mathematics

MDPI

Article

Arbitrary Coefficient Assignment by Static Output Feedback for Linear Differential Equations with Non-Commensurate Lumped and Distributed Delays

Vasilii Zaitsev *,† and Inna Kim †

Laboratory of Mathematical Control Theory, Udmurt State University, 426034 Izhevsk, Russia; kimingeral@gmail.com
* Correspondence: verba@udm.ru
† These authors contributed equally to this work.

Abstract: We consider a linear control system defined by a scalar stationary linear differential equation in the real or complex space with multiple non-commensurate lumped and distributed delays in the state. In the system, the input is a linear combination of multiple variables and its derivatives, and the output is a multidimensional vector of linear combinations of the state and its derivatives. For this system, we study the problem of arbitrary coefficient assignment for the characteristic function by linear static output feedback with lumped and distributed delays. We obtain necessary and sufficient conditions for the solvability of the arbitrary coefficient assignment problem by the static output feedback controller. Corollaries on arbitrary finite spectrum assignment and on stabilization of the system are obtained. We provide an example illustrating our results.

Keywords: linear differential equation; time-delay system; lumped delay; distributed delay; characteristic function coefficient assignment; stabilization; linear static output feedback

check for updates

Citation: Zaitsev, V.; Kim, I. Arbitrary Coefficient Assignment by Static Output Feedback for Linear Differential Equations with Non-Commensurate Lumped and Distributed Delays. *Mathematics* **2021**, *9*, 2158. https://doi.org/10.3390/math9172158

Academic Editor: Juan R. Torregrosa

Received: 2 August 2021
Accepted: 1 September 2021
Published: 4 September 2021

1. Introduction

A large number of works have been devoted to the problem of stability of time-delay systems and problems of stabilization for control systems with delays (see reviews [1–4]). A number of methods have been developed to solve this problem. One of the methods, known as the Lyapunov–Krasovskii functional approach [5], allows one to obtain sufficient conditions for asymptotic and exponential stabilization of delayed systems [6–8]. This method historically traces back to the second Lyapunov method. Another approach to studying problems of stability and stabilization of time-delay systems is an eigenvalue-based approach [9]. This approach traces back to the first Lyapunov method. Here, the goal is to find conditions providing the desired assignment of the spectrum of the system, that is, the set of zeros of the characteristic function of the system.

The classical problem of spectrum assignment (for systems without delays) is usually studied as the problem of coefficient assignment for characteristic function (in another terminology as the problem of modal control) and is as follows: consider a linear time-invariant control system

$$\dot{x} = Ax + Bu, \tag{1}$$

$x \in \mathbb{K}^n$, $u \in \mathbb{K}^m$ (here $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$). Let the controller have the form of linear static state feedback

$$u = Qx. \tag{2}$$

The closed-loop system (1), (2) takes the form

$$\dot{x} = (A + BQ)x. \tag{3}$$

Definition 1. *System* (1) *is called arbitrary spectrum assignable by means of static state feedback* (2) *if, for any $\gamma_i \in \mathbb{K}$, $i = \overline{1,n}$, there exists a gain matrix Q such that the characteristic polynomial of the matrix $A + BQ$ of the system* (3) *coincides with the polynomial*

$$\lambda^n + \gamma_1\lambda^{n-1} + \ldots + \gamma_n. \tag{4}$$

The problem of arbitrary spectrum assignment was solved in [10] (for $\mathbb{K} = \mathbb{C}$) and in [11] (for $\mathbb{K} = \mathbb{R}$)—namely, it has been proven that complete controllability of (1) is a necessary and sufficient condition for arbitrary spectrum assignability (i.e., for modal controllability) of system (3). This property is a sufficient condition for exponential stabilization of the system (with any given decay of rate).

For systems with delays, the spectrum is in general infinite. The spectrum depends on coefficients of the characteristic function. The problem of arbitrary spectrum assignment of linear time-delay systems (in contrast to systems without delays) is not equivalent to the problem of arbitrary coefficient assignment (ACA) for the characteristic function of the closed-loop system. A number of early works dealt with coefficient assignment, spectrum assignment, and stabilizability problems for linear time-delay systems by means of static state feedback with delays: sufficient conditions for ACA were obtained in [12] for systems with multiple lumped delays; decoupling and canonical forms were used for coefficient assignment in [13,14]; stabilizability and spectrum assignment for linear autonomous systems with general time delays were studied in [15]; problems of stabilizability independent of delay were developed for the class of delay differential systems of the retarded type with commensurate time delays in [16]; spectrum placement problem was studied in [17] by using a ring of delay operators.

In a study [18], an approach was developed for assigning an arbitrary *finite* spectrum for linear systems with delays. Later, the finite spectrum assignment problem for time-delay systems by linear state feedback was studied in [19] for systems with one lumped delay in the states with the scalar controller, in [20] for systems with multiple commensurate lumped delays in the states with the scalar controller, in [21] for systems with multiple commensurate lumped delays in the states and control with the scalar controller, in [22] for systems with multiple commensurate lumped delays in the states and control with the multidimensional controller, in [23] for systems with multiple commensurate lumped and distributed delays in the states with the scalar controller, and in [24] for systems of neutral type with multiple commensurate lumped delays in the states with the scalar controller. In [25], the ACA problem was studied for single-input single-output (SISO) systems with commensurate lumped delays in the states by the dynamic output feedback controller. In [26], the problem of stabilization of linear systems with both input and state delays by observer–predictors was studied. In the paper [27], the assignment of the poles of a second-order vibrating system through state feedback with one lumped delay was studied by means of the linear matrix inequality (LMI) approach. A partial pole assignment approach was presented in [28] for second-order systems with time delay.

Some recent important works on stochastic time-delay systems have been reported in [29–31]. In [29], the event-triggered control problem of stochastic nonlinear delay systems with exogenous disturbances and the event-triggered feedback control was studied. In [30], the stability problem for a class of stochastic delay nonlinear systems driven by G-Brownian motion was studied. The global stabilization of stochastic nonlinear systems with time-varying delay, unknown powers, and SISS stochastic inverse dynamics was studied in [31].

The problem of spectrum assignment by static output feedback (for systems without delays) is as follows: consider a linear time-invariant control system

$$\dot{x} = Fx + Gu, \quad y = Hx, \tag{5}$$

$x \in \mathbb{K}^n$, $u \in \mathbb{K}^m$, $y \in \mathbb{K}^k$. Let the controller have the form of linear static output feedback where

$$u = Ky. \tag{6}$$

The closed-loop system (5), (6) takes the form

$$\dot{x} = (F + GKH)x. \tag{7}$$

Definition 2. *System (5) is called arbitrary spectrum assignable by means of static output feedback (6) if, for any $\gamma_i \in \mathbb{K}$, $i = \overline{1,n}$, there exists a gain matrix K such that the characteristic polynomial of the matrix $F + GKH$ of the system (7) coincides with the polynomial (4).*

The static output feedback problem of eigenvalue assignment (in particular, of stabilization) is one of the most important open questions in control theory; see reviews [32,33]. This problem has been studied for over 40 years by various authors. The most significant results have been obtained in [34] for $\mathbb{K} = \mathbb{C}$, in [35–37] for $\mathbb{K} = \mathbb{R}$.

The problems of stabilization and spectrum assignment by static output feedback for time-delay systems are more difficult to study. A study [38] considered the problem of stabilization of system (5) by static output feedback with a delay $u(t) = Ky(t - \tau)$. For SISO systems with delays, necessary conditions for the existence of static output feedback stabilizing controllers were derived in [39]. Another study [40] considered the output feedback stabilization problem for a class of linear SISO systems with I/O network delays. The problem of stabilization of linear time-varying systems with input delays via delayed static output feedback is studied in [41].

Consider a control system defined by a linear differential equation of nth order where the input is a linear combination of m variables and its derivatives of order $\leq n - p$, and the output is a k-dimensional vector of linear combinations of the state x and its derivatives of order $\leq p - 1$ $(1 \leq p \leq n)$:

$$\begin{gathered} x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n x = \\ = b_{p1} u_1^{(n-p)} + b_{p+1,1} u_1^{(n-p-1)} + \ldots + b_{n1} u_1 + \ldots \\ + b_{pm} u_m^{(n-p)} + b_{p+1,m} u_m^{(n-p-1)} + \ldots + b_{nm} u_m, \end{gathered} \tag{8}$$

$$\begin{gathered} y_1 = c_{11} x + c_{21} x' + \ldots + c_{p1} x^{(p-1)}, \quad \ldots, \\ y_k = c_{1k} x + c_{2k} x' + \ldots + c_{pk} x^{(p-1)}. \end{gathered} \tag{9}$$

Here, $x \in \mathbb{K}$ is a state variable, $u_\alpha \in \mathbb{K}$ are control variables, $y_\beta \in \mathbb{K}$ are output variables, a_i, $b_{l\alpha}$, $c_{\nu\beta} \in \mathbb{K}$, $i = \overline{1,n}$, $l = \overline{p,n}$, $\nu = \overline{1,p}$, $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$. Construct the vectors $u = \operatorname{col}(u_1, \ldots, u_m) \in \mathbb{K}^m$, $y = \operatorname{col}(y_1, \ldots, y_k) \in \mathbb{K}^k$. Let the control in system (8), (9) have the form of linear static output feedback:

$$u = Qy. \tag{10}$$

System (8), (9), (10) can be rewritten in the form (5), (6). We say that *system (8), (9) is arbitrary coefficient assignable by linear static output feedback* (10) if for any $\gamma_i \in \mathbb{K}$, $i = \overline{1,n}$, there exists a linear static output feedback control (10) such that the characteristic polynomial of the closed-loop system (8), (9), (10) has the form (4). The conditions imposed on the orders of derivatives in (8) and (9) are natural because one needs the orders of the derivatives on the right-hand side of the closed-loop system to be less than n.

For the scalar system (8), (9), (10), the arbitrary coefficient assignment problem by static output feedback has solved in [42]. Construct the matrices $B = \{b_{l\alpha}\}$, $l = \overline{1,n}$, $\alpha = \overline{1,m}$, and $C = \{c_{\nu\beta}\}$, $\nu = \overline{1,n}$, $\beta = \overline{1,k}$, where $b_{l\alpha} := 0$ for $l < p$ and $c_{\nu\beta} := 0$ for $\nu > p$. Let $J := \{\vartheta_{ij}\} \in M_n(\mathbb{R})$ where $\vartheta_{ij} = 1$ for $j = i + 1$ and $\vartheta_{ij} = 0$ for $j \neq i + 1$. Let T denote the transposition of a matrix. The following theorem holds [42]:

Theorem 1. *System (8), (9) is arbitrary coefficient assignable by linear static output feedback (10) if and only if the matrices*

$$C^T B,\ C^T J B,\ \dots,\ C^T J^{n-1} B$$

are linearly independent.

Extension of Theorem 1 to the case when the state, input, and output are multidimensional (i.e., $x \in \mathbb{K}^s$, $u_\alpha \in \mathbb{K}^s$, $y_\beta \in \mathbb{K}^s$, $s \geq 1$) was obtained in [43].

In the present paper, we extend Theorem 1 on arbitrary coefficient assignment by static output feedback to systems with non-commensurate lumped and distributed delays in the state variable.

Notation. Denote $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$; $\mathbb{K}^n = \{x = \operatorname{col}(x_1, \dots, x_n) \colon x_i \in \mathbb{K}\}$ is the linear space of column vectors over $\mathbb{K}$; $M_{m,n}(\mathbb{K})$ is the space of $m \times n$-matrices over $\mathbb{K}$; $M_n(\mathbb{K}) := M_{n,n}(\mathbb{K})$; $I \in M_n(\mathbb{K})$ is the identity matrix; $\overline{a}$ is the complex conjugation of a; T is the transposition of a vector or a matrix; $*$ is the Hermitian conjugation, i.e., $A^* = \overline{A}^T$; $\operatorname{Sp} H$ is the trace of a matrix $H \in M_n(\mathbb{K})$; for a matrix $H \in M_n(\mathbb{K})$, we use the denotation $H^0 := I$; denote $J := \{\epsilon_{ij}\} \in M_n(\mathbb{R})$ where $\epsilon_{ij} = 1$ for $j = i+1$ and $\epsilon_{ij} = 0$ for $j \neq i+1$; denote by $\operatorname{vec} \colon M_{p,q}(\mathbb{K}) \to \mathbb{K}^{pq}$ the mapping, which "unrolls" a matrix $Z = \{z_{ij}\}$, $i = \overline{1,p}$, $j = \overline{1,q}$, by rows into the column vector $\operatorname{vec} Z = \operatorname{col}(z_{11}, \dots, z_{1q}, \dots, z_{p1}, \dots, z_{pq}) \in \mathbb{K}^{pq}$.

2. Main Results

Consider a control system defined by a linear time-invariant differential equation of nth order with multiple non-commensurate lumped and distributed delays in the state variable $x \in \mathbb{K}$; the input is a linear combination of m variables and its derivatives of order $\leq n - p$; the output is a k-dimensional vector of linear combinations of the state x and its derivatives of order $\leq p - 1$,

$$\begin{aligned}
&x^{(n)}(t) + a_{10}x^{(n-1)}(t) + a_{11}x^{(n-1)}(t-h_1) + \ldots + a_{1s}x^{(n-1)}(t-h_s) + \ldots \\
&+ a_{n0}x(t) + a_{n1}x(t-h_1) + \ldots + a_{ns}x(t-h_s) \\
&+ \sum_{i=1}^{n}\sum_{\eta=1}^{s}\int_{-h_\eta}^{-h_{\eta-1}} g_{i\eta}(\tau)x^{(n-i)}(t+\tau)\,d\tau \\
&= b_{p1}u_1^{(n-p)}(t) + b_{p+1,1}u_1^{(n-p-1)}(t) + \ldots + b_{n1}u_1(t) \\
&+ \ldots + b_{pm}u_m^{(n-p)}(t) + \ldots + b_{nm}u_m(t), \quad t > 0,
\end{aligned} \tag{11}$$

$$\begin{aligned}
y_1(t) &= \overline{c}_{11}x(t) + \overline{c}_{21}x'(t) + \ldots + \overline{c}_{p1}x^{(p-1)}(t), \ldots, \\
y_k(t) &= \overline{c}_{1k}x(t) + \overline{c}_{2k}x'(t) + \ldots + \overline{c}_{pk}x^{(p-1)}(t),
\end{aligned} \tag{12}$$

with initial conditions $x^{(n-i)}(\tau) = \phi_i(\tau)$, $\tau \in [-h_s, 0]$; here $0 = h_0 < h_1 < \ldots < h_s$ are constant delays, $\phi_i \colon [-h_s, 0] \to \mathbb{K}$ are continuous functions; a_{ij}, $b_{l\alpha}$, $c_{\nu\beta} \in \mathbb{K}$, $i = \overline{1,n}$, $j = \overline{0,s}$, $l = \overline{p,n}$, $\alpha = \overline{1,m}$, $\nu = \overline{1,p}$, $\beta = \overline{1,k}$; $g_{i\eta} \colon [-h_\eta, -h_{\eta-1}] \to \mathbb{K}$ are integrable functions ($i = \overline{1,n}$, $\eta = \overline{1,s}$); $u = \operatorname{col}(u_1, \dots, u_m) \in \mathbb{K}^m$ is a control vector and $y = \operatorname{col}(y_1, \dots, y_k) \in \mathbb{K}^k$ is an output vector; $p \in \{\overline{1,n}\}$; the complex conjugation to $c_{\nu\beta}$ is used for convenience of notation (for consistency with previous works).

Let the controller in system (11), (12) have the form of linear static output feedback with lumped and distributed delays:

$$u(t) = \sum_{\rho=0}^{\theta} Q_\rho y(t-\sigma_\rho) + \sum_{\varkappa=1}^{\theta}\int_{-\sigma_\varkappa}^{-\sigma_{\varkappa-1}} R_\varkappa(\tau)y(t+\tau)\,d\tau, \tag{13}$$

$y(t) = 0$, $t < -h_s$. Here $\theta \geq 0$ is an integer, $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_\theta$ are constant delays, $Q_\rho = \{q_{\alpha\beta}^{\rho}\} \in M_{m,k}(\mathbb{K})$ are constant matrices ($\rho = \overline{0,\theta}$), $R_\varkappa(\tau) = \{r_{\alpha\beta}^{\varkappa}(\tau)\} \in M_{m,k}(\mathbb{K})$,

$r^{\varkappa}_{\alpha\beta} : [-\sigma_{\varkappa}, -\sigma_{\varkappa-1}] \to \mathbb{K}$ are integrable functions ($\varkappa = \overline{1,\theta}$), $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$. By (12), we have

$$y_\beta(t) = \sum_{\nu=1}^{p} \bar{c}_{\nu\beta} x^{(\nu-1)}(t), \quad \beta = \overline{1,k}.$$

Hence, for any $\alpha = \overline{1,m}$,

$$u_\alpha(t) = \sum_{\beta=1}^{k} \Big[\sum_{\rho=0}^{\theta} q^{\rho}_{\alpha\beta} \big(\sum_{\nu=1}^{p} \bar{c}_{\nu\beta} x^{(\nu-1)}(t-\sigma_\rho) \big) \\ + \sum_{\varkappa=1}^{\theta} \int_{-\sigma_\varkappa}^{-\sigma_{\varkappa-1}} r^{\varkappa}_{\alpha\beta}(\tau) \big(\sum_{\nu=1}^{p} \bar{c}_{\nu\beta} x^{(\nu-1)}(t+\tau) \big) d\tau \Big].$$

The closed-loop system (11), (12), (13) takes the form

$$\begin{aligned} &x^{(n)}(t) + \sum_{i=1}^{n}\sum_{j=0}^{s} a_{ij} x^{(n-i)}(t-h_j) \\ &+ \sum_{i=1}^{n}\sum_{\eta=1}^{s} \int_{-h_\eta}^{-h_{\eta-1}} g_{i\eta}(\tau) x^{(n-i)}(t+\tau)\, d\tau \\ &- \sum_{\alpha=1}^{m}\sum_{l=p}^{n} b_{l\alpha} \Big(\sum_{\beta=1}^{k} \Big[\sum_{\rho=0}^{\theta} q^{\rho}_{\alpha\beta} \big(\sum_{\nu=1}^{p} \bar{c}_{\nu\beta} x^{(\nu-1)}(t-\sigma_\rho) \big) \\ &+ \sum_{\varkappa=1}^{\theta} \int_{-\sigma_\varkappa}^{-\sigma_{\varkappa-1}} r^{\varkappa}_{\alpha\beta}(\tau) \big(\sum_{\nu=1}^{p} \bar{c}_{\nu\beta} x^{(\nu-1)}(t+\tau) \big)\, d\tau \Big] \Big)^{(n-l)} = 0. \end{aligned} \tag{14}$$

Denote by $\varphi(\lambda)$ the characteristic function of the closed-loop system (14). Then

$$\varphi(\lambda) = \lambda^n + \sum_{i=1}^{n} \lambda^{n-i} \Big(\sum_{j=0}^{s} a_{ij} e^{-\lambda h_j} + \sum_{\eta=1}^{s} \int_{-h_\eta}^{-h_{\eta-1}} g_{i\eta}(\tau) e^{\lambda\tau}\, d\tau \Big) \tag{15}$$
$$- \sum_{\alpha=1}^{m}\sum_{l=p}^{n} b_{l\alpha} \Big(\sum_{\nu=1}^{p} \Big[\sum_{\beta=1}^{k} \big(\sum_{\rho=0}^{\theta} q^{\rho}_{\alpha\beta} \bar{c}_{\nu\beta} e^{-\lambda\sigma_\rho} \big) + \sum_{\varkappa=1}^{\theta} \int_{-\sigma_\varkappa}^{-\sigma_{\varkappa-1}} r^{\varkappa}_{\alpha\beta}(\tau) \bar{c}_{\nu\beta} e^{\lambda\tau}\, d\tau \Big] \lambda^{n-l+\nu-1} \Big).$$

The set $\Lambda = \{\lambda \in \mathbb{C} \colon \varphi(\lambda) = 0\}$ is the spectrum of system (14). If Λ is contained in the open left half-plane, then system (14) is exponentially stable. The spectrum of system (14) is uniquely determined by the coefficients of system (14). We study the problem of assigning an arbitrary coefficients to the characteristic function (15) of the closed-loop system.

Definition 3. *System (11), (12) is said to be arbitrary coefficient assignable by static output feedback (13) if, for any integer $\ell \geq 0$, for any given $0 = \omega_0 < \omega_1 < \ldots < \omega_\ell$, for any numbers $\gamma_{i\mu} \in \mathbb{K}$, $i = \overline{1,n}$, $\mu = \overline{0,\ell}$, and for any integrable functions $\delta_{i\xi} \colon [-\omega_\xi, -\omega_{\xi-1}] \to \mathbb{K}$, $i = \overline{1,n}$, $\xi = \overline{1,\ell}$, there exist an integer $\theta \geq 0$, numbers $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_\theta$, constant matrices $Q_\rho \in M_{m,k}(\mathbb{K})$, $\rho = \overline{0,\theta}$, and integrable matrix functions $R_\varkappa \colon [-\sigma_\varkappa, -\sigma_{\varkappa-1}] \to M_{m,k}(\mathbb{K})$, $\varkappa = \overline{1,\theta}$, such that the characteristic function (15) of the closed-loop system (14) satisfies the equality*

$$\varphi(\lambda) = \lambda^n + \sum_{i=1}^{n} \lambda^{n-i} \Bigg(\sum_{\mu=0}^{\ell} \gamma_{i\mu} e^{-\lambda\omega_\mu} + \sum_{\xi=1}^{\ell} \int_{-\omega_\xi}^{-\omega_{\xi-1}} \delta_{i\xi}(\tau) e^{\lambda\tau} d\tau \Bigg).$$

Remark 1. *The problem of arbitrary coefficient assignment was studied and solved in [44] for system (11), (12), (13) with only lumped delays ($g_{i\eta}(\tau) \equiv 0$, $i = \overline{1,n}$, $\eta = \overline{1,s}$; $R_\varkappa(\tau) \equiv 0 \in M_{m,k}(\mathbb{K})$, $\varkappa = \overline{1,\theta}$); in [45] for systems with only one lumped and one distributed delays ($s = 1$, $\theta = 1$, $h_1 = \sigma_1 = h > 0$); in [46] for systems with multiple commensurate delays ($h_j = jh$, $j = \overline{0,s}$; $\sigma_\rho = \rho h$, $\rho = \overline{0,\theta}$; $h > 0$). Here, we consider a more general case.*

From system (11), (12), construct the matrices $B = \{b_{l\alpha}\}$, $l = \overline{1,n}$, $\alpha = \overline{1,m}$, and $C = \{c_{\nu\beta}\}$, $\nu = \overline{1,n}$, $\beta = \overline{1,k}$, where $b_{l\alpha} := 0$ for $l < p$ and $c_{\nu\beta} := 0$ for $\nu > p$.

Theorem 2. *System (11), (12) is arbitrary coefficient assignable by the static output feedback controller (13) if and only if the matrices*

$$C^*B, \quad C^*JB, \quad ,\ldots, \quad C^*J^{n-1}B \tag{16}$$

are linearly independent.

Proof. Let a function

$$\psi(\lambda) = \lambda^n + \sum_{i=1}^{n} \lambda^{n-i} \left(\sum_{\mu=0}^{\ell} \gamma_{i\mu} e^{-\lambda\omega_\mu} + \sum_{\xi=1}^{\ell} \int_{-\omega_\xi}^{-\omega_{\xi-1}} \delta_{i\xi}(\tau) e^{\lambda\tau} d\tau \right) \tag{17}$$

be given, where $\ell \geq 0$ is an arbitrary integer, $0 = \omega_0 < \omega_1 < \ldots < \omega_\ell$ are arbitrary given delays, $\gamma_{i\mu} \in \mathbb{K}$ are arbitrary numbers, and $\delta_{i\xi} \colon [-\omega_\xi, -\omega_{\xi-1}] \to \mathbb{K}$ are arbitrary integrable functions. One needs to construct a number $\theta \geq 0$, numbers $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_\theta$, matrices $Q_\rho \in M_{m,k}(\mathbb{K})$, $\rho = \overline{0,\theta}$, and integrable functions $R_\varkappa \colon [-\sigma_\varkappa, -\sigma_{\varkappa-1}] \to M_{m,k}(\mathbb{K})$, $\varkappa = \overline{1,\theta}$, such that the characteristic function (15) of the closed-loop system (14) satisfies the equality

$$\varphi(\lambda) = \psi(\lambda). \tag{18}$$

Let us write the characteristic function (15) of the closed-loop system (14) in the form

$$\varphi(\lambda) = \lambda^n + \sum_{i=1}^{n} \lambda^{n-i} \Big(\sum_{j=0}^{s} a_{ij} e^{-\lambda h_j} + \sum_{\eta=1}^{s} \int_{-h_\eta}^{-h_{\eta-1}} g_{i\eta}(\tau) e^{\lambda\tau}\, d\tau \Big) - \Delta, \tag{19}$$

where $\Delta = \Delta_1 + \Delta_2$, and

$$\Delta_1 = \sum_{\rho=0}^{\theta} \sum_{\alpha=1}^{m} \sum_{\beta=1}^{k} \sum_{l=p}^{n} \sum_{\nu=1}^{p} b_{l\alpha} \bar{c}_{\nu\beta} q^{\rho}_{\alpha\beta} \lambda^{n-l+\nu-1} e^{-\lambda\sigma_\rho},$$

$$\Delta_2 = \sum_{\varkappa=1}^{\theta} \sum_{\alpha=1}^{m} \sum_{\beta=1}^{k} \sum_{l=p}^{n} \sum_{\nu=1}^{p} b_{l\alpha} \bar{c}_{\nu\beta} \int_{-\sigma_\varkappa}^{-\sigma_{\varkappa-1}} r^{\varkappa}_{\alpha\beta}(\tau) \lambda^{n-l+\nu-1} e^{\lambda\tau} d\tau.$$

By using the proof of [44] (Theorem 2) (see the reasoning from Formula (11) to Formula (15) in [44]), we obtain that

$$\Delta_1 = \sum_{\rho=0}^{\theta} \sum_{i=1}^{n} \mathrm{Sp}\,(C^* J^{i-1} B Q_\rho) \lambda^{n-i} e^{-\lambda\sigma_\rho}.$$

By the same reasoning, we obtain that

$$\Delta_2 = \sum_{\varkappa=1}^{\theta} \sum_{i=1}^{n} \int_{-\sigma_\varkappa}^{-\sigma_{\varkappa-1}} \mathrm{Sp}\,(C^* J^{i-1} B R_\varkappa(\tau)) \lambda^{n-i} e^{\lambda\tau} d\tau.$$

Substituting Δ_1 and Δ_2 in (19), we obtain

$$\begin{aligned} \varphi(\lambda) &= \lambda^n + \sum_{i=1}^{n} \lambda^{n-i} \Big(\sum_{j=0}^{s} a_{ij} e^{-\lambda h_j} + \sum_{\eta=1}^{s} \int_{-h_\eta}^{-h_{\eta-1}} g_{i\eta}(\tau) e^{\lambda\tau}\, d\tau \Big) \\ &- \sum_{i=1}^{n} \lambda^{n-i} \left(\sum_{\rho=0}^{\theta} \mathrm{Sp}\,(C^* J^{i-1} B Q_\rho) e^{-\lambda\sigma_\rho} + \sum_{\varkappa=1}^{\theta} \int_{-\sigma_\varkappa}^{-\sigma_{\varkappa-1}} \mathrm{Sp}\,(C^* J^{i-1} B R_\varkappa(\tau)) e^{\lambda\tau} d\tau \right). \end{aligned} \tag{20}$$

Taking into account (20), (18), and (17), we obtain that system (11), (12) is arbitrary coefficient assignable by (13) iff there exist $\theta \geq 0$, numbers $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_\theta$, matrices $Q_\rho \in M_{m,k}(\mathbb{K})$, $\rho = \overline{0,\theta}$, and integrable matrix functions $R_\varkappa \colon [-\sigma_\varkappa, -\sigma_{\varkappa-1}] \to M_{m,k}(\mathbb{K})$, $\varkappa = \overline{1,\theta}$, such that for all $i = \overline{1,n}$ the following equalities hold:

$$\sum_{\mu=0}^{\ell} \gamma_{i\mu} e^{-\lambda\omega_\mu} = \sum_{j=0}^{s} a_{ij} e^{-\lambda h_j} - \sum_{\rho=0}^{\theta} \mathrm{Sp}\,(C^* J^{i-1} B Q_\rho) e^{-\lambda\sigma_\rho}, \tag{21}$$

$$\begin{aligned} \sum_{\xi=1}^{\ell} \int_{-\omega_\xi}^{-\omega_{\xi-1}} \delta_{i\xi}(\tau) e^{\lambda\tau} d\tau &= \sum_{\eta=1}^{s} \int_{-h_\eta}^{-h_{\eta-1}} g_{i\eta}(\tau) e^{\lambda\tau} d\tau \\ &- \sum_{\varkappa=1}^{\theta} \int_{-\sigma_\varkappa}^{-\sigma_{\varkappa-1}} \mathrm{Sp}\,\big(C^* J^{i-1} B R_\varkappa(\tau)\big) e^{\lambda\tau} d\tau. \end{aligned} \tag{22}$$

Denote

$$T_1 := \{\omega_1, \ldots, \omega_\ell\}, \quad T_2 := \{h_1, \ldots, h_s\},$$
$$S_1 := \{\mu \in \{\overline{1,\ell}\} \colon \omega_\mu \in T_1 \setminus T_2\}, \quad S_2 := \{\mu \in \{\overline{1,\ell}\} \colon \omega_\mu \in T_1 \cap T_2\},$$
$$S_3 := \{j \in \{\overline{1,s}\} \colon h_j \in T_1 \cap T_2\}, \quad S_4 := \{j \in \{\overline{1,s}\} \colon h_j \in T_2 \setminus T_1\}.$$

Set $T := T_1 \cup T_2$, $\theta := |T|$. Set $\sigma_0 := 0$. Let us denote the elements of the set T as $\sigma_1 < \sigma_2 < \ldots < \sigma_\theta$.

Let

$$K_1 := \{\rho \in \{\overline{1,\theta}\} \colon \exists\, \mu \in S_1 \quad \sigma_\rho = \omega_\mu\},$$
$$K_2 := \{\rho \in \{\overline{1,\theta}\} \colon \exists\, j \in S_3 \quad \sigma_\rho = h_j\},$$
$$K_3 := \{\rho \in \{\overline{1,\theta}\} \colon \exists\, j \in S_4 \quad \sigma_\rho = h_j\}.$$

Then, equalities (21) take the form

$$\begin{aligned} &\gamma_{i0} + \Big(\sum_{\mu\in S_1} + \sum_{\mu\in S_2}\Big) \gamma_{i\mu} e^{-\lambda\omega_\mu} \\ &= a_{i0} - \mathrm{Sp}\,(C^* J^{i-1} B Q_0) + \Big(\sum_{j\in S_3} + \sum_{j\in S_4}\Big) a_{ij} e^{-\lambda h_j} \\ &- \Big(\sum_{\rho\in K_1} + \sum_{\rho\in K_2} + \sum_{\rho\in K_3}\Big) \mathrm{Sp}\,(C^* J^{i-1} B Q_\rho) e^{-\lambda\sigma_\rho}. \end{aligned} \tag{23}$$

Denote $R \colon [-\sigma_\theta, 0] \to M_{m,k}(\mathbb{K})$:

$$R(\tau) := \begin{cases} R_1(\tau), & \tau \in [-\sigma_1, 0], \\ R_2(\tau), & \tau \in [-\sigma_2, -\sigma_1), \\ \ldots\ldots, \\ R_\theta(\tau), & \tau \in [-\sigma_\theta, -\sigma_{\theta-1}). \end{cases}$$

Set

$$\delta_i(\tau) := \begin{cases} \delta_{i1}(\tau), & \tau \in [-\omega_1, 0], \\ \delta_{i2}(\tau), & \tau \in [-\omega_2, -\omega_1), \\ \ldots\ldots, \\ \delta_{i\ell}(\tau), & \tau \in [-\omega_\ell, -\omega_{\ell-1}), \\ 0, & \tau \in [-\sigma_\theta, -\omega_\ell), \end{cases}$$

$$g_i(\tau) := \begin{cases} g_{i1}(\tau), & \tau \in [-h_1, 0], \\ g_{i2}(\tau), & \tau \in [-h_2, -h_1), \\ \ldots\ldots, \\ g_{is}(\tau), & \tau \in [-h_s, -h_{s-1}), \\ 0, & \tau \in [-\sigma_\theta, -h_s), \end{cases}$$

Then, equalities (22) take the form

$$\int_{-\sigma_\theta}^{0} \delta_i(\tau) e^{\lambda\tau} d\tau = \int_{-\sigma_\theta}^{0} g_i(\tau) e^{\lambda\tau} d\tau - \int_{-\sigma_\theta}^{0} \mathrm{Sp}\left(C^* J^{i-1} B R(\tau)\right) e^{\lambda\tau} d\tau. \tag{24}$$

Equalities (23) hold for all $i = \overline{1,n}$ iff for all $i = \overline{1,n}$ the following equalities hold:

$$\begin{aligned} \gamma_{i0} &= a_{i0} - \mathrm{Sp}\,(C^* J^{i-1} B Q_0); \\ \gamma_{i\mu} &= -\,\mathrm{Sp}\,(C^* J^{i-1} B Q_\rho), \quad \mu \in S_1, \quad \rho \in K_1, \quad \sigma_\rho = \omega_\mu; \\ \gamma_{i\mu} &= a_{ij} - \mathrm{Sp}\,(C^* J^{i-1} B Q_\rho), \quad \mu \in S_2, \quad j \in S_3, \quad \rho \in K_2, \quad \sigma_\rho = h_j = \omega_\mu; \\ 0 &= a_{ij} - \mathrm{Sp}\,(C^* J^{i-1} B Q_\rho), \quad j \in S_4, \quad \rho \in K_3, \quad \sigma_\rho = h_j. \end{aligned} \tag{25}$$

Equalities (24) hold for all $i = \overline{1,n}$ iff for a.e. $\tau \in [-\sigma_\theta, 0]$ the following equalities hold:

$$\delta_i(\tau) = g_i(\tau) - \mathrm{Sp}\left(C^* J^{i-1} B R(\tau)\right), \quad i = \overline{1,n}. \tag{26}$$

Every ρth system of (25) consists of n equations with mk unknown entries of the matrix Q_ρ, $\rho = \overline{0,\theta}$. System (26) consists of n equations with mk unknown entries of the matrix function $R(\tau)$, $\tau \in [\sigma_\theta, 0]$. Let us rewrite systems (25), (26) in the vector form. By definition of the mapping vec, we have $\mathrm{Sp}\,(XY) = (\mathrm{vec}\,X)^T \cdot (\mathrm{vec}\,Y^T)$ for any $X \in M_{p,q}(\mathbb{K})$, $Y \in M_{q,p}(\mathbb{K})$. Let us apply this equality in system (25), for every $i = \overline{1,n}$, to the matrix $X = C^* J^{i-1} B$ and to the matrices $Y = Q_\rho$, $\rho = \overline{0,\theta}$, and in system (26), for every $i = \overline{1,n}$, to the matrix $X = C^* J^{i-1} B$ and to $Y = R(\tau)$. Let us construct the $mk \times n$-matrix

$$P = [\mathrm{vec}(C^* B), \mathrm{vec}(C^* J B), \ldots, \mathrm{vec}(C^* J^{n-1} B)]. \tag{27}$$

Denote $v_\rho := \mathrm{vec}\,(Q_\rho^T) \in \mathbb{K}^{mk}$, $\rho = \overline{0,\theta}$, $f(\tau) := \mathrm{vec}\,(R^T(\tau)) \in \mathbb{K}^{mk}$, $\tau \in [-\sigma_\theta, 0]$,

$$\begin{aligned} w_0 &:= \mathrm{col}\,(a_{10} - \gamma_{10}, \ldots, a_{n0} - \gamma_{n0}) \in \mathbb{K}^n; \\ w_\rho &:= \mathrm{col}\,(-\gamma_{1\mu}, \ldots, -\gamma_{n\mu}) \in \mathbb{K}^n, \quad \mu \in S_1, \quad \rho \in K_1, \quad \sigma_\rho = \omega_\mu; \\ w_\rho &:= \mathrm{col}\,(a_{1j} - \gamma_{1\mu}, \ldots, a_{nj} - \gamma_{n\mu}) \in \mathbb{K}^n, \quad \mu \in S_2, \quad j \in S_3, \quad \rho \in K_2, \quad \sigma_\rho = h_j = \omega_\mu; \\ w_\rho &:= \mathrm{col}\,(a_{1j}, \ldots, a_{nj}) \in \mathbb{K}^n, \quad j \in S_4, \quad \rho \in K_3, \quad \sigma_\rho = h_j; \end{aligned}$$

$$\vartheta(\tau) := \mathrm{col}\left(g_1(\tau) - \delta_1(\tau), \ldots, g_n(\tau) - \delta_n(\tau)\right) \in \mathbb{K}^n.$$

Then, one can rewrite systems (25), (26) in the vector form

$$P^T v_\rho = w_\rho, \quad \rho = \overline{0,\theta}, \tag{28}$$

$$P^T f(\tau) = \vartheta(\tau) \ \text{ a.e. } \ \tau \in [-\sigma_\theta, 0]. \tag{29}$$

System (11), (12) is arbitrary coefficient assignable by feedback (13) if and only if system (28), (29) is solvable with respect to v_ρ, $\rho = \overline{0,\theta}$, and $f(\tau)$, $\tau \in [-\sigma_\theta, 0]$, for any $0 = \omega_0 < \omega_1 < \ldots < \omega_\ell$, any numbers $\gamma_{i\mu} \in \mathbb{K}$, $\mu = \overline{0,\ell}$, and any integrable functions $\delta_{i\xi} \colon [-\omega_\xi, -\omega_{\xi-1}] \to \mathbb{K}$, $\xi = \overline{1,\ell}$, $i = \overline{1,n}$. The condition of linear independency of the matrices (16) is necessary and sufficient for solvability of system (28), (29). In that case, system (28), (29) has the particular solution

$$v_\rho = P(P^T P)^{-1} w_\rho, \quad \rho = \overline{0,\theta}, \tag{30}$$

$$f(\tau) = P(P^T P)^{-1} \vartheta(\tau), \quad \tau \in [-\sigma_\theta, 0]. \tag{31}$$

The required matrices Q_ρ, $\rho = \overline{0,\theta}$, and $R(\tau)$, $\tau \in [-\sigma_\theta, 0]$, can be found from the equalities

$$Q_\rho = (\mathrm{vec}^{-1} v_\rho)^T, \; \rho = \overline{0,\theta}, \qquad R(\tau) = (\mathrm{vec}^{-1} f(\tau))^T.$$

□

3. Corollaries

If the characteristic function of the closed-loop system (14) turns into a polynomial, then the spectrum Λ of system (14) is finite. We say that system (11), (12) is *arbitrary finite spectrum assignable by linear static output feedback* (13) if for any $\zeta_i \in \mathbb{K}$, $i = \overline{1,n}$, there exist an integer $\theta \geq 0$, numbers $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_\theta$, constant matrices $Q_\rho \in M_{m,k}(\mathbb{K})$, $\rho = \overline{0,\theta}$, and integrable matrix functions $R_\varkappa \colon [-\sigma_\varkappa, -\sigma_{\varkappa-1}] \to M_{m,k}(\mathbb{K})$, $\varkappa = \overline{1,\theta}$, such that the characteristic function (15) of the closed-loop system (14) satisfies the equality

$$\varphi(\lambda) = \lambda^n + \zeta_1 \lambda^{n-1} + \ldots + \zeta_n.$$

Corollary 1. *System* (11), (12) *is arbitrary finite spectrum assignable by linear static output feedback* (13) *if and only if the matrices* (16) *are linearly independent.*

The proof of Corollary 1 follows from the proof of Theorem 2: the problem under consideration is equivalent to solvability of system (28), (29), where $\ell = 0$, $\gamma_{i0} = \zeta_i$ ($i = \overline{1,n}$), $T_1 = \varnothing$, $T_2 = \{h_1, \ldots, h_s\}$, $S_1 = S_2 = S_3 = \varnothing$, $S_4 = \{\overline{1,s}\}$, $\theta = s$, $\sigma_\rho = h_\rho$ ($\rho = \overline{1,s}$), $K_1 = K_2 = \varnothing$, $K_3 = S_4$.

Corollary 2. *System* (11), (12) *is exponentially stabilizable with an arbitrary pregiven decay rate by linear static output feedback* (13) *if the matrices* (16) *are linearly independent.*

Corollary 2 follows from Corollary 1.

Consider system (11), (12) containing only lumped delays, i.e., suppose that

$$g_{i\eta}(\tau) \equiv 0, \; \tau \in [-h_\eta, -h_{\eta-1}], \; i = \overline{1,n}, \; \eta = \overline{1,s}. \tag{32}$$

Let the controller (13) also contains only lumped delays, i.e.,

$$R_\varkappa(\tau) \equiv 0, \quad \tau \in [-\sigma_\varkappa, -\sigma_{\varkappa-1}], \quad \varkappa = \overline{1,\theta}. \tag{33}$$

In this case, the closed-loop system (14) does not contain distributed delays. Under conditions (32) and (33), the statements of the problem is as follows: system (11), (12) is said to be *arbitrary coefficient assignable by static output feedback* (13) if, for any integer $\ell \geq 0$, for any given $0 = \omega_0 < \omega_1 < \ldots < \omega_\ell$, and for any numbers $\gamma_{i\mu} \in \mathbb{K}$, $i = \overline{1,n}$, $\mu = \overline{0,\ell}$, there exist an integer $\theta \geq 0$, numbers $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_\theta$, and constant matrices $Q_\rho \in M_{m,k}(\mathbb{K})$, $\rho = \overline{0,\theta}$, such that the characteristic function (15) of the closed-loop system (14) satisfies the equality

$$\varphi(\lambda) = \lambda^n + \sum_{i=1}^{n} \sum_{\mu=0}^{\ell} \gamma_{i\mu} \lambda^{n-i} e^{-\lambda \omega_\mu}$$

Corollary 3. *Under conditions* (32) *and* (33)*, system* (11)*,* (12) *is arbitrary coefficient assignable by linear static output feedback* (13) *if and only if the matrices* (16) *are linearly independent.*

The proof of Corollary 3 repeats the proof of Theorem 2, under conditions (32), (33) and condition $\delta_i(\tau) \equiv 0$, $\tau \in [-\omega_\ell, 0]$, $i = \overline{1,n}$. In fact, this proof was carried out in [44] (Theorem 2). So, Theorem 2 is an extension of [44] (Theorem 2) from systems with only lumped delays to systems with lumped and distributed delays.

Suppose that the delays in system (11), (12) and in feedback (13) are commensurate, i.e., for some $h > 0$,

$$h_j = jh, \quad j = \overline{0,s}, \tag{34}$$

$$\sigma_\rho = \rho h, \quad \rho = \overline{0,\theta}. \tag{35}$$

In this case, the closed-loop system (14) contains only commensurate delays. Under conditions (34) and (35), the statements of the problem is as follows: system (11), (12) is said to be *arbitrary coefficient assignable by static output feedback* (13) if, for any integer $\ell \geq 0$, for any numbers $\gamma_{i\mu} \in \mathbb{K}$, $i = \overline{1,n}$, $\mu = \overline{0,\ell}$, and for any integrable functions $\delta_{i\xi}\colon [-\xi h, -(\xi-1)h] \to \mathbb{K}$, $i = \overline{1,n}$, $\xi = \overline{1,\ell}$, there exist an integer $\theta \geq 0$, constant matrices $Q_\rho \in M_{m,k}(\mathbb{K})$, $\rho = \overline{0,\theta}$, and integrable matrix functions $R_\varkappa\colon [-\varkappa h, -(\varkappa-1)h] \to M_{m,k}(\mathbb{K})$, $\varkappa = \overline{1,\theta}$, such that the characteristic function (15) of the closed-loop system (14) satisfies the equality

$$\varphi(\lambda) = \lambda^n + \sum_{i=1}^{n} \lambda^{n-i} \left(\sum_{\mu=0}^{\ell} \gamma_{i\mu} e^{-\lambda\mu h} + \sum_{\xi=1}^{\ell} \int_{-\xi h}^{-(\xi-1)h} \delta_{i\xi}(\tau) e^{\lambda\tau} d\tau \right). \tag{36}$$

Corollary 4. *Under conditions* (34) *and* (35)*, system* (11)*,* (12) *is arbitrary coefficient assignable by linear static output feedback* (13) *if and only if the matrices* (16) *are linearly independent.*

The proof of Corollary 4 repeats the proof of Theorem 2, under conditions (34), (35) and condition $\omega_\mu = \mu h$, $\mu = \overline{0,\ell}$. In fact, this proof was carried out in [46] (Theorem 1). So, Theorem 2 is an extension of [46] (Theorem 1) from systems with commensurate delays to systems with non-commensurate delays.

Remark 2. *Let us indicate to differences between systems with commensurate and non-commensurate delays. In systems with commensurate delays, one needs, for a given triplet*

$$\mathcal{T}_1 = \Big(\ell, \{\gamma_{i\mu},\, i = \overline{1,n},\, \mu = \overline{0,\ell}\}, \{\delta_{i\xi}(\cdot),\, i = \overline{1,n},\, \xi = \overline{1,\ell}\}\Big),$$

to construct a triplet

$$\mathcal{T}_2 = \Big(\theta, \{Q_\rho,\, \rho = \overline{0,\theta}\}, \{R_\varkappa(\cdot),\, \varkappa = \overline{1,\theta}\}\Big)$$

ensuring (36)*, while in systems with non-commensurate delays, one needs, for a given quadruple*

$$\mathfrak{Q}_1 = \Big(\ell, \{\omega_\mu,\, \mu = \overline{0,\ell}\}, \{\gamma_{i\mu},\, i = \overline{1,n},\, \mu = \overline{0,\ell}\}, \{\delta_{i\xi}(\cdot),\, i = \overline{1,n},\, \xi = \overline{1,\ell}\}\Big),$$

to construct a quadruple

$$\mathfrak{Q}_2 = \Big(\theta, \{\sigma_\rho,\, \rho = \overline{0,\theta}\}, \{Q_\rho,\, \rho = \overline{0,\theta}\}, \{R_\varkappa(\cdot),\, \varkappa = \overline{1,\theta}\}\Big),$$

ensuring the equality from Definition 3. Thus, the problem statements are different. Corollary 4, for systems with commensurate delays, was proved in [46] (Theorem 1). Here we prove a more general result. Difference and difficulty here, with respect to [46], is in choosing the required numbers σ_ρ. The proof given in [46] does not provide an algorithm for constructing the indicated numbers σ_ρ and *the corresponding gain coefficients Q_ρ and $R_\varkappa(\cdot)$. Here, we overcome these difficulties. It provides the novelty of the results obtained.*

4. Modeling Example

Let $h_1 = 1$, $h_2 = \sqrt{2}$. Consider the system

$$\begin{aligned} & x'''(t) + x''(t-h_1) + x''(t-h_2) + x'(t) + x(t) \\ & + x(t-h_1) - x(t-h_2) + \int_{-h_1}^{0} x''(t+\tau)\cos\tau\,d\tau \\ & - 2\int_{-h_2}^{-h_1} x'(t+\tau)\sin\tau\,d\tau - \int_{-h_1}^{0} x(t+\tau)\cos 2\tau\,d\tau \\ & = u_1'(t) + u_1(t) + u_2'(t), \end{aligned} \tag{37}$$

$$y_1(t) = -x(t) - x'(t), \quad y_2(t) = x'(t), \tag{38}$$

$x \in \mathbb{R}$, $u = \mathrm{col}\,(u_1, u_2) \in \mathbb{R}^2$, $y = \mathrm{col}\,(y_1, y_2) \in \mathbb{R}^2$. System (37), (38) has the form (11), (12) where $n = 3$, $m = 2$, $k = 2$, $p = 2$, $s = 2$;

$$\begin{gathered} a_{10} = 0,\ a_{11} = 1,\ a_{12} = 1,\ a_{20} = 1,\ a_{21} = 0,\ a_{22} = 0, \\ a_{30} = 1,\ a_{31} = 1,\ a_{32} = -1; \\ g_{11}(\tau) = \cos\tau,\ g_{12}(\tau) = 0,\ g_{21}(\tau) = 0,\ g_{22}(\tau) = -2\sin\tau, \\ g_{31}(\tau) = -\cos 2\tau,\ g_{32}(\tau) = 0; \\ b_{21} = 1,\ \ b_{22} = 1,\ \ b_{31} = 1,\ \ b_{32} = 0; \\ c_{11} = -1,\ \ c_{21} = -1,\ \ c_{12} = 0,\ \ c_{22} = 1. \end{gathered}$$

From system (37), (38), construct the matrices B, C: we obtain $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$. Hence,

$$C^*B = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \quad C^*JB = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}, \quad C^*J^2B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{39}$$

Construct the matrix (27):

$$P = \begin{bmatrix} -1 & -2 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We have rank $P = 3 = n$, hence, the matrices (39) are linearly independent. Therefore, by Theorem 2, system (37), (38) is arbitrary coefficient assignable by linear static output feedback (13). Let us construct such a controller. Suppose, for example, that $\ell = 2$, $\omega_1 = 1$, $\omega_2 = \sqrt{3}$, and

$$\begin{aligned} \varphi(\lambda) = \lambda^3 + \lambda^2 \Big(2 + e^{-\lambda\omega_2} - \int_{-\omega_1}^{0} e^{\lambda\tau} (\sin\tau - \cos\tau)\,d\tau \Big) \\ + \lambda \Big(1 + 2e^{-\lambda\omega_1} - \int_{-\omega_2}^{-\omega_1} e^{\lambda\tau} \sin 2\tau\,d\tau \Big) + e^{-\lambda\omega_1}. \end{aligned} \tag{40}$$

Then

$$\begin{gathered}
T_1 = \{\omega_1, \omega_2\} = \{1, \sqrt{3}\}, \quad T_2 = \{h_1, h_2\} = \{1, \sqrt{2}\}; \\
\gamma_{10} = 2, \quad \gamma_{11} = 0, \quad \gamma_{12} = 1, \quad \gamma_{20} = 1, \quad \gamma_{21} = 2, \\
\gamma_{22} = 0, \quad \gamma_{30} = 0, \quad \gamma_{31} = 1, \quad \gamma_{32} = 0; \\
\delta_{11}(\tau) = \cos\tau - \sin\tau, \quad \delta_{12}(\tau) = 0, \quad \delta_{21}(\tau) = 0, \\
\delta_{22}(\tau) = -\sin 2\tau, \quad \delta_{31}(\tau) = 0, \quad \delta_{32}(\tau) = 0.
\end{gathered}$$

From the proof of Theorem 2, we obtain

$$\begin{gathered}
S_1 = \{2\}, \quad S_2 = \{1\}, \quad S_3 = \{1\}, \quad S_4 = \{2\}, \\
T = \{h_1, h_2, \omega_2\} = \{1, \sqrt{2}, \sqrt{3}\}, \quad \theta = 3, \\
\sigma_1 = h_1 = \omega_1 = 1, \quad \sigma_2 = h_2 = \sqrt{2}, \quad \sigma_3 = \omega_2 = \sqrt{3}, \\
K_1 = \{3\}, \quad K_2 = \{1\}, \quad K_3 = \{2\}.
\end{gathered}$$

We have

$$g_1(\tau) = \begin{cases} \cos\tau, & \tau \in [-\sigma_1, 0], \\ 0, & \tau \in [-\sigma_2, -\sigma_1), \\ 0, & \tau \in [-\sigma_3, -\sigma_2), \end{cases}$$

$$g_2(\tau) = \begin{cases} 0, & \tau \in [-\sigma_1, 0], \\ -2\sin\tau, & \tau \in [-\sigma_2, -\sigma_1), \\ 0, & \tau \in [-\sigma_3, -\sigma_2), \end{cases}$$

$$g_3(\tau) = \begin{cases} -\cos 2\tau, & \tau \in [-\sigma_1, 0], \\ 0, & \tau \in [-\sigma_2, -\sigma_1), \\ 0, & \tau \in [-\sigma_3, -\sigma_2), \end{cases}$$

$$\delta_1(\tau) = \begin{cases} \cos\tau - \sin\tau, & \tau \in [-\sigma_1, 0], \\ 0, & \tau \in [-\sigma_2, -\sigma_1), \\ 0, & \tau \in [-\sigma_3, -\sigma_2), \end{cases}$$

$$\delta_2(\tau) = \begin{cases} 0, & \tau \in [-\sigma_1, 0], \\ -\sin 2\tau, & \tau \in [-\sigma_2, -\sigma_1), \\ -\sin 2\tau, & \tau \in [-\sigma_3, -\sigma_2), \end{cases}$$

$$\delta_3(\tau) = \begin{cases} 0, & \tau \in [-\sigma_1, 0], \\ 0, & \tau \in [-\sigma_2, -\sigma_1), \\ 0, & \tau \in [-\sigma_3, -\sigma_2). \end{cases}$$

Next, we have

$$\begin{aligned}
w_0 &= \operatorname{col}(a_{10} - \gamma_{10}, a_{20} - \gamma_{20}, a_{30} - \gamma_{30}) = (-2, 0, 1), \\
w_1 &= \operatorname{col}(a_{11} - \gamma_{11}, a_{21} - \gamma_{21}, a_{31} - \gamma_{31}) = (1, -2, 0), \\
w_2 &= \operatorname{col}(a_{12}, a_{22}, a_{32}) = (1, 0, -1), \\
w_3 &= \operatorname{col}(-\gamma_{12}, -\gamma_{22}, -\gamma_{32}) = (-1, 0, 0), \\
\vartheta(\tau) &= \operatorname{col}\big(g_1(\tau) - \delta_1(\tau), g_2(\tau) - \delta_2(\tau), g_3(\tau) - \delta_3(\tau)\big) \\
&= \begin{cases} \operatorname{col}(\sin\tau, 0, -\cos 2\tau), & \tau \in [-\sigma_1, 0], \\ \operatorname{col}(0, -2\sin\tau + \sin 2\tau, 0), & \tau \in [-\sigma_2, -\sigma_1), \\ \operatorname{col}(0, \sin 2\tau, 0), & \tau \in [-\sigma_3, -\sigma_2). \end{cases}
\end{aligned}$$

Calculating v_0, v_1, v_2, v_3, and $f(\tau)$ by formulas (30), (31), we obtain

$$v_0 = \operatorname{col}(-1,1,-1,-1), \qquad v_1 = \operatorname{col}(0,1,-1,3),$$
$$v_2 = \operatorname{col}(1,-1,1,0), \qquad v_3 = \operatorname{col}(0,0,0,-1);$$

$$f(\tau) = \begin{cases} \operatorname{col}(\cos 2\tau, -\cos 2\tau, \cos 2\tau, \sin\tau - \cos 2\tau), \\ \tau \in [-\sigma_1, 0], \\ \operatorname{col}(0, \sin\tau - \sin\tau\cos\tau, \\ -\sin\tau + \sin\tau\cos\tau, 2\sin\tau - \sin 2\tau), \\ \tau \in [-\sigma_2, -\sigma_1), \\ \operatorname{col}(0, -\sin\tau\cos\tau, \sin\tau\cos\tau, -\sin 2\tau), \\ \tau \in [-\sigma_3, -\sigma_2). \end{cases}$$

From this, it follows that

$$Q_0 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$R(\tau) = \begin{cases} \begin{bmatrix} \cos 2\tau & \cos 2\tau \\ -\cos 2\tau & \sin\tau - \cos 2\tau \end{bmatrix}, \\ \tau \in [-\sigma_1, 0], \\ \begin{bmatrix} 0 & -\sin\tau + \sin\tau\cos\tau \\ \sin\tau - \sin\tau\cos\tau & 2\sin\tau - \sin 2\tau \end{bmatrix}, \\ \tau \in [-\sigma_2, -\sigma_1), \\ \begin{bmatrix} 0 & \sin\tau\cos\tau \\ -\sin\tau\cos\tau & -\sin 2\tau \end{bmatrix}, \\ \tau \in [-\sigma_3, -\sigma_2). \end{cases}$$

The controller (13)

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = Q_0 \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + Q_1 \begin{bmatrix} y_1(t-\sigma_1) \\ y_2(t-\sigma_1) \end{bmatrix}$$
$$+ Q_2 \begin{bmatrix} y_1(t-\sigma_2) \\ y_2(t-\sigma_2) \end{bmatrix} + Q_3 \begin{bmatrix} y_1(t-\sigma_3) \\ y_2(t-\sigma_3) \end{bmatrix} + \int_{-\sigma_3}^{0} R(\tau) y(t+\tau) d\tau \tag{41}$$

has the components

$$u_1(t) = -x'(t-h_1) + x(t) - x(t-h_2) - \int_{-h_1}^{0} x(t+\tau)\cos 2\tau\, d\tau$$
$$+ \int_{-h_2}^{-h_1} x'(t+\tau)(\sin\tau\cos\tau - \sin\tau)\, d\tau + \int_{-\omega_2}^{-h_2} x'(t+\tau)\sin\tau\cos\tau\, d\tau,$$
$$u_2(t) = -2x'(t) + 2x'(t-h_1) + x'(t-h_2) - x'(t-\omega_2)$$
$$- x(t) - x(t-h_1) + x(t-h_2) + \int_{-h_1}^{0} x(t+\tau)\cos 2\tau\, d\tau + \int_{-h_1}^{0} x'(t+\tau)\sin\tau\, d\tau$$
$$+ \int_{-h_2}^{-h_1} x(t+\tau)(\sin\tau\cos\tau - \sin\tau)\, d\tau + \int_{-h_2}^{-h_1} x'(t+\tau)(\sin\tau - \sin\tau\cos\tau)\, d\tau$$
$$+ \int_{-\omega_2}^{-h_2} x(t+\tau)\sin\tau\cos\tau\, d\tau - \int_{-\omega_2}^{-h_2} x'(t+\tau)\sin\tau\cos\tau\, d\tau.$$

System (37), (38) closed-loop by feedback (41) takes the form

$$\begin{aligned} &x'''(t) + 2x''(t) + x''(t-\omega_2) + x'(t) + 2x'(t-\omega_1) \\ &+ x(t-\omega_1) - \int_{-\omega_1}^{0} x''(t+\tau)(\sin\tau - \cos\tau)\,d\tau - \int_{-\omega_2}^{-\omega_1} x'(t+\tau)\sin 2\tau\,d\tau = 0. \end{aligned} \tag{42}$$

The characteristic function of system (42) is equal to (40). In particular, system (42) (with $\omega_1 = 1, \omega_2 = \sqrt{3}$) is exponentially stable. This is confirmed by Figures 1 and 2. Figure 1 shows the spectrum of system (42) with $\omega_1 = 1$, $\omega_2 = \sqrt{3}$. The spectrum is in the left half-plane. Figure 2 shows solutions of system (42) with $\omega_1 = 1$, $\omega_2 = \sqrt{3}$, with the initial functions $x(\tau) = 1$ (blue plot), $x(\tau) = \tau$ (violet plot), $x(\tau) = \tau^2$ (green plot) for all $\tau \in [-\sqrt{2}, 0]$. Solutions tend to zero exponentially.

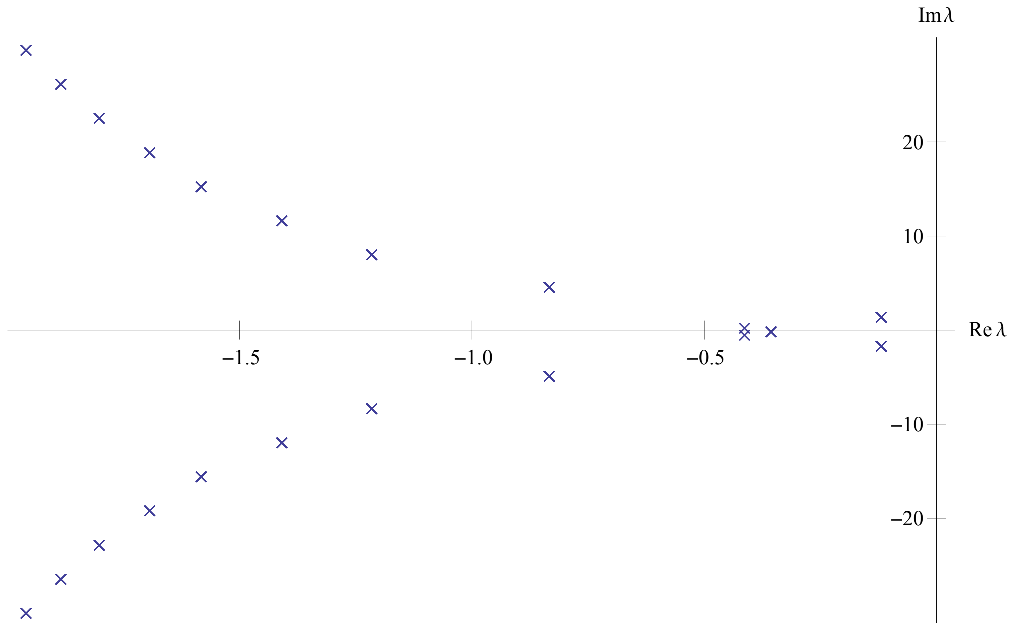

Figure 1. The spectrum of system (42) with $\omega_1 = 1, \omega_2 = \sqrt{3}$.

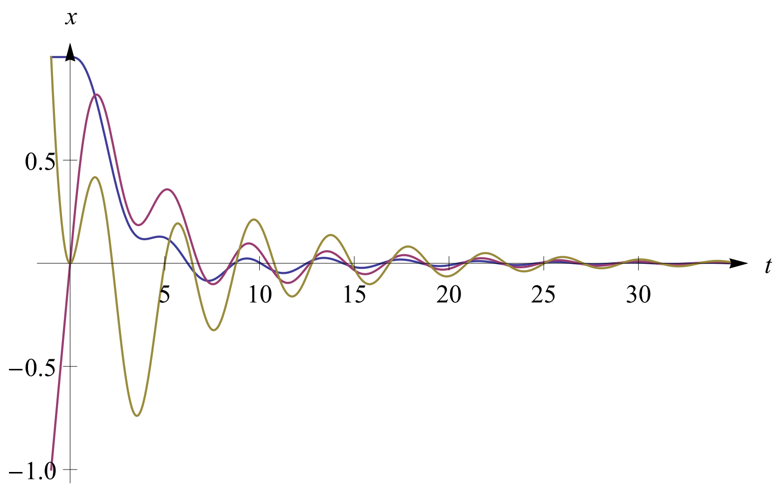

Figure 2. Solutions of system (42) with $\omega_1 = 1, \omega_2 = \sqrt{3}$.

5. Conclusions

Necessary and sufficient conditions were obtained for the problem of arbitrary coefficient assignment for the characteristic function of the closed-loop system by static output feedback for a linear differential equation with non-commensurate lumped and distributed delays. The obtained results extend the earlier corresponding results for systems with commensurate delays and for systems with only lumped delays. Corollaries on arbitrary finite spectrum assignment and on stabilization were stated. We provided an example illustrating our results. In future works, we expect to extend these results to control systems defined by non-scalar systems of differential equations. Moreover, this approach could be applied to problems of stabilization by static output feedback for linear quasi-differential equations and for nonlinear differential equations with delays based on linear approximation.

Author Contributions: Conceptualization, V.Z. and I.K.; methodology, V.Z. and I.K.; software, V.Z. and I.K.; validation, V.Z. and I.K.; formal analysis, V.Z. and I.K.; investigation, V.Z. and I.K.; resources, V.Z. and I.K.; data curation, V.Z. and I.K.; writing—original draft preparation, V.Z. and I.K.; writing—review and editing, V.Z. and I.K.; visualization, V.Z. and I.K.; supervision, V.Z.; project administration, V.Z.; funding acquisition, V.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Ministry of Science and Higher Education of the Russian Federation in the framework of state assignment no. 075-00928-21-01 (project FEWS-2020-0010) and by the Russian Foundation for Basic Research under Grant 20-01-00293.

Acknowledgments: The research was performed using computing resources of the collective use center of the Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences "Supercomputer center of IMM UB RAS".

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Gu, K.; Niculescu, S.I. Survey on recent results in the stability and control of time-delay systems. *J. Dyn. Syst. Meas. Control* **2003**, *125*, 158–165. [CrossRef]
2. Richard, J.-P. Time-delay systems: An overview of some recent advances and open problems. *Automatica* **2003**, *39*, 1667–1694. [CrossRef]
3. Sipahi, R.; Niculescu, S.; Abdallah, C.T.; Michiels, W.; Gu, K. Stability and stabilization of systems with time delay. *IEEE Control Syst.* **2011**, *31*, 38–65. [CrossRef]
4. Pekař, L.; Gao, Q. Spectrum analysis of LTI continuous-time systems with constant delays: A literature overview of some recent results. *IEEE Access* **2018**, *6*, 35457–35491. [CrossRef]
5. Krasovskii, N.-N. *Stability of Motion: Applications of Lyapunov's Second Method to Differential Systems and Equations With Delay*; Stanford Univerity Press: Stanford, CA, USA, 1964.
6. Kharitonov, V.L.; Zhabko, A.P. Lyapunov–Krasovskii approach to the robust stability analysis of time-delay systems. *Automatica* **2003**, *39*, 15–20. [CrossRef]
7. Kharitonov, V.L. *Time-Delay Systems*; Birkäuser: Boston, MA, USA, 2013. [CrossRef]
8. Egorov, A.V.; Cuvas, C.; Mondié, S. Necessary and sufficient stability conditions for linear systems with pointwise and distributed delays. *Automatica* **2017**, *80*, 218–224. [CrossRef]
9. Michiels, W.; Niculescu, S.I. *Stability and Stabilization of Time-Delay Systems. An Eigenvalue-Based Approach*; SIAM: Philadelphia, PA, USA, 2007. [CrossRef]
10. Popov, V.M. Hyperstability and optimality of automatic systems with several control functions. *Rev. Roumaine Sci. Tech. Ser. Electrotech. Energy* **1964**, *9*, 629–690.
11. Wonham, W.M. On pole assignment in multi-input controllable linear systems. *IEEE Trans. Automat. Control* **1967**, *12*, 660–665. [CrossRef]
12. Asmykovich, I.K.; Marchenko, V.M. Modal control of multiinput linear delayed systems. *Autom. Remote Control* **1980**, *41*, 1–5.
13. Kono, M. Decoupling and arbitrary coefficient assignment in time-delay systems. *Syst. Control Lett.* **1983**, *3*, 349–354. [CrossRef]
14. Lee, E.; Lu, W. Coefficient assignability for linear systems with delays. *IEEE Trans. Automat. Control* **1984**, *29*, 1048–1052. [CrossRef]
15. Olbrot, A. Stabilizability, detectability, and spectrum assignment for linear autonomous systems with general time delays. *IEEE Trans. Automat. Control* **1978**, *23*, 887–890. [CrossRef]
16. Kamen, E. Linear systems with commensurate time delays: Stability and stabilization independent of delay. *IEEE Trans. Automat. Control* **1982**, *27*, 367–375. [CrossRef]

17. Lee, E.; Zak, S. On spectrum placement for linear time invariant delay systems. *IEEE Trans. Automat. Control* **1982**, *27*, 446–449. [CrossRef]
18. Manitius, A.; Olbrot, A. Finite spectrum assignment problem for systems with delays. *IEEE Trans. Automat. Control* **1979**, *24*, 541–552. [CrossRef]
19. Watanabe, K.; Ito, M.; Kaneko, M.; Ouchi, T. Finite spectrum assignment problem for systems with delay in state variables. *IEEE Trans. Automat. Control* **1983**, *28*, 506–508. [CrossRef]
20. Watanabe, K.; Ito, M.; Kaneko, M. Finite spectrum assignment problem for systems with multiple commensurate delays in state variables. *Int. J. Control* **1983**, *38*, 913–926. [CrossRef]
21. Watanabe, K.; Ito, M.; Kaneko, M. Finite spectrum assignment problem of systems with multiple commensurate delays in states and control. *Int. J. Control* **1984**, *39*, 1073–1082. [CrossRef]
22. Watanabe, K. Finite spectrum assignment and observer for multivariable systems with commensurate delays. *IEEE Trans. Automat. Control* **1986**, *31*, 543–550. [CrossRef]
23. Metel'skii, A.V. Finite spectrum assignment problem for a delay type system. *Differ. Equ.* **2014**, *50*, 689–699. [CrossRef]
24. Metel'skii, A.V. Finite spectrum assignment problem for a differential system of neutral type. *Differ. Equ.* **2015**, *51*, 69–82. [CrossRef]
25. Metel'skii, A.V. Modal controllability of a delay differential system by an incomplete output. *Differ. Equ.* **2018**, *54*, 1483–1493. [CrossRef]
26. Zhou, B.; Liu, Q.; Mazenc, F. Stabilization of linear systems with both input and state delays by observer–predictors. *Automatica* **2017**, *83*, 368–377. [CrossRef]
27. Belotti, R.; Richiedei, D. Pole assignment in vibrating systems with time delay: An approach embedding an a priori stability condition based on Linear Matrix Inequality. *Mech. Syst. Signal Process.* **2020**, *137*, 106396. [CrossRef]
28. Dantas, N.J.B.; Dorea, C.E.T.; Araujo, J.M. Partial pole assignment using rank-one control and receptance in second-order systems with time delay. *Meccanica* **2021**, *56*, 287–302. [CrossRef]
29. Zhu, Q. Stabilization of stochastic nonlinear delay systems with exogenous disturbances and the event-triggered feedback control. *IEEE Trans. Automat. Control* **2019**, *64*, 3764–3771. [CrossRef]
30. Zhu, Q.; Huang, T. Stability analysis for a class of stochastic delay nonlinear systems driven by G-Brownian motion. *Syst. Control Lett.* **2020**, *140*, 104699. [CrossRef]
31. Wang, H.; Zhu, Q. Global stabilization of a class of stochastic nonlinear time-delay systems with SISS inverse dynamics. *IEEE Trans. Automat. Control* **2020**, *65*, 4448–4455. [CrossRef]
32. Syrmos, V.L.; Abdallah, C.T.; Dorato, P.; Grigoriadis, K. Static output feedback–A survey. *Automatica* **1997**, *33*, 125–137. [CrossRef]
33. Sadabadi, M.S.; Peaucelle, D. From static output feedback to structured robust static output feedback: A survey. *Annu. Rev. Control* **2016**, *42*, 11–26. [CrossRef]
34. Brockett, R.; Byrnes, C. Multivariable Nyquist criteria, root loci, and pole placement: A geometric viewpoint. *IEEE Trans. Automat. Control* **1981**, *26*, 271–284. [CrossRef]
35. Wang, X. Pole placement by static output feedback. *J. Math. Syst., Estim. Control* **1992**, *2*, 205–218.
36. Wang, X.A. Grassmannian, central projection, and output feedback pole assignment of linear systems. *IEEE Trans. Automat. Control* **1996**, *41*, 786–794. [CrossRef]
37. Rosenthal, J.; Schumacher, J.M.; Willems, J.C. Generic eigenvalue assignment by memoryless real output feedback. *Syst. Control Lett.* **1995**, *26*, 253–260. [CrossRef]
38. Niculescu, S.I.; Abdallah, C.T. Delay effects on static output feedback stabilization. In Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No.00CH37187), Sydney, NSW, Australia, 12–15 December 2000; pp. 2811–2816. [CrossRef]
39. Kharitonov, V.L.; Niculescu, S.I.; Moreno, J.; Michiels, W. Static output feedback stabilization: Necessary conditions for multiple delay controllers. *IEEE Trans. Automat. Control* **2005**, *50*, 82–86. [CrossRef]
40. Méndez-Barrios, C.-F.; Niculescu, S.I.; Chen, J.; Maya-Méndez, M. Output feedback stabilisation of single-input single-output linear systems with I/O network-induced delays. An eigenvalue-based approach. *Int. J. Control* **2013**, *87*, 346–362. [CrossRef]
41. Mazenc, F.; Niculescu, S.; Chen, J.; Bekiaris-Liberis, N. Asymptotic stabilization of linear time-varying systems with input delays via delayed static output feedback. In Proceedings of the 2015 American Control Conference (ACC), Chicago, IL, USA, 1–3 July 2015. [CrossRef]
42. Zaitsev, V.A. Modal control of a linear differential equation with incomplete feedback. *Differ. Equ.* **2003**, *39*, 145–148. [CrossRef]
43. Zaitsev, V.; Kim, I. Matrix eigenvalue spectrum assignment for linear control systems by static output feedback. *Linear Algebra Appl.* **2021**, *613*, 115–150. [CrossRef]
44. Zaitsev, V.A.; Kim, I.G. Arbitrary spectrum assignment by static output feedback for linear differential equations with state variable delays. *IFAC-PapersOnLine* **2018**, *51*, 810–814. [CrossRef]
45. Zaitsev, V.A.; Kim, I.G. Spectrum assignment and stabilization of linear differential equations with delay by static output feedback with delay. *Vestn. Udmurt. Univ. Mat. Mekh. Komp'yut. Nauki* **2020**, *30*, 208–220. [CrossRef]
46. Zaitsev, V.A.; Kim, I.G. Spectrum assignment in linear systems with several commensurate lumped and distributed delays in state by means of static output feedback. *Izv. Inst. Mat. Inform. Udmurt. Gos. Univ.* **2020**, *56*, 5–19. [CrossRef]