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INTERACTION BETWEEN SUBBANDS IN A QUASI-ONE-DIMENSIONAL SUPERCONDUCTOR

Yu. P. Chuburin* **and T. S. Tinyukova**†

In the framework of the Bogoliubov–de Gennes equation, we study the spinless p-wave superconductor in an infinite strip in the presence of some impurity. We analytically determine the wave functions of stable bound states with energies close to edge points of the energy gap. We prove that for a small impurity potential, the contribution of the nearest subbands to the wave functions in the case of energy values close to edge points is very small, and these energy levels are significantly closer to the gap edge than in the one-dimensional case. We also study the bound states with nearly zero energy values; in contrast to the one-dimensional case, they do not have the "particle–hole" symmetry. In the cases under study, in addition to the bound states, there also exit resonance states related to them.

Keywords: Bogoliubov–de Gennes Hamiltonian, superconducting gap, Andreev bound state, Majorana bound state, resonance state, subband

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1. Introduction

The topological superconductors and Andreev bound states (ABSs) with energies in the energy gap (including Majorana bound states (MBSs), which are quasiparticles of "electron–hole" type, have zero energy, and obey non-Abelian quantum statistics) have been actively investigated in the last 15 to 20 years. The non-Abelian statistics of MBSs opens up prospects for their use in quantum computations (see, e.g., reviews [1]–[3]).

The existence of MBSs is related to the presence of a zero-bias conductance peak in the topological phase [4]–[7]. However, such peaks can also be generated by ABSs with nearly zero energy, also in the nontopological mode [8], [9]. In theoretical studies of ABSs and MBSs, one-dimensional models are typically used, but when the nearest subbands are sufficiently populated, their influence must be taken into account [10]. The interaction of subbands increases the value of the conductance peak [11], [12]; the peak can then be caused by ABSs due to the repulsion of levels near the subbands [13]. The problem of distinguishing between ABSs and MBSs is very important; its solution can be facilitated, in particular, by theoretical studies of the localization of quasiparticles and of the presence of "particle–hole" symmetry in them (the conjugation conditions in the spinless case, see (37) below) [13], [14].

*Udmurt Federal Research Center, Ural Branch, RAS, Izhevsk, Russia, e-mail: chuburin@ftiudm.ru (corresponding author).

†Udmurt State University, Izhevsk, Russia, e-mail: ttinyukova@mail.ru.

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In this paper, the Bogoliubov–de Gennes (BdG) equation is used to investigate a model of a spinless superconductor with p-wave order in an infinite strip in the presence of an impurity (cf. the finite model with p-wave superconductivity studied in [15]). The energies and wave functions of ABSs arising in low-energy subbands and interacting with each other due to the impurity potential are studied analytically. The most interesting are the bound and associated resonance (decaying) states with energies near the edges of the superconducting gap (cf. [16], [17]), as well as the states with nearly zero energy, among which, generally speaking, there may be states close to Majorana states. We here prove that for a small potential, the contribution of the nearest subbands to the wave functions in the case of energy values close to the gap edge is very small, and these energy levels are significantly closer to the gap edge than in the one-dimensional case. The ABSs with nearly zero energies, unlike in the one-dimensional case, do not have "particle–hole" symmetry (cf. the numerical results in [13]), and the contribution from the nearest subbands to their wave functions can be large. We show that in the topologically nontrivial phase, ABSs can turn into resonance states as the system parameters change.

2. ABSs with energy values near the superconducting gap edges

In the case of a two-dimensional p-wave superconductor, the BdG Hamiltonian has the form [3], [18]

$$H = \begin{pmatrix} -\partial_x^2 - \partial_y^2 - \mu & \Delta(-\partial_x + i\partial_y) \\ \Delta(\partial_x + i\partial_y) & \partial_x^2 + \partial_y^2 + \mu \end{pmatrix}, \tag{1}$$

where Δ is a real pairwise interaction strength and μ is the chemical potential. Hamiltonian (1) acts on functions of the form

$$\Psi(x,y) = (\psi_{\mathrm{e}}(x,y), \psi_{\mathrm{h}}(x,y))^{\mathrm{T}},$$

where $\psi_{\mathrm{e}}(x,y)$ and $\psi_{\mathrm{h}}(x,y)$ are the electron and hole components and the superscript T denotes transposition. We assume that the function $\Psi(x,y)$ is defined in the strip $-\infty < x < \infty$, $0 < y < l$ and satisfies periodic boundary conditions in y:

$$\Psi(x,l) = \Psi(x,0), \qquad \partial_y \Psi(x,l) = \partial_y \Psi(x,0).$$

The spectrum of H is described by the inequality $|E| \geqslant |\mu|$ (see the Appendix).

The following representation holds for $\Psi(x,y)$:

$$\Psi(x,y) = \frac{1}{\sqrt{l}} \sum_{n=-\infty}^{\infty} \Psi_n(x) e^{-2\pi i n y/l}, \tag{2}$$

where

$$\Psi_n(x) = (\psi_{\mathrm{e}}^{(n)}(x), \psi_{\mathrm{h}}^{(n)}(x))^{\mathrm{T}} = \frac{1}{\sqrt{l}} \int_0^l \Psi(x,y) e^{2\pi i n y/l}\, dy.$$

The operator H in the nth subband, i.e., in the subspace of functions $\Psi(x)e^{-2\pi i n y/l}$, has the form

$$H^{(n)} = \begin{pmatrix} -\partial_x^2 + \left(\dfrac{2\pi n}{l}\right)^2 - \mu & \Delta\left(-\partial_x + \dfrac{2\pi n}{l}\right) \\ \Delta\left(\partial_x + \dfrac{2\pi n}{l}\right) & \partial_x^2 - \left(\dfrac{2\pi n}{l}\right)^2 + \mu \end{pmatrix}.$$

We further consider the perturbed Hamiltonian $H + V$, where $V = V(x,y)$ is the impurity potential

$$V = Z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (e^{-2\pi i y/l} + e^{2\pi i y/l}) \delta(x), \tag{3}$$

where Z is a real parameter and $\delta(x)$ is the Dirac delta function. To obtain analytic results, we consider only the low-energy subbands corresponding to the values $n = 0, \pm 1$. We are interested in stable ABSs with energies E that are close to edge points of the energy gap, and also in ABSs with nearly zero energy (cf. the one-dimensional case [16] and the finite model with an MBS at the ends of the strip [15]). The wave functions of the considered quasiparticles satisfy the BdG equation

$$(H+V)\Psi = E\Psi. \tag{4}$$

We use (1)–(3) to write Eq. (4) as

$$\begin{aligned}
(H^{(0)}-E)\begin{pmatrix}\psi_{\mathrm{e}}^{(0)}(x)\\ \psi_{\mathrm{h}}^{(0)}(x)\end{pmatrix} &= -Z\begin{pmatrix}\psi_{\mathrm{e}}^{(1)}(x)+\psi_{\mathrm{e}}^{(-1)}(x)\\ -(\psi_{\mathrm{h}}^{(1)}(x)+\psi_{\mathrm{h}}^{(-1)}(x))\end{pmatrix}\delta(x),\\
(H^{(\pm1)}-E)\begin{pmatrix}\psi_{\mathrm{e}}^{(\pm1)}(x)\\ \psi_{\mathrm{h}}^{(\pm1)}(x)\end{pmatrix} &= -Z\begin{pmatrix}\psi_{\mathrm{e}}^{(0)}(x)\\ -\psi_{\mathrm{h}}^{(0)}(x)\end{pmatrix}\delta(x).
\end{aligned} \tag{5}$$

The Green's function $G^{(0)}(x-x', E)$ of the Hamiltonian $H^{(0)}$, in other words, the kernel of the resolvent $(H^{(0)}-E)^{-1}$, is (see [16])

$$G^{(0)}(x-x',E) = \begin{pmatrix} -g_+^{(1)}e^{ip_+^{(0)}|x-x'|} - g_-^{(2)}e^{ip_-^{(0)}|x-x'|} & \dfrac{\Delta}{4a}A\,\mathrm{sgn}(x-x') \\ -\dfrac{\Delta}{4a}A\,\mathrm{sgn}(x-x') & g_-^{(1)}e^{ip_+^{(0)}|x-x'|} + g_+^{(2)}e^{ip_-^{(0)}|x-x'|} \end{pmatrix}, \tag{6}$$

where

$$A = e^{ip_+^{(0)}|x-x'|} - e^{ip_-^{(0)}|x-x'|}, \qquad a = \sqrt{E^2 + \frac{\Delta^4}{4} - \mu\Delta^2}, \qquad p_\pm^{(0)} = \sqrt{\mu \pm a - \frac{\Delta^2}{2}},$$

$$g_\pm^{(1)} = \frac{a - \Delta^2/2 \pm E}{4ip_+^{(0)}a}, \qquad g_\pm^{(2)} = \frac{a + \Delta^2/2 \pm E}{4ip_-^{(0)}a}$$

($p_\pm^{(0)}$ coincides with $p_\pm$ in (A.7) for $n = 0$).

We first consider the case of a topologically nontrivial phase $\mu > 0$ (see [1], [2]) and the energy $E = \mu - \varepsilon$, where $0 < \varepsilon \ll \mu$, near the upper edge of the gap. To calculate $(H^{(\pm1)} - E)^{-1}$ (see the Appendix), we assume that $|\mu| \ll \min\{|\Delta|, \Delta^2\}$ and the quantities μ and $1/l$ have the same order of magnitude. Using (6) and (A.9), we rewrite the first equation in (5) as

$$\begin{aligned}
\psi_{\mathrm{e}}^{(0)}(x) &= \frac{Z}{2|\Delta|}\left(\sqrt{\frac{\varepsilon}{2\mu}}e^{-\sqrt{2\mu\varepsilon}|x/\Delta|} - e^{-|\Delta x|}\right)(\psi_{\mathrm{e}}^{(1)}(0)+\psi_{\mathrm{e}}^{(-1)}(0)) + \\
&\quad + \frac{Z}{2\Delta}\left(e^{-\sqrt{2\mu\varepsilon}|x/\Delta|} - e^{-|\Delta x|}\right)\mathrm{sgn}(x)(\psi_{\mathrm{h}}^{(1)}(0)+\psi_{\mathrm{h}}^{(-1)}(0)),\\
\psi_{\mathrm{h}}^{(0)}(x) &= \frac{Z}{2\Delta}\left(e^{-\sqrt{2\mu\varepsilon}|x/\Delta|} - e^{-|\Delta x|}\right)\mathrm{sgn}(x)(\psi_{\mathrm{e}}^{(1)}(0)+\psi_{\mathrm{e}}^{(-1)}(0)) + \\
&\quad + \frac{Z}{2|\Delta|}\left(\sqrt{\frac{2\mu}{\varepsilon}}e^{-\sqrt{2\mu\varepsilon}|x/\Delta|} - e^{-|\Delta x|}\right)(\psi_{\mathrm{h}}^{(1)}(0)+\psi_{\mathrm{h}}^{(-1)}(0)).
\end{aligned} \tag{7}$$

We put $x_{\mathrm{e(h)}}^{(n)} = \psi_{\mathrm{e(h)}}^{(n)}(0)$. From (7) with $x = 0$, we approximately derive the equations

$$x_{\mathrm{e}}^{(0)} = -\frac{Z}{2|\Delta|}(x_{\mathrm{e}}^{(1)} + x_{\mathrm{e}}^{(-1)}), \qquad x_{\mathrm{h}}^{(0)} = \frac{Z}{2|\Delta|}\sqrt{\frac{2\mu}{\varepsilon}}(x_{\mathrm{h}}^{(1)} + x_{\mathrm{h}}^{(-1)}). \tag{8}$$

Using (8) and (A.11), we write the second equation in (5) as

$$\begin{aligned}
\psi_{\mathrm{e}}^{(\pm 1)}(x) &= \frac{Z^2 \operatorname{sgn}\Delta}{4\Delta^2}\biggl(\operatorname{sgn}(\Delta)e^{-|\Delta x|}(x_{\mathrm{e}}^{(1)}+x_{\mathrm{e}}^{(-1)}) \pm \sqrt{\frac{2\mu}{\varepsilon}}e^{-(2\pi/l)|x|}(x_{\mathrm{h}}^{(1)}+x_{\mathrm{h}}^{(-1)}) + \\
&\qquad + \sqrt{\frac{2\mu}{\varepsilon}}(e^{-(2\pi/l)|x|}-e^{-|\Delta x|})(x_{\mathrm{h}}^{(1)}+x_{\mathrm{h}}^{(-1)})\operatorname{sgn}(x)\biggr), \\
\psi_{\mathrm{h}}^{(\pm 1)}(x) &= \frac{Z^2 \operatorname{sgn}\Delta}{4\Delta^2}\biggl(\pm e^{-(2\pi/l)|x|}(x_{\mathrm{e}}^{(1)}+x_{\mathrm{e}}^{(-1)}) - \sqrt{\frac{2\mu}{\varepsilon}}\operatorname{sgn}(\Delta)e^{-|\Delta x|}(x_{\mathrm{h}}^{(1)}+x_{\mathrm{h}}^{(-1)}) - \\
&\qquad - (e^{-(2\pi/l)|x|}-e^{-|\Delta x|})(x_{\mathrm{e}}^{(1)}+x_{\mathrm{e}}^{(-1)})\operatorname{sgn}(x) + \frac{\sqrt{2}\mu^{3/2}}{\Delta\sqrt{\varepsilon}(\pi/l)}e^{-(2\pi/l)|x|}(x_{\mathrm{h}}^{(1)}+x_{\mathrm{h}}^{(-1)})\biggr),
\end{aligned} \tag{9}$$

whence, for $x = 0$, we obtain

$$\begin{aligned}
x_{\mathrm{e}}^{(\pm 1)} &= \frac{Z^2}{4|\Delta|}\biggl(\operatorname{sgn}(\Delta)(x_{\mathrm{e}}^{(1)}+x_{\mathrm{e}}^{(-1)}) \pm \sqrt{\frac{2\mu}{\varepsilon}}(x_{\mathrm{h}}^{(1)}+x_{\mathrm{h}}^{(-1)})\biggr), \\
x_{\mathrm{h}}^{(\pm 1)} &= \frac{Z^2}{4|\Delta|}\biggl(\pm(x_{\mathrm{e}}^{(1)}+x_{\mathrm{e}}^{(-1)}) - \sqrt{\frac{2\mu}{\varepsilon}}\operatorname{sgn}(\Delta)(x_{\mathrm{h}}^{(1)}+x_{\mathrm{h}}^{(-1)}) + \frac{\sqrt{2}\mu^{3/2}}{\Delta\sqrt{\varepsilon}(\pi/l)}(x_{\mathrm{h}}^{(1)}+x_{\mathrm{h}}^{(-1)})\biggr).
\end{aligned} \tag{10}$$

We now approximately rewrite (10) as the system

$$\begin{aligned}
x_{\mathrm{e}}^{(\pm 1)} &= \pm\frac{Z^2 \operatorname{sgn}(\Delta)}{4\Delta^2}\sqrt{\frac{2\mu}{\varepsilon}}(x_{\mathrm{h}}^{(1)}+x_{\mathrm{h}}^{(-1)}), \\
x_{\mathrm{h}}^{(\pm 1)} &= \frac{Z^2}{4\Delta^2}\sqrt{\frac{2\mu}{\varepsilon}}\biggl(\frac{\mu}{|\Delta|(\pi/l)}-1\biggr)(x_{\mathrm{h}}^{(1)}+x_{\mathrm{h}}^{(-1)}).
\end{aligned} \tag{11}$$

We put

$$\sigma = \frac{Z^2}{4\Delta^2}\sqrt{\frac{2\mu}{\varepsilon}}\biggl(\frac{\mu}{|\Delta|(\pi/l)}-1\biggr),$$

then the second equation in (11) becomes

$$\begin{pmatrix} 1-\sigma & -\sigma \\ -\sigma & 1-\sigma \end{pmatrix}\begin{pmatrix} x_{\mathrm{h}}^{(1)} \\ x_{\mathrm{h}}^{(-1)} \end{pmatrix} = 0, \tag{12}$$

whence we obtain the existence condition for the solution of this equation: $\sigma = 1/2$. Thus, we have $x_{\mathrm{h}}^{(1)} = x_{\mathrm{h}}^{(-1)} = C = \text{const}$. From the first equation in (11), we derive

$$x_{\mathrm{e}}^{(\pm 1)} = \pm\frac{CZ^2 \operatorname{sgn}(\Delta)}{2\Delta^2}\sqrt{\frac{2\mu}{\varepsilon}}. \tag{13}$$

The relation $\sigma = 1/2$, which means that there are ABSs with energy close to the upper edge of the superconducting gap, can be written in the form that shows the dependence of the level on the system parameters:

$$\sqrt{\varepsilon} = \frac{Z^2\sqrt{2\mu}}{2\Delta^2}\biggl(\frac{\mu}{|\Delta|(\pi/l)}-1\biggr). \tag{14}$$

In particular, a bound state exists only if $|\Delta| < \mu/(\pi/l)$ (see below for the case $|\Delta| > \mu/(\pi/l)$).

Taking into account that the norm $||e^{-2\mu\varepsilon|x/\Delta|}||$ in the space $L^2(-\infty,\infty)$ of square integrable functions on the real axis, which is proportional to $\varepsilon^{-1/2}$, is much greater than the norm $||e^{-|\Delta x|}||$ for small ε, from (7) we approximately obtain

$$\psi_{\mathrm{e}}^{(0)}(x) = \frac{CZ}{\Delta}\,e^{-\sqrt{2\mu\varepsilon}|x/\Delta|}\operatorname{sgn}(x), \qquad \psi_{\mathrm{h}}^{(0)}(x) = \frac{CZ}{|\Delta|}\sqrt{\frac{2\mu}{\varepsilon}}\,e^{-\sqrt{2\mu\varepsilon}|x/\Delta|}. \tag{15}$$

Taking (13) and the inequality $||e^{-|\Delta x|}|| \ll ||e^{-(2\pi/l)|x|}||$ into account, from (9) for small ε, we have

$$\begin{aligned} \psi_{\mathrm{e}}^{(\pm 1)}(x) &= \frac{CZ^2\,\mathrm{sgn}(\Delta)}{2\Delta^2}\sqrt{\frac{2\mu}{\varepsilon}}\,(\mathrm{sgn}(\Delta)\pm 1)e^{-(2\pi/l)|x|}, \\ \psi_{\mathrm{h}}^{(\pm 1)}(x) &= \frac{CZ^2\mu\,\mathrm{sgn}(\Delta)}{2(\pi/l)\Delta^3}\sqrt{\frac{2\mu}{\varepsilon}}\,e^{-(2\pi/l)|x|}. \end{aligned} \tag{16}$$

Using (2), (15), and (16), we determine the form of the wave function

$$\begin{aligned} \Psi(x,y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\sqrt{2\mu\varepsilon}\,|x/\Delta|} + \frac{Z}{2\Delta}\Bigg(\begin{pmatrix} 2\theta(x) \\ \mu/(\Delta(\pi/l)) \end{pmatrix} e^{-2\pi i y/l} + \\ + \begin{pmatrix} -2\theta(-x) \\ \mu/(\Delta(\pi/l)) \end{pmatrix} e^{2\pi i y/l}\Bigg) e^{-(2\pi/l)|x|}. \end{aligned} \tag{17}$$

Similarly, using (A.12), we study the case $E = -\mu + \varepsilon$; the existence condition for ABSs then coincides with (14) and the wave function becomes

$$\begin{aligned} \Phi(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\sqrt{2\mu\varepsilon}\,|x/\Delta|} + \frac{Z}{2\Delta}\Bigg(\begin{pmatrix} \mu/(\Delta(\pi/l)) \\ -2\theta(-x) \end{pmatrix} e^{-2\pi i y/l} + \\ + \begin{pmatrix} \mu/(\Delta(\pi/l)) \\ 2\theta(x) \end{pmatrix} e^{2\pi i y/l}\Bigg) e^{-(2\pi/l)|x|}. \end{aligned} \tag{18}$$

We now consider the case of a topologically trivial phase $\mu < 0$ (see [1], [2]); for the energies close to the upper edge of the gap, we then have $E = |\mu| - \varepsilon = -\mu - \varepsilon$, where $0 < \varepsilon \ll |\mu|$, and it is necessary to use Eqs. (A.12) (instead of (A.11) in the case $\mu > 0$; we use (A.11) for $E = -|\mu| + \varepsilon = \mu + \varepsilon$). We write the ABS existence condition

$$\sqrt{\varepsilon} = \frac{Z^2\sqrt{2|\mu|}}{2\Delta^2}\left(\frac{|\mu|}{|\Delta|(\pi/l)} + 1\right) \tag{19}$$

and the wave function

$$\begin{aligned} \Psi(x,y) = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\sqrt{2\mu\varepsilon}\,|x/\Delta|} + \frac{Z}{2\Delta}\Bigg(\begin{pmatrix} |\mu|/(\Delta(\pi/l)) \\ 2\theta(-x) \end{pmatrix} e^{-2\pi i y/l} + \\ + \begin{pmatrix} |\mu|/(\Delta(\pi/l)) \\ -2\theta(x) \end{pmatrix} e^{2\pi i y/l}\Bigg) e^{-(2\pi/l)|x|}. \end{aligned} \tag{20}$$

For $E = \mu + \varepsilon \approx -|\mu|$, the existence condition for an ABS coincides with (19), and the wave function becomes

$$\begin{aligned} \Phi(x,y) = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\sqrt{2\mu\varepsilon}\,|x/\Delta|} + \frac{Z}{2\Delta}\Bigg(\begin{pmatrix} -2\theta(x) \\ |\mu|/(\Delta(\pi/l)) \end{pmatrix} e^{-2\pi i y/l} + \\ + \begin{pmatrix} 2\theta(-x) \\ |\mu|/(\Delta(\pi/l)) \end{pmatrix} e^{2\pi i y/l}\Bigg) e^{-(2\pi/l)|x|}. \end{aligned} \tag{21}$$

We next consider the case $\mu > 0$ and, for example, $E = \mu - \varepsilon$. Then up to factors, the Green's function of the one-dimensional BdG Hamiltonian contains the Green's function of the Hamiltonian $-\partial_x^2$ of the form $\sqrt{2\mu/\varepsilon}\, e^{-\sqrt{2\mu\varepsilon}|x/\Delta|}$ (see (7)), where ε is the energy of a quasiparticle referenced to the gap edge. In the transition to the second ("nonphysical") sheet of the Riemann surface of the square root $\sqrt{\varepsilon}$ (or, which is the same, the Green's function (6)), the sign of the square root is reversed, and relation (14) is satisfied for $|\Delta| > \mu/(\pi/l)$. Simultaneously, by (7), the wave function becomes exponentially increasing (in the stationary approach), i.e., describes a resonance state. At the time of transition to the second sheet, $\varepsilon = 0$ and hence $E = \mu$. Thus, when the level corresponding to an ABS reflects from the gap edge, it is already a resonance state (a similar effect occurs in the one-dimensional case when Z changes sign [17]). The life time of the considered resonance state is inversely proportional to $\sqrt{2\mu\varepsilon}/|\Delta|$ [19] and is therefore large. We note that as follows from (14), the topological phase transition, i.e., a decrease in μ to negative values, results in the change of the sign of ε. This means that the resonance state turns into a continuous spectrum.

In the trivial phase, $\mu < 0$ for small Z, there always exist ABSs with energy values close to the edge.

3. ABSs with nearly zero energy values

Under the same conditions as above, we consider the case of small energies $E = \varepsilon$, $|\varepsilon| \ll |\mu|$. In the case of a topologically nontrivial phase $\mu > 0$, using (6) and (A.9), we write the first equation in (5) in the form

$$\begin{aligned}
\psi_{\mathrm{e}}^{(0)}(x) &= \frac{Z}{2|\Delta|}\left(\sqrt{\frac{\mu-\varepsilon}{\mu+\varepsilon}}e^{(-\sqrt{\mu^2-E^2}/|\Delta|)|x|} - e^{-|\Delta x|}\right)(x_{\mathrm{e}}^{(1)} + x_{\mathrm{e}}^{(-1)}) + \\
&\quad + \frac{Z}{2\Delta}(e^{-|\mu x/\Delta|} - e^{-|\Delta x|})\operatorname{sgn}(x)(x_{\mathrm{h}}^{(1)} + x_{\mathrm{h}}^{(-1)}), \\
\psi_{\mathrm{h}}^{(0)}(x) &= \frac{Z}{2\Delta}(e^{-|\mu x/\Delta|} - e^{-|\Delta x|})\operatorname{sgn}(x)(x_{\mathrm{e}}^{(1)} + x_{\mathrm{e}}^{(-1)}) + \\
&\quad + \frac{Z}{2|\Delta|}\left(\sqrt{\frac{\mu+\varepsilon}{\mu-\varepsilon}}e^{(-\sqrt{\mu^2-E^2}/|\Delta|)|x|} - e^{-|\Delta x|}\right)(x_{\mathrm{h}}^{(1)} + x_{\mathrm{h}}^{(-1)}).
\end{aligned} \tag{22}$$

From (22) for $x = 0$, using the approximate relation

$$\sqrt{\frac{\mu \mp \varepsilon}{\mu \pm \varepsilon}} = 1 \mp \frac{\varepsilon}{\mu}, \tag{23}$$

we derive the equations

$$x_{\mathrm{e}}^{(0)} = -\frac{\varepsilon Z}{2|\Delta|\mu}(x_{\mathrm{e}}^{(1)} + x_{\mathrm{e}}^{(-1)}), \qquad x_{\mathrm{h}}^{(0)} = \frac{\varepsilon Z}{2|\Delta|\mu}(x_{\mathrm{h}}^{(1)} + x_{\mathrm{h}}^{(-1)}). \tag{24}$$

Using (A.13) and (24), from the second equation in (5), we obtain

$$\begin{aligned}
\psi_{\mathrm{e}}^{(\pm1)}(x) &= \frac{\varepsilon Z^2 \operatorname{sgn}(\Delta)}{4\Delta^2\mu}\biggl(\operatorname{sgn}(\Delta)e^{-|\Delta x|}(x_{\mathrm{e}}^{(1)} + x_{\mathrm{e}}^{(-1)}) \pm e^{-(2\pi/l)|x|}(x_{\mathrm{h}}^{(1)} + x_{\mathrm{h}}^{(-1)}) - \\
&\quad - \frac{\mu}{2\Delta(\pi/l)}e^{-(2\pi/l)|x|}(x_{\mathrm{e}}^{(1)} + x_{\mathrm{e}}^{(-1)}) + (e^{-(2\pi/l)|x|} - e^{-|\Delta x|})(x_{\mathrm{h}}^{(1)} + x_{\mathrm{h}}^{(-1)})\operatorname{sgn}(x)\biggr), \\
\psi_{\mathrm{h}}^{(\pm1)}(x) &= \frac{\varepsilon Z^2 \operatorname{sgn}(\Delta)}{4\Delta^2\mu}\biggl(-\operatorname{sgn}(\Delta)e^{-|\Delta x|}(x_{\mathrm{h}}^{(1)} + x_{\mathrm{h}}^{(-1)}) \pm e^{-(2\pi/l)|x|}(x_{\mathrm{e}}^{(1)} + x_{\mathrm{e}}^{(-1)}) + \\
&\quad + \frac{\mu}{2\Delta(\pi/l)}e^{-(2\pi/l)|x|}(x_{\mathrm{h}}^{(1)} + x_{\mathrm{h}}^{(-1)}) - (e^{-(2\pi/l)|x|} - e^{-|\Delta x|})(x_{\mathrm{e}}^{(1)} + x_{\mathrm{e}}^{(-1)})\operatorname{sgn}(x)\biggr).
\end{aligned} \tag{25}$$

We put

$$\sigma_1 = \frac{\varepsilon Z^2 \operatorname{sgn}(\Delta)}{4\Delta^2\mu}, \qquad \sigma_2 = \sigma_1\left(\frac{\mu}{2\Delta(\pi/l)} - \operatorname{sgn}(\Delta)\right). \tag{26}$$

From (25) and (26), we have

$$x_{\rm e}^{(\pm1)} = -\sigma_2(x_{\rm e}^{(1)} + x_{\rm e}^{(-1)}) \pm \sigma_1(x_{\rm h}^{(1)} + x_{\rm h}^{(-1)}),$$
$$x_{\rm h}^{(\pm1)} = \sigma_2(x_{\rm h}^{(1)} + x_{\rm h}^{(-1)}) \pm \sigma_1(x_{\rm e}^{(1)} + x_{\rm e}^{(-1)})$$

or

$$\begin{pmatrix} 1+\sigma_2 & \sigma_2 & -\sigma_1 & -\sigma_1 \\ \sigma_2 & 1+\sigma_2 & \sigma_1 & \sigma_1 \\ -\sigma_1 & -\sigma_1 & 1-\sigma_2 & -\sigma_2 \\ \sigma_1 & \sigma_1 & -\sigma_2 & 1-\sigma_2 \end{pmatrix} \begin{pmatrix} x_{\rm e}^{(1)} \\ x_{\rm e}^{(-1)} \\ x_{\rm h}^{(1)} \\ x_{\rm h}^{(-1)} \end{pmatrix} = 0. \tag{27}$$

System (27) has a nonzero solution if $\sigma_2 = \pm 1/2$ or

$$\varepsilon = \pm\frac{4\mu|\Delta|^3(\pi/l)}{Z^2(\mu - 2|\Delta|(\pi/l))}, \tag{28}$$

and in this case,

$$\sigma_1 = \pm\frac{\Delta(\pi/l)}{\mu - 2|\Delta|(\pi/l)}. \tag{29}$$

For definiteness, let $\sigma_2 = 1/2$ (for $\sigma_2 = -1/2$, the sign of ε is reversed); it then follows from (27) that $x_{\rm e}^{(1)} = -x_{\rm e}^{(-1)} = 2\sigma_1 x_{\rm h}^{(1)}$ and $x_{\rm h}^{(-1)} = x_{\rm h}^{(1)} = C$, where C is an arbitrary constant. Because the condition $\varepsilon \ll \mu$ must be satisfied, we assume that Z is sufficiently large. Let $C = 1$. From (22) and (25), we approximately obtain the wave function in the form

$$\Psi(x,y) = \begin{pmatrix} \operatorname{sgn}(x) \\ \operatorname{sgn}(\Delta) \end{pmatrix} e^{-|\mu x/\Delta|} + \frac{\varepsilon Z}{2|\Delta|\mu}\left(\begin{pmatrix} 2\theta(x) \\ \mu/(2\Delta(\pi/l)) \end{pmatrix} e^{-2\pi i y/l} + \begin{pmatrix} -2\theta(-x) \\ \mu/(2\Delta(\pi/l)) \end{pmatrix} e^{2\pi i y/l} \right) e^{-(2\pi/l)|x|}. \tag{30}$$

It follows from this and from (28) and (29) that the contribution of the subbands with numbers $n = \pm 1$ is generally not small.

In the case of a topologically trivial phase, i.e., for $\mu < 0$, we similarly obtain

$$\begin{aligned} x_{\rm e}^{(0)} &= -\frac{Z}{2|\Delta|}\left(2 - \frac{\varepsilon}{\mu}\right)(x_{\rm e}^{(1)} + x_{\rm e}^{(-1)}), \\ x_{\rm h}^{(0)} &= -\frac{Z}{2|\Delta|}\left(2 + \frac{\varepsilon}{\mu}\right)(x_{\rm h}^{(1)} + x_{\rm h}^{(-1)}). \end{aligned} \tag{31}$$

From the second equation in (5), (31), and (A.13), we have

$$\begin{aligned} \psi_{\rm e}^{(\pm1)}(x) = \frac{Z^2 \operatorname{sgn}(\Delta)}{4\Delta^2}\bigg(&-\frac{\mu}{2\Delta(\pi/l)}\left(2 - \frac{\varepsilon}{\mu}\right)(x_{\rm e}^{(1)} + x_{\rm e}^{(-1)})e^{-(2\pi/l)|x|} + \\ &+ \operatorname{sgn}(\Delta)\left(2 - \frac{\varepsilon}{\mu}\right)(x_{\rm e}^{(1)} + x_{\rm e}^{(-1)})e^{-|\Delta x|} - \left(2 + \frac{\varepsilon}{\mu}\right)(x_{\rm h}^{(1)} + x_{\rm h}^{(-1)}) \times \\ &\times (e^{-(2\pi/l)|x|} - e^{-|\Delta x|}) \operatorname{sgn}(x) \mp \left(2 + \frac{\varepsilon}{\mu}\right)(x_{\rm h}^{(1)} + x_{\rm h}^{(-1)})e^{-(2\pi/l)|x|}\bigg), \\ \psi_{\rm h}^{(\pm1)}(x) = \frac{Z^2 \operatorname{sgn}(\Delta)}{4\Delta^2}\bigg(&-\frac{\mu}{2\Delta(\pi/l)}\left(2 + \frac{\varepsilon}{\mu}\right)(x_{\rm h}^{(1)} + x_{\rm h}^{(-1)})e^{-(2\pi/l)|x|} + \\ &+ \operatorname{sgn}(\Delta)\left(2 + \frac{\varepsilon}{\mu}\right)(x_{\rm h}^{(1)} + x_{\rm h}^{(-1)})e^{-|\Delta||x|} \pm \left(2 - \frac{\varepsilon}{\mu}\right)(x_{\rm e}^{(1)} + x_{\rm e}^{(-1)}) \times \\ &\times e^{-(2\pi/l)|x|} - \left(2 - \frac{\varepsilon}{\mu}\right)(x_{\rm e}^{(1)} + x_{\rm e}^{(-1)})(e^{-(2\pi/l)|x|} - e^{-|\Delta x|}) \operatorname{sgn}(x)\bigg). \end{aligned} \tag{32}$$

From (32), introducing the notation

$$\sigma_3 = \frac{Z^2 \operatorname{sgn}(\Delta)}{4\Delta^2}, \qquad \sigma_4 = \sigma_3\left(\frac{\mu}{2\Delta(\pi/l)} - \operatorname{sgn}(\Delta)\right),$$

we obtain

$$\begin{pmatrix} 1+\sigma_4\left(2-\frac{\varepsilon}{\mu}\right) & \sigma_4\left(2-\frac{\varepsilon}{\mu}\right) & \sigma_3\left(2+\frac{\varepsilon}{\mu}\right) & \sigma_3\left(2+\frac{\varepsilon}{\mu}\right) \\ \sigma_4\left(2-\frac{\varepsilon}{\mu}\right) & 1+\sigma_4\left(2-\frac{\varepsilon}{\mu}\right) & -\sigma_3\left(2+\frac{\varepsilon}{\mu}\right) & -\sigma_3\left(2+\frac{\varepsilon}{\mu}\right) \\ -\sigma_3\left(2-\frac{\varepsilon}{\mu}\right) & -\sigma_3\left(2-\frac{\varepsilon}{\mu}\right) & 1+\sigma_4\left(2+\frac{\varepsilon}{\mu}\right) & \sigma_4\left(2+\frac{\varepsilon}{\mu}\right) \\ \sigma_3\left(2-\frac{\varepsilon}{\mu}\right) & \sigma_3\left(2-\frac{\varepsilon}{\mu}\right) & \sigma_4\left(2+\frac{\varepsilon}{\mu}\right) & 1+\sigma_4\left(2+\frac{\varepsilon}{\mu}\right) \end{pmatrix} \begin{pmatrix} x_{\mathrm{e}}^{(1)} \\ x_{\mathrm{e}}^{(-1)} \\ x_{\mathrm{h}}^{(1)} \\ x_{\mathrm{h}}^{(-1)} \end{pmatrix} = 0. \qquad (33)$$

The determinant of system (33) is equal to

$$d = \left(1 + 2\sigma_4\left(2 - \frac{\varepsilon}{\mu}\right)\right)\left(1 + 2\sigma_4\left(2 + \frac{\varepsilon}{\mu}\right)\right). \qquad (34)$$

Thus, system (33) has a nonzero solution if $\sigma_4 = -1/2(2 \pm \varepsilon/\mu)$, i.e.,

$$\varepsilon = \pm 2\mu\left(1 - \frac{\Delta^2}{Z^2(1 - \mu/(2|\Delta|(\pi/l)))}\right). \qquad (35)$$

The following condition must be satisfied to ensure the smallness of ε:

$$Z^2 \approx \frac{\Delta^2}{1 - \mu/(2|\Delta|(\pi/l))}.$$

For example, let $\sigma_4 = -1/2(2 + \varepsilon/\mu)$. We can then write system (33) as

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -\sigma_3\left(2+\frac{\varepsilon}{\mu}\right) & -\sigma_3\left(2+\frac{\varepsilon}{\mu}\right) \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{\mathrm{e}}^{(1)} \\ x_{\mathrm{e}}^{(-1)} \\ x_{\mathrm{h}}^{(1)} \\ x_{\mathrm{h}}^{(-1)} \end{pmatrix} = 0.$$

Therefore,

$$x_{\mathrm{e}}^{(1)} = -x_{\mathrm{e}}^{(-1)} = -2\sigma_3\left(2 + \frac{\varepsilon}{\mu}\right)x_{\mathrm{h}}^{(1)}, \qquad x_{\mathrm{h}}^{(1)} = x_{\mathrm{h}}^{(-1)} = C.$$

Let $C = 1$. Then

$$\Phi(x,y) = \begin{pmatrix} \operatorname{sgn}(x) \\ -\operatorname{sgn}(\Delta) \end{pmatrix} e^{-|\mu x/\Delta|} - \\ - \frac{Z}{|\Delta|}\left(\begin{pmatrix} 2\theta(x) \\ \mu/(2\Delta(\pi/l)) \end{pmatrix} e^{-2\pi i y/l} + \begin{pmatrix} -2\theta(-x) \\ \mu/(2\Delta(\pi/l)) \end{pmatrix} e^{2\pi i y/l}\right) e^{-(2\pi/l)|x|}. \qquad (36)$$

In contrast to the one-dimensional case [16], the conjugation conditions for ABSs with nearly zero energy, which have the form [18], [20]

$$(\psi_{\mathrm{e}}(x,y))^* = \psi_{\mathrm{h}}(x,y) \tag{37}$$

in the spinless quasi-one-dimensional case (and are typical for MBSs), are not satisfied.

In the case $E=\varepsilon\approx 0$, up to factors, the Green's function (6), (22) contains two Green's functions of the Hamiltonian $-\partial_x^2$ of the form

$$\frac{1}{\sqrt{\mu\pm\varepsilon}}e^{-(\sqrt{\mu}/|\Delta|)\sqrt{\mu\pm\varepsilon}|x|}.$$

As above, the change in the sign of the square root $\sqrt{\mu\pm\varepsilon}$ means the transition of the (shifted) energy to the second sheet of the Riemann surface of the function $G^{(0)}(x-x',E)$, i.e., the transformation of a bound state into a resonance state. In the case under study, as is easy to see, this transformation means that quantities (23) and the exponent $(-\sqrt{\mu^2-E^2}/|\Delta|)|x|$ are multiplied by -1 in key equations (22). In the transition to the second sheet, instead of (24), we have (31), and conversely. Thus, taking both bound and resonance states into account, we obtain a symmetric picture.

4. Conclusion

In this paper, we use the BdG equation to investigate the model of a spinless superconductor with p-wave order in an infinite strip in the presence of an impurity. We analytically study the wave functions of ABSs generated by the impurity with energies close to the edges of the superconducting gap. We prove the existence of both ABSs and resonance states related to them. For such energies, the contribution of the nearest subbands to the wave functions is very small. In this case, the energies close to the edge are significantly closer to the gap edge than in the one-dimensional case. We also study ABSs with nearly zero energies. In contrast to the one-dimensional case, they do not have the "particle–hole" symmetry (cf. the numerical results obtained in [13]), and the contribution of the nearest subbands can be large.

Appendix

To determine the Green's function of the Hamiltonian $H^{(n)}$, we solve the equation

$$(H^{(n)}-E)\Psi^{(n)} = \Phi^{(n)} \tag{A.1}$$

for $\Psi^{(n)}$. Using (1), we rewrite Eq. (A.1) as

$$\begin{pmatrix} -\partial_x^2+\left(\dfrac{2\pi n}{l}\right)^2-\mu-E & \Delta\left(\dfrac{-\partial_x+2\pi n}{l}\right) \\ \Delta\left(\partial_x+\dfrac{2\pi n}{l}\right) & \partial_x^2-\left(\dfrac{2\pi n}{l}\right)^2+\mu-E \end{pmatrix}\begin{pmatrix}\psi_{\mathrm{e}}^{(n)}(x)\\ \psi_{\mathrm{h}}^{(n)}(x)\end{pmatrix} = \begin{pmatrix}\varphi_{\mathrm{e}}^{(n)}(x)\\ \varphi_{\mathrm{h}}^{(n)}(x)\end{pmatrix}, \tag{A.2}$$

$n=0,\pm1,\dots$. After the Fourier transformation with respect to x, Eq. (A.2) becomes

$$\begin{pmatrix} p^2+\left(\dfrac{2\pi n}{l}\right)^2-\mu-E & \Delta\left(-ip+\dfrac{2\pi n}{l}\right) \\ \Delta\left(ip+\dfrac{2\pi n}{l}\right) & -p^2-\left(\dfrac{2\pi n}{l}\right)^2+\mu-E \end{pmatrix}\begin{pmatrix}\widetilde{\psi}_{\mathrm{e}}^{(n)}(p)\\ \widetilde{\psi}_{\mathrm{h}}^{(n)}(p)\end{pmatrix} = \begin{pmatrix}\widetilde{\varphi}_{\mathrm{e}}^{(n)}(p)\\ \widetilde{\varphi}_{\mathrm{h}}^{(n)}(p)\end{pmatrix}, \tag{A.3}$$

$n=0,\pm1,\dots$. The determinant of matrix (A.3) is

$$d_n = E^2-\left(p^2+\left(\frac{2\pi n}{l}\right)^2-\mu\right)^2-\Delta^2\left(p^2+\left(\frac{2\pi n}{l}\right)^2\right). \tag{A.4}$$

By (A.4), the dispersion law $d_n = 0$ is

$$\left(p^2 + \left(\frac{2\pi n}{l}\right)^2 + \frac{\Delta^2}{2} - \mu\right)^2 - \frac{\Delta^4}{4} + \Delta^2\mu - E^2 = 0. \tag{A.5}$$

Using the assumption $|\mu| \ll \min\{|\Delta|, \Delta^2\}$ in (A.5), we obtain that the spectrum of the operator $H^{(n)}$, which coincides with matrix (A.2), is determined by the inequality

$$E^2 \geqslant \left(\frac{2\pi n}{l}\right)^4 + 2\left(\frac{\Delta^2}{2} - \mu\right)\left(\frac{2\pi n}{l}\right)^2 + \mu^2 \geqslant \mu^2$$

and decreases as $|n|$ increases. Therefore, the spectrum of the Hamiltonian H, which coincides with the union of the spectra of $H^{(n)}$, $n = 0, \pm 1, \dots$, is described by the inequality $|E| \geqslant |\mu|$.

From (A.4) and (A.5), we have

$$\frac{1}{d_n} = -\frac{1}{2a}\left(\frac{1}{p^2 - p_+^2} - \frac{1}{p^2 - p_-^2}\right), \tag{A.6}$$

where

$$a = \sqrt{\frac{\Delta^4}{4} - \Delta^2\mu + E^2}, \qquad p_\pm = \sqrt{\pm a + \mu - \frac{\Delta^2}{2} - \left(\frac{2\pi n}{l}\right)^2}. \tag{A.7}$$

In what follows, we only consider the case $n = \pm 1$ and use the assumption that μ and $O(1/l)$ are of the same order of magnitude. We obtain the Green's function of the operators $H^{(\pm 1)}$ near the edge points of the superconducting gap $(-|\mu|, |\mu|)$. By (A.7), we approximately have

$$a = \frac{\Delta^2}{2} - \mu - \frac{\mu^2 - E^2}{\Delta^2}, \qquad p_+ = \frac{2\pi i n}{l}, \qquad p_- = i|\Delta|. \tag{A.8}$$

We note that

$$p_+ = \frac{i\sqrt{\mu^2 - E^2}}{|\Delta|}, \qquad n = 0. \tag{A.9}$$

For definiteness, we consider the case of a topological phase $\mu > 0$. For the energies $E = \mu - \varepsilon$, where $0 < \varepsilon \ll \mu$, we have $a = \Delta^2/2 - \mu - 2\mu\varepsilon/(\Delta^2)$. From (A.3), (A.6), and (A.8), we obtain

$$\begin{aligned}
\widetilde{\psi}_{\mathrm{e}}^{(\pm 1)}(p) &= \frac{1}{\Delta^2}\left(\left(p^2 + \left(\frac{2\pi}{l}\right)^2\right)\widetilde{\varphi}_{\mathrm{e}}^{(\pm 1)}(p) - \Delta\left(ip \mp \frac{2\pi}{l}\right)\widetilde{\varphi}_{\mathrm{h}}^{(\pm 1)}(p)\right) \times \\
&\quad \times \left(\frac{1}{p^2 + (2\pi/l)^2} - \frac{1}{p^2 + \Delta^2}\right), \\
\widetilde{\psi}_{\mathrm{h}}^{(\pm 1)}(p) &= \frac{1}{\Delta^2}\left(\Delta\left(ip \pm \frac{2\pi}{l}\right)\widetilde{\varphi}_{\mathrm{e}}^{(\pm 1)}(p) - \left(p^2 + \left(\frac{2\pi}{l}\right)^2 - 2\mu\right)\widetilde{\varphi}_{\mathrm{h}}^{(\pm 1)}(p)\right) \times \\
&\quad \times \left(\frac{1}{p^2 + (2\pi/l)^2} - \frac{1}{p^2 + \Delta^2}\right).
\end{aligned} \tag{A.10}$$

From this, by using the well-known relations

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ipx}\widetilde{\varphi}(p)\,dp}{p^2-p_0^2} = -\frac{1}{2ip_0}\int_{-\infty}^{\infty}e^{ip_0|x-x'|}\varphi(x')\,dx',$$
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{pe^{ipx}\widetilde{\varphi}(p)\,dp}{p^2-p_0^2} = -\frac{1}{2i}\int_{-\infty}^{\infty}e^{ip_0|x-x'|}\operatorname{sgn}(x-x')\varphi(x')\,dx'$$

we obtain $\Psi^{(\pm1)} = (H^{(\pm1)}-E)^{-1}\Phi^{(\pm1)}$ (i.e., in fact, the resolvent):

$$\begin{aligned}
\psi_{\mathrm{e}}^{(\pm1)}(x) = \frac{1}{2\Delta}\Bigg(&\operatorname{sgn}(\Delta)\int_{-\infty}^{\infty}e^{-|\Delta||x-x'|}\varphi_{\mathrm{e}}^{(\pm1)}(x')\,dx' + \\
&+\int_{-\infty}^{\infty}e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{h}}^{(\pm1)}(x')\operatorname{sgn}(x-x')\,dx' \pm \\
&\pm\int_{-\infty}^{\infty}e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{h}}^{(\pm1)}(x')\,dx' - \\
&-\int_{-\infty}^{\infty}e^{-|\Delta||x-x'|}\varphi_{\mathrm{h}}^{(\pm1)}(x')\operatorname{sgn}(x-x')\,dx'\Bigg), \\
\psi_{\mathrm{h}}^{(\pm1)}(x) = \frac{1}{2\Delta}\Bigg(&\frac{\mu}{\Delta(\pi/l)}\int_{-\infty}^{\infty}e^{-|2\pi/l||x-x'|}\varphi_{\mathrm{h}}^{(\pm1)}(x')\,dx' - \\
&-\operatorname{sgn}(\Delta)\int_{-\infty}^{\infty}e^{-|\Delta||x-x'|}\varphi_{\mathrm{h}}^{(\pm1)}(x')\,dx' - \\
&-\int_{-\infty}^{\infty}e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{e}}^{(\pm1)}(x')\operatorname{sgn}(x-x')\,dx' \pm \\
&\pm\int_{-\infty}^{\infty}e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{e}}^{(\pm1)}(x')\,dx' + \\
&+\int_{-\infty}^{\infty}e^{-|\Delta||x-x'|}\varphi_{\mathrm{e}}^{(\pm1)}(x')\operatorname{sgn}(x-x')\,dx'\Bigg).
\end{aligned} \tag{A.11}$$

In the case $E = -\mu+\varepsilon$, $\varepsilon > 0$, we similarly obtain

$$\begin{aligned}
\psi_{\mathrm{e}}^{(\pm1)}(x) = \frac{1}{2\Delta}\Bigg(&-\frac{\mu}{\Delta(\pi/l)}\int_{-\infty}^{\infty}e^{-|2\pi/l||x-x'|}\varphi_{\mathrm{e}}^{(\pm1)}(x')\,dx' + \\
&+\operatorname{sgn}(\Delta)\int_{-\infty}^{\infty}e^{-|\Delta||x-x'|}\varphi_{\mathrm{e}}^{(\pm1)}(x')\,dx' + \\
&+\int_{-\infty}^{\infty}e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{h}}^{(\pm1)}(x')\operatorname{sgn}(x-x')\,dx' \pm \\
&\pm\int_{-\infty}^{\infty}e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{h}}^{(\pm1)}(x')\,dx' - \\
&-\int_{-\infty}^{\infty}e^{-|\Delta||x-x'|}\varphi_{\mathrm{h}}^{(\pm1)}(x')\operatorname{sgn}(x-x')\,dx'\Bigg), \\
\psi_{\mathrm{h}}^{(\pm1)}(x) = \frac{1}{2\Delta}\Bigg(&-\operatorname{sgn}(\Delta)\int_{-\infty}^{\infty}e^{-|\Delta||x-x'|}\varphi_{\mathrm{h}}^{(\pm1)}(x')\,dx' - \\
&-\int_{-\infty}^{\infty}e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{e}}^{(\pm1)}(x')\operatorname{sgn}(x-x')\,dx' \pm \\
&\pm\int_{-\infty}^{\infty}e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{e}}^{(\pm1)}(x')\,dx' + \\
&+\int_{-\infty}^{\infty}e^{-|\Delta||x-x'|}\varphi_{\mathrm{e}}^{(\pm1)}(x')\operatorname{sgn}(x-x')\,dx'\Bigg).
\end{aligned} \tag{A.12}$$

We now assume that $|E| \ll \mu$. As above, we obtain $\Psi^{(\pm 1)} = (H^{(\pm 1)} - E)^{-1}\Phi^{(\pm 1)}$, and in this case, we have

$$\begin{aligned}
\psi_{\mathrm{e}}^{(\pm 1)}(x) = \frac{1}{2\Delta}\biggl(&-\frac{\mu}{2\Delta(\pi/l)}\int_{-\infty}^{\infty} e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{e}}^{(\pm 1)}(x')\,dx' + \\
&+ \operatorname{sgn}(\Delta)\int_{-\infty}^{\infty} e^{-|\Delta||x-x'|}\varphi_{\mathrm{e}}^{(\pm 1)}(x')\,dx' + \\
&+ \int_{-\infty}^{\infty} e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{h}}^{(\pm 1)}(x')\operatorname{sgn}(x-x')\,dx' \pm \\
&\pm \int_{-\infty}^{\infty} e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{h}}^{(\pm 1)}(x')\,dx' - \\
&- \int_{-\infty}^{\infty} e^{-|\Delta||x-x'|}\varphi_{\mathrm{h}}^{(\pm 1)}(x')\operatorname{sgn}(x-x')\,dx'\biggr), \\
\psi_{\mathrm{h}}^{(\pm 1)}(x) = \frac{1}{2\Delta}\biggl(&\frac{\mu}{2\Delta(\pi/l)}\int_{-\infty}^{\infty} e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{h}}^{(\pm 1)}(x')\,dx' - \\
&- \operatorname{sgn}(\Delta)\int_{-\infty}^{\infty} -e^{-|\Delta||x-x'|}\varphi_{\mathrm{h}}^{(\pm 1)}(x')\,dx' - \\
&- \int_{-\infty}^{\infty} e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{e}}^{(\pm 1)}(x')\operatorname{sgn}(x-x')\,dx' \pm \\
&\pm \int_{-\infty}^{\infty} e^{-(2\pi/l)|x-x'|}\varphi_{\mathrm{e}}^{(\pm 1)}(x')\,dx' + \\
&+ \int_{-\infty}^{\infty} e^{-|\Delta||x-x'|}\varphi_{\mathrm{e}}^{(\pm 1)}(x')\operatorname{sgn}(x-x')\,dx'\biggr).
\end{aligned} \tag{A.13}$$

Conflicts of interest. The authors declare no conflicts of interest.

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