$$

§ 3. The uniqueness of the solution of the problem (1.1)–(1.4)

Let condition (2.17) be violated. Then we suppose that

$$\begin{aligned}\sigma_{n_1,\dots,n_m}(\omega)=T^{1-\theta}E_{(\alpha_2-\beta_2,\alpha_2),1}\left(-\varepsilon\mu^2_{n_1,\dots,n_m}T^{\alpha_2-\beta_2},-\mu^2_{n_1,\dots,n_m}\omega^2T^{\alpha_2}\right)\times\\ \times E_{(\alpha_2-\beta_2,\alpha_2),2-\theta}\left(-\varepsilon\mu^2_{n_1,\dots,n_m}T^{\alpha_2-\beta_2},-\mu^2_{n_1,\dots,n_m}\omega^2T^{\alpha_2}\right)-\\ -T^{1-\theta}E_{(\alpha_2-\beta_2,\alpha_2),2}\left(-\varepsilon\mu^2_{n_1,\dots,n_m}T^{\alpha_2-\beta_2},-\mu^2_{n_1,\dots,n_m}\omega^2T^{\alpha_2}\right)\times\\ \times\left[E_{(\alpha_2-\beta_2,\alpha_2),1-\theta}\left(-\varepsilon\,\mu^2_{n_1,\dots,n_m}T^{\alpha_2-\beta_2},-\mu^2_{n_1,\dots,n_m}\omega^2T^{\alpha_2}\right)-\frac{1}{\Gamma(1-\theta)}\right]=0\end{aligned} \tag{3.1}$$

for some values of ω, where $\mu_{n_1,\dots,n_m}=\frac{\pi}{l}\sqrt{n_1^2+\ldots+n_m^2}$.

The set of positive solutions of this equation (3.1) with respect to the spectral parameter ω is denoted by $\Im_1$. We call the values $\omega\in\Im_1$ irregular because the condition (2.17) is violated for them. The set $\Lambda_1=(0;\infty)\backslash\Im_1$ is called the set of regular values of the spectral parameter ω, for which condition (2.17) is fulfilled.

If conditions in (2.22) are violated, then the kernels of the mixed integro-differential equation (1.1) have for each value of $n_1,\dots,n_m$ two values of ν_1 and ν_2

$$\nu_1=\frac{1}{\int_0^T b_1(s)N_{13n_1,\dots,n_m}(s,\varepsilon,\omega)\,ds},\quad \nu_2=\frac{1}{\int_{-T}^0 b_2(s)N_{23n_1,\dots,n_m}(s,\varepsilon,\omega)\,ds},$$

where

$$\int_0^T b_1(s)\,N_{13n_1,\dots,n_m}(s,\varepsilon,\omega)\,ds\neq0,\quad \int_{-T}^0 b_2(s)N_{23n_1,\dots,n_m}(s,\varepsilon,\omega)\,ds\neq0.$$

We regard these real nonzero numbers as irregular kernel numbers of the mixed integro-differential equation (1.1) and denote their set $\{\nu_1,\nu_2\}$ by $\Im_2$. We take away the values ν_1 and ν_2 of the spectral parameter ν from the set of nonzero real numbers $(-\infty;0)\cup(0;\infty)$. The resulting set $\Lambda_2=(-\infty;0)\cup(0;\infty)\backslash\Im_2$ is called a set of regular values of the parameter ν. For all values of $\nu\in\Lambda_2$ condition (2.22) is satisfied. We use the following notation

$$\aleph=\{n_1,\dots,n_m\in\mathbb{N};\ \omega\in\Lambda_1;\nu\in\Lambda_2\},$$

where $\mathbb{N}$ is the set of natural numbers. This is the case when all values of the spectral parameters ω and ν are regular. Therefore, in this case, the solution of the problem (1.1)–(1.4) in the domain Ω is represented in the form of series (2.25) and (2.26).

To establish the uniqueness of the function $U(t,x,\varepsilon,\omega,\nu)$ we show that, under the zero conditions $\varphi_i(x)\equiv 0$, $x\in\Omega_l^m$, $i=1,2$, the problem (1.1)–(1.4) has only a trivial solution. We suppose that $\varphi_i(x)\equiv 0$, $i=1,2$. Then $\varphi_{in_1,\dots,n_m}=0$, $i=1,2$, and from formulas (2.25) and (2.26) it implies that

$$\int_{\Omega_l^m} U(t,x,\varepsilon,\omega,\nu)\vartheta_{n_1,\dots,n_m}(x)\,dx=0.$$

Hence, by virtue of completeness of systems of the eigenfunctions

$$\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi n_1}{l}x_1\right\},\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi n_2}{l}x_2\right\},\dots,\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi n_m}{l}x_m\right\}$$

in the space $L_2\left(\Omega_l^m\right)$ we deduce that

$$U(t,x,\varepsilon,\omega,\nu)\equiv 0$$

for all $x\in[0,l]^m$ and $t\in[-T;T]$. Therefore, for the $(n_1,\dots,n_m,\omega,\nu)\in\aleph$ the solution of the problem (1.1)–(1.4) is unique, if this solution exists in the domain Ω.

§ 4. Convergence of series (2.25) and (2.26)

We show that under certain conditions with respect to the functions $\varphi_i(x)$ $(i=1,2)$ the series (2.25) and (2.26) converge absolutely and uniformly in the domain $\overline{\Omega}$. In this order we use the following well known properties of the Mittag–Leffler function:

1) for all $k>0$, α, β, $\gamma\in(0;\,2]$, $\beta\leqslant\alpha\leqslant\gamma$, $t\geqslant 0$ the function $t^{\alpha-1}E_{(\beta,\alpha),\gamma}\left(-kt^\beta,-kt^\alpha\right)$ is completely monotonous and there holds

$$(-1)^s\left[t^{\alpha-1}E_{(\beta,\alpha),\gamma}\left(-kt^\beta,-k\,t^\alpha\right)\right]^{(s)}\geqslant 0,\quad s=0,1,2,\dots;\tag{4.1}$$

2) for all $\alpha,\beta\in(0,2)$, $\gamma\in\mathbb{R}$ and $\arg z_1=\pi$ there takes place the following estimate

$$\left|E_{(\beta,\alpha),\gamma}\left(z_1,z_2\right)\right|\leqslant\frac{C_1}{1+|z_1|},\tag{4.2}$$

$$\left|E_{(\beta,\alpha),\gamma}\left(\varepsilon_1 z_1,z_2\right)-E_{(\beta,\alpha),\gamma}\left(\varepsilon_2 z_1,z_2\right)\right|\leqslant|\varepsilon_1-\varepsilon_2|\frac{C_2}{1+|z_1|},\tag{4.3}$$

where $0<C_i=\text{const}$ do not depend from z, $\varepsilon_i\in(0;\varepsilon_0)$, $0<\varepsilon_0=\text{const}$, $i=1,2$.

Indeed, according to the properties of the Mittag–Leffler function (formulas (4.1) and (4.2)) the functions $Q_{ijn_1,\dots,n_m}(t,\,\varepsilon,\omega,\nu)$ $(i,j=1,2)$ are uniformly bounded on the segment $[-T;T]$. So for any positive integers $n_1,\dots,n_m$ there exist finite constant numbers C_{0i} $(i=1,2)$, that the following estimates take place

$$\max_{n_1,\dots,n_m\in\mathbb{N}}\left\{\max_{t\in[0;T]}\left|Q_{11n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\right|;\ \max_{t\in[0;T]}\left|Q_{12n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\right|\right\}\leqslant C_{01},\tag{4.4}$$

$$\max_{n_1,\dots,n_m\in\mathbb{N}}\left\{\max_{t\in[-T;0]}\left|Q_{21n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\right|;\ \max_{t\in[-T;0]}\left|Q_{22n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\right|\right\}\leqslant C_{02},$$

where $C_{0i}=\text{const}$, $i=1,2$.

Condition A. We suppose that the functions $\varphi_i(x) \in C^2[0;l]^m$, $i=1,2$, on the domain $[0;l]^m$ have piecewise continuous third order derivatives. Then integrating by parts the following integrals three times with respect to each of variables $x_1,\ x_2,\dots,x_m$ we derive [28]

$$|\varphi_{i\,n_1,\dots,n_m}| \leqslant \left(\frac{l}{\pi}\right)^{3m} \frac{\left|\varphi_{in_1,\dots,n_m}^{(3m)}\right|}{n_1^3,\dots,n_m^3}, \tag{4.5}$$

where

$$\varphi_{in_1,\dots,n_m}^{(3m)} = \int_{\Omega_l^m} \frac{\partial^{3m}\varphi_i(x)}{\partial x_1^3 \partial x_2^3 \dots \partial x_m^3} \vartheta_{n_1,\dots,n_m}(x)\, dx, \quad i=1,2.$$

Here the Bessel inequalities are true

$$\sum_{n_1,\dots,n_m=1}^{\infty} \left[\varphi_{in_1,\dots,n_m}^{(3m)}\right]^2 \leqslant \left(\frac{2}{l}\right)^m \int_{\Omega_l^m} \left[\frac{\partial^{3m}\varphi_i(x)}{\partial x_1^3 \partial x_2^3 \dots \partial x_m^3}\right]^2 dx, \quad i=1,2. \tag{4.6}$$

Taking formulas (4.4)–(4.6) into account and applying the Cauchy–Schwarz inequality and Bessel inequality, for series (2.25) and (2.26) we obtain

$$\begin{aligned} |U(t,x,\varepsilon,\omega,\nu)| &\leqslant \sum_{n_1,\dots,n_m=1}^{\infty} \left|u_{n_1,\dots,n_m}^{\pm}(t,\varepsilon,\omega,\nu)\right| \cdot |\vartheta_{n_1,\dots\, n_m}(x)| \leqslant \\ &\leqslant \left(\sqrt{\frac{2}{l}}\right)^m C_{0i} \sum_{n_1,\dots,n_m=1}^{\infty} \left[|\varphi_{1n_1,\dots,n_m}| + |\varphi_{2n_1,\dots,n_m}|\right] \leqslant \\ &\leqslant \gamma_{1i}\left[\sum_{n_1,\dots,n_m=1}^{\infty} \frac{1}{n_1^3\dots n_m^3}\left|\varphi_{1n_1,\dots,n_m}^{(3m)}\right| + \sum_{n_1,\dots,n_m=1}^{\infty} \frac{1}{n_1^3\dots n_m^3}\left|\varphi_{2n_1,\dots,n_m}^{(3m)}\right|\right] \leqslant \\ &\leqslant \left(\sqrt{\frac{2}{l}}\right)^m \gamma_{1i} \sqrt{\sum_{n_1,\dots,n_m=1}^{\infty} \frac{1}{n_1^6\dots n_m^6}} \left[\sqrt{\int_{\Omega_l^m}\left[\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3\partial x_2^3\dots\partial x_m^3}\right]^2 dx} + \right. \\ &\left. + \sqrt{\int_{\Omega_l^m}\left[\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3\partial x_2^3\dots\partial x_m^3}\right]^2 dx}\right] < \infty, \end{aligned} \tag{4.7}$$

where $\gamma_{1i} = \left(\sqrt{\frac{2}{l}}\right)^m C_{0i}\left(\frac{l}{\pi}\right)^{3m}$, $i=1,2$.

It follows from estimate (4.7) that the series (2.25) and (2.26) are convergent absolutely and uniformly in the domain $\overline{\Omega}$ for the $(n_1,\dots,n_m,\ \omega,\ \nu) \in \aleph$.

§ 5. Possibility of term differentiation of series (2.25) and (2.26)

For the $(n_1,\dots,n_m,\omega,\nu) \in \aleph$ functions (2.25) and (2.26) formally differentiate in $\overline{\Omega}$ the required number of times

$$\begin{aligned} {}_CD_{0t}^{\alpha_1}U(t,x,\varepsilon,\omega,\nu) &= \sum_{n_1,\dots,n_m=1}^{\infty} \vartheta_{n_1,\dots,n_m}(x) \times \\ \times \left[\varphi_{1n_1,\dots,n_m}{}_CD_{0t}^{\alpha_1}Q_{11n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) + \varphi_{2n_1,\dots,n_m}{}_CD_{0t}^{\alpha_1}Q_{12n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\right], &\quad t>0, \end{aligned} \tag{5.1}$$

$$\begin{aligned} {}_CD_{0t}^{\alpha_2}U(t,x,\varepsilon,\omega,\nu) &= \sum_{n_1,\dots,n_m=1}^{\infty} \vartheta_{n_1,\dots,n_m}(x) \times \\ \times \left[\varphi_{1n_1,\dots,n_m}{}_CD_{0t}^{\alpha_2}Q_{21n_1,\dots,n_m}(t,\varepsilon,\omega,\nu) + \varphi_{2n_1,\dots,n_m}{}_CD_{0t}^{\alpha_2}Q_{22n_1,\dots,n_m}(t,\varepsilon,\omega,\nu)\right], &\quad t<0, \end{aligned} \tag{5.2}$$

$$U_{x_1x_1}(t,x,\varepsilon,\omega,\nu)=-\sum_{n_1,\ldots n_m=1}^{\infty}\left(\frac{\pi n_1}{l}\right)^2\vartheta_{n_1,\ldots,n_m}(x)\times$$
$$\times\left[\varphi_{1n_1,\ldots,n_m}Q_{i1n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)+\varphi_{2n_1,\ldots,n_m}Q_{i2n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\right],\quad -T<t<T, \tag{5.3}$$

$$U_{x_2x_2}(t,x,\varepsilon,\omega,\nu)=-\sum_{n_1,\ldots,n_m=1}^{\infty}\left(\frac{\pi n_2}{l}\right)^2\vartheta_{n_1,\ldots,n_m}(x)\times$$
$$\times[\varphi_{1n_1,\ldots,n_m}Q_{i1n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)+$$
$$+\varphi_{2n_1,\ldots,n_m}Q_{i2n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)],\quad -T<t<T,\quad i=1,2. \tag{5.4}$$

The expansions of the following functions into Fourier series are defined in the domain Ω in a similar way

$$U_{x_3x_3}(t,x,\varepsilon,\omega,\nu),\ldots,U_{x_mx_m}(t,x,\varepsilon,\omega,\nu),{}_CD_{0\,t}^{\alpha_1}U_{x_1x_1}(t,x,\varepsilon,\omega,\nu),$$

$${}_CD_{0\,t}^{\alpha_2}U_{x_1x_1}(t,x,\varepsilon,\omega,\nu),\ {}_CD_{0\,t}^{\alpha_1}U_{x_2x_2}(t,x,\varepsilon,\omega,\nu),\ldots,{}_CD_{0\,t}^{\alpha_2}U_{x_2x_2}(t,x,\varepsilon,\omega,\nu),\ldots,$$

$${}_CD_{0\,t}^{\alpha_1}U_{x_mx_m}(t,x,\varepsilon,\omega,\nu),\ {}_CD_{0\,t}^{\alpha_2}U_{x_mx_m}(t,x,\varepsilon,\omega,\nu).$$

The convergence of series (5.1) and (5.2) is proved similarly to the proof of the convergence of series (2.25) and (2.26). So, we show the convergence of series (5.3) and (5.4). Taking into account formulas (4.4)–(4.6) and estimate (4.7) and applying the Cauchy–Schwarz inequality and Bessel inequality, we obtain

$$|U_{x_1x_1}(t,x,\varepsilon,\omega,\nu)|\leqslant\sum_{n_1,\ldots n_m=1}^{\infty}\left(\frac{\pi n_1}{l}\right)^2\left|u^{\pm}_{n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\right|\cdot|\vartheta_{n_1,\ldots,n_m}(x)|\leqslant$$
$$\leqslant\left(\sqrt{\frac{2}{l}}\right)^m\left(\frac{\pi}{l}\right)^2C_{0i}\sum_{n_1,\ldots,n_m=1}^{\infty}n_1^2\left[|\varphi_{1n_1,\ldots,n_m}|+|\varphi_{2n_1,\ldots,n_m}|\right]\leqslant$$
$$\leqslant\gamma_{2i}\left[\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1n_2^3\ldots n_m^3}\left|\varphi^{(3m)}_{1n_1,\ldots,n_m}\right|+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1n_2^3\ldots n_m^3}\left|\varphi^{(3m)}_{2n_1,\ldots,n_m}\right|\right]\leqslant$$
$$\leqslant\left(\sqrt{\frac{2}{l}}\right)^m\gamma_{2i}\sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^2n_2^6\ldots n_m^6}}\left[\sqrt{\int_{\Omega_l^m}\left[\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}+\right.$$
$$\left.+\sqrt{\int_{\Omega_l^m}\left[\frac{\partial^{3\,m}\varphi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}\right]<\infty,$$

where $\gamma_{2i}=\left(\sqrt{\frac{2}{l}}\right)^mC_{0i}\left(\frac{l}{\pi}\right)^{3\,m-2}$, $i=1,2$;

$$|U_{x_2x_2}(t,x,\varepsilon,\omega,\nu)|\leqslant\sum_{n_1,\ldots,n_m=1}^{\infty}\left(\frac{\pi n_2}{l}\right)^2\left|u^{\pm}_{n_1,\ldots,n_m}(t,\varepsilon,\omega,\nu)\right|\cdot|\vartheta_{n_1,\ldots,n_m}(x)|\leqslant$$
$$\leqslant\left(\sqrt{\frac{2}{l}}\right)^m\left(\frac{\pi}{l}\right)^2C_{0i}\sum_{n_1,\ldots,n_m=1}^{\infty}n_2^2\left[|\varphi_{1n_1,\ldots,n_m}|+|\varphi_{2n_1,\ldots,n_m}|\right]\leqslant$$
$$\leqslant\gamma_{2i}\left[\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^3n_2n_3^3\ldots n_m^3}\left|\varphi^{(3m)}_{1n_1,\ldots,n_m}\right|+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^3n_2n_3^3\ldots n_m^3}\left|\varphi^{(3m)}_{2n_1,\ldots n_m}\right|\right]\leqslant$$

$$\leqslant \left(\sqrt{\frac{2}{l}}\right)^m \gamma_{2i} \sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^6 n_2^2 n_3^6 \ldots n_m^6}} \left[\sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \right]^2 dx} + \right.$$
$$\left. + \sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \right]^2 dx} \right] < \infty.$$

The convergence of Fourier series for functions

$$U_{x_3x_3}(t,x,\varepsilon,\omega,\nu), \ldots, U_{x_mx_m}(t,x,\varepsilon,\omega,\nu), {}_CD_{0\,t}^{\alpha_1} U_{x_1x_1}(t,x,\varepsilon,\omega,\nu),$$
$${}_CD_{0\,t}^{\alpha_2} U_{x_1x_1}(t,x,\varepsilon,\omega,\nu), {}_CD_{0\,t}^{\alpha_1} U_{x_2x_2}(t,x,\varepsilon,\omega,\nu), \ldots, {}_CD_{0\,t}^{\alpha_2} U_{x_2x_2}(t,x,\varepsilon,\omega,\nu), \ldots$$
$${}_CD_{0\,t}^{\alpha_1} U_{x_mx_m}(t,x,\varepsilon,\omega,\nu), {}_CD_{0\,t}^{\alpha_2} U_{x_mx_m}(t,x,\varepsilon,\omega,\nu).$$

is proved in a similar way in the domain Ω. It follows from these last estimates that the functions (2.25) and (2.26) possess properties (1.2) for regular values of spectral parameters ω and ν.

§ 6. Continuous dependence of the solution on a small parameter

We consider the continuous dependence of the solution of the problem (1.1)–(1.4) on a small parameter ε for regular values of spectral parameters ω and ν. Let ε_1 and ε_2 be two different values of a small positive parameter ε. It is easy to check from (4.3), that the following estimates hold

$$\max_{n_1,\ldots,n_m \in \mathbb{N}} \max_{t\in[0;T]} \left| Q_{1in_1,\ldots,n_m}(t,\varepsilon_1,\omega,\nu) - Q_{1in_1,\ldots,n_m}(t,\varepsilon_2,\omega,\nu) \right| \leqslant C_{1i} \left| \varepsilon_1 - \varepsilon_2 \right|, \tag{6.1}$$

$$\max_{n_1,\ldots\, n_m \in \mathbb{N}} \max_{t\in[-T;0]} \left| Q_{2in_1,\ldots,n_m}(t,\varepsilon_1,\omega,\nu) - Q_{2in_1,\ldots\, n_m}(t,\varepsilon_2,\omega,\nu) \right| \leqslant C_{2i} \left| \varepsilon_1 - \varepsilon_2 \right|, \tag{6.2}$$

where $C_{ji} = \text{const},\ j,i = 1,2,\ \varepsilon_i \in (0;\varepsilon_0),\ 0 < \varepsilon_0 = \text{const},\ i = 1,2.$

Then, taking formulas (4.5), (4.6) and estimates (6.1), (6.2) into account and applying the Cauchy–Schwarz inequality and Bessel inequality, for series (2.25) and (2.26) we obtain

$$|U(t,x,\varepsilon_1,\omega,\nu) - U(t,x,\varepsilon_2,\omega,\nu| \leqslant$$
$$\leqslant \sum_{n_1,\ldots,n_m=1}^{\infty} \left| u_{n_1,\ldots,n_m}^{\pm}(t,\varepsilon_1,\omega,\nu) - u_{n_1,\ldots,n_m}^{\pm}(t,\varepsilon_2,\omega,\nu) \right| \cdot \left| \vartheta_{n_1,\ldots,n_m}(x) \right| \leqslant$$
$$\leqslant \left(\sqrt{\frac{2}{l}}\right)^m (C_{i1} + C_{i2}) \left| \varepsilon_1 - \varepsilon_2 \right| \sum_{n_1,\ldots,n_m=1}^{\infty} \left[|\varphi_{1n_1,\ldots,n_m}| + |\varphi_{2n_1,\ldots,n_m}| \right] \leqslant$$
$$\leqslant \gamma_{3i} \left| \varepsilon_1 - \varepsilon_2 \right| \left[\sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^3 \ldots n_m^3} \left| \varphi_{1n_1,\ldots\, n_m}^{(3m)} \right| + \sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^3 \ldots n_m^3} \left| \varphi_{2n_1,\ldots,n_m}^{(3m)} \right| \right] \leqslant \tag{6.3}$$
$$\leqslant \left(\sqrt{\frac{2}{l}}\right)^m \left| \varepsilon_1 - \varepsilon_2 \right| \gamma_{3i} \sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1^6 \ldots n_m^6}} \left[\sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \right]^2 dx} + \right.$$
$$\left. + \sqrt{\int_{\Omega_l^m} \left[\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3 \partial x_2^3 \ldots \partial x_m^3} \right]^2 dx} \right] = \left| \varepsilon_1 - \varepsilon_2 \right| \cdot \Delta,$$

where $\gamma_{3i} = \left(\sqrt{\frac{2}{l}}\right)^m (C_{i1} + C_{i2}) \left(\frac{l}{\pi}\right)^{3m},\ i = 1,2,\ \Delta = \text{const} < \infty.$

It follows from estimate (6.3) that $|U(t,x,\varepsilon_1,\omega,\nu) - U(t,x,\varepsilon_2,\omega,\nu)\,|$ is small, if $|\varepsilon_1 - \varepsilon_2|$ is small in the domain $\overline{\Omega}$ for the $(n_1,\ldots,n_m,\omega,\nu) \in \aleph$.

§ 7. Statement of the theorem

As a conclusion, we will formulate a theorem that we have already proved in this article. Thus, the following theorem is true.

T h e o r e m 7.1. *Let conditions of **A** be fulfilled. Then for the possible numbers* $n_1, \ldots, n_m$ *and regular values of spectral parameters* ω *and* ν *from the set* $\aleph$ *the problem* (1.1)–(1.4) *is uniquely solvable in the domain* Ω *and this solution is represented in the form of series* (2.25) *and* (2.26). *Moreover, it is true that*

$$\lim_{\varepsilon\to 0} U(t, x, \varepsilon, \omega, \nu) = U(t, x, 0, \omega, \nu),$$

where $U(t, x, 0, \omega, \nu)$ *is the solution of a mixed fractional integro-differential equation of the form*

$$A_0(U) - B_\omega(U) = \begin{cases} \nu \displaystyle\int_0^T K_1(t,s)U(s,x)\,ds, & t > 0, \\ \nu \displaystyle\int_{-T}^0 K_2(t,s)U(s,x)\,ds, & t < 0, \end{cases}$$

$$A_0(U) = \left[\frac{1+\operatorname{sgn}(t)}{2}\,{}_C D_{0t}^{\alpha_1} + \frac{1-\operatorname{sgn}(t)}{2}\,{}_C D_{0t}^{\alpha_2}\right] U(t,x), \quad B_\omega(U) = \begin{cases} \displaystyle\sum_{i=1}^m U_{x_i x_i}, & t > 0, \\ \omega^2 \displaystyle\sum_{i=1}^m U_{x_i x_i}, & t < 0, \end{cases}$$

with boundary value conditions (1.3) *and* (1.4) *under consideration.*

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Смешанные интегро-дифференциальные уравнения с операторами Капуто дробного порядка и спектральными параметрами

Ключевые слова: интегро-дифференциальное уравнение, уравнение смешанного типа, малый параметр, спектральные параметры, дробные операторы Капуто, однозначная разрешимость.

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Рассматриваются вопросы однозначной разрешимости краевой задачи для интегро-дифференциального уравнения смешанного типа с двумя операторами Капуто дробного порядка и спектральными параметрами. Интегро-дифференциальное уравнение смешанного типа является интегро-дифференциальным уравнением с частными производными дробного порядка как в положительной, так и в отрицательной частях рассматриваемой многомерной прямоугольной области. Порядок дробного оператора Капуто меньше в положительной части области, чем порядок соответствующего оператора в отрицательной части области. Используя метод рядов Фурье, получены две системы счетных систем обыкновенных дробных интегро-дифференциальных уравнений с вырожденными ядрами. Далее используется метод вырожденных ядер. Для определения произвольных постоянных интегрирования получена система алгебраических уравнений. Из этой системы были вычислены регулярные и нерегулярные значения спектральных параметров. Решение рассматриваемой задачи получено в виде рядов Фурье. Доказана однозначная разрешимость задачи для регулярных значений спектральных параметров. При доказательстве сходимости рядов Фурье используются свойства функции Миттаг–Леффлера, неравенство Коши–Шварца и неравенство Бесселя. Также изучена непрерывная зависимость решения задачи от малого параметра при регулярных значениях спектральных параметров. Результаты сформулированы в виде теоремы.

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