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CONTROL THEORY

Estimate of the Capture Time and Construction of the Pursuer's Strategy in a Nonlinear Two-Person Differential Game

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Abstract— In a finite-dimensional Euclidean space, we consider a differential game of two persons—a pursuer and an evader—described by a nonlinear autonomous controlled system of differential equations in normal form the right-hand side of which is the sum of two functions, one of which depends only on the state variable and the pursuer's control and the other, only on the state variable and the evader's control. The set of values of the pursuer's control is finite, and the set of values of the evader's control is compact. The goal of the pursuer is to bring the trajectory of the system from the initial position to any predetermined neighborhood of zero in finite time. The pursuer strategy is constructed as a piecewise constant function with values in a given finite set. To construct the pursuer control, it is allowed to use only information about the value of the current state coordinates. The evader's control is a measurable function for the construction of which there are no constraints on available information. It is shown that, to transfer the system to any predetermined neighborhood of zero, it is sufficient for the pursuer to use a strategy with a constant step of partitioning the time interval. The value of the fixed partitioning step is found in closed form. A class of systems is singled out for which an estimate of the transfer time from an arbitrary initial position to a given neighborhood of zero is obtained. The estimate is sharp in some well-defined sense. The solution essentially uses the notion of a positive basis in a vector space.

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INTRODUCTION

Two-person differential games, originally considered by R. Isaacs [1], are now a fairly developed theory with numerous practical applications [2–7]. It developed methods for solving various classes of game problems: the Isaacs method based on the analysis of a certain partial differential equation and its characteristics, the Krasovsky extreme aiming method, the Pontryagin method, and others. Krasovskii and his scientific school created the theory of positional games, which is based on the concept of the maximum stable bridge and the extreme aiming rule. However, the efficient construction of such bridges for real conflict-controlled processes, primarily for nonlinear differential games, is very difficult or even impossible. It is more convenient to build bridges that are not maximal but have the property of stability and provide efficiently implemented control procedures for individual classes of games. Sufficient conditions for the solvability of the pursuit problem in Pontryagin's nonlinear example were obtained in [8]. Sufficient conditions for the solvability of the pursuit problem in a nonlinear differential game are presented in [9] under some additional conditions on the system's vectogram and the terminal set. Approximate (in particular, numerical) construction of stable bridges in nonlinear differential games is considered, e.g., in [10, 11].

The paper [12] introduced the notion of a positive basis of a vector space, which was efficiently used in the papers [12, 13] to study the controllability property of nonlinear systems described by differential equations in a finite-dimensional Euclidean space. The properties of a positive basis were used in the papers [14–16] to study control systems on manifolds and in the papers [17–21] to study the problem of pursuit by a group of pursuers of one or more evaders in linear differential games with equal opportunities of players. Sufficient conditions for the solvability of the capture problem for a two-player differential game described by a first-order nonlinear differential system under discrete control and with incomplete information were obtained in [22]. It was proved that there exists a neighborhood of zero from each point of which a capture occurs.

In continuation of the study in [22], the following results are obtained in the present paper. It is shown that to transfer the system to any predetermined neighborhood of zero, it suffices to use a strategy with a constant step of partitioning the time interval. A class of systems is distinguished for which an estimate of the capture time from a given initial position is obtained. This estimate is sharp in a certain sense described in the paper. The properties of a positive basis of a vector space play an essential role in what follows.

1. STATEMENT OF THE PROBLEM

In the space $\mathbb{R}^k$ ($k \geq 2$), we consider a differential game of two persons, a pursuer P and an evader E. The game dynamics is described by the system of differential equations

$$\dot{x} = f(x,u) + g(x,v), \quad u \in U, \quad v \in V, \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^k$ is the state vector and u and v are the controls. The set $U = \{u_1, \dots, u_m\}$ is finite, $u_i \in \mathbb{R}^l$, $i = 1, \dots, m$; the set $V \subset \mathbb{R}^s$ is a compact set. For each $u \in U$, the function $f : \mathbb{R}^k \times U \to \mathbb{R}^k$ is Lipschitz in x. The function $g : \mathbb{R}^k \times V \to \mathbb{R}^k$ is jointly Lipschitz in all the variables; i.e., there exist positive numbers $\overline{L}_1, \dots, \overline{L}_m$ and L_2 such that

$$\begin{aligned} &\big\|f(x^1, u_i) - f(x^2, u_i)\big\| \leq \overline{L}_i \|x^1 - x^2\|, && x^1, x^2 \in \mathbb{R}^k, \quad i = 1, \dots, m, \\ &\big\|g(x^1, v^1) - g(x^2, v^2)\big\| \leq L_2\big(\|x^1 - x^2\| + \|v^1 - v^2\|\big), && x^1, x^2 \in \mathbb{R}^k, \quad v^1, v^2 \in V. \end{aligned} \tag{2}$$

Here and in the following, the norm is assumed to be Euclidean. Set $L_1 = \max\{\overline{L}_1, \dots, \overline{L}_m\}$.

By a partition σ of the interval $[0,T]$ we mean a finite set $\{\tau_q\}_{q=0}^{\eta}$ of points of this interval such that $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_\eta = T$.

Definition 1. A *piecewise constant strategy* W of the pursuer P is a pair (σ, W_σ), where $\sigma = \{\tau_q\}_{q=0}^{n}$ is a partition of the interval $[0,T]$ and W_σ is a family of mappings d_r, $r = 0, \dots, \eta - 1$, that take pairs $(\tau_r, x(\tau_r)) \in [0,T] \times \mathbb{R}^k$ to the constant control $\overline{u}_r(t) \equiv \overline{u}_r \in U$, $t \in [\tau_r, \tau_{r+1})$.

By an *evader's control* we mean an arbitrary measurable function $v : [0, \infty) \to V$.

Denote this game by $\Gamma(x_0)$.

Definition 2. We say that an *ε-capture occurs* in the game $\Gamma(x_0)$ if there exists a $T > 0$ such that for each $\hat{\varepsilon} > 0$ there exists a piecewise constant strategy W of the pursuer P such that for each admissible evader's control $v(\cdot)$ the inequality $\|x(\tau)\| < \hat{\varepsilon}$ holds for some $\tau \in [0,T]$.

The pursuer's goal is to perform an ε-capture.

The goal of the evader is to prevent this.

Definition 3 [12]. A set of vectors $a_1, \dots, a_n \in \mathbb{R}^k$ is called a *positive basis* in $\mathbb{R}^k$ if for each point $\xi \in \mathbb{R}^k$ there exist nonnegative numbers $\mu_1, \dots, \mu_n$ such that $\xi = \sum_{i=1}^{n} \mu_i a_i$.

We use the following notation: $\operatorname{Int} A$ is the interior of a set A; $\operatorname{co} A$ is a convex hull of the set A; $O_\varepsilon(x)$ is the ε-neighborhood of a point x; $D_\varepsilon(x)$ is the closed ball of radius ε centered at x.

The following capture theorem holds true [22].

Theorem 1 [22]. *Let vectors $f(0,u_1), \dots, f(0,u_m)$ form a positive basis, and let the inclusions $-g(0,V) \subset \operatorname{Int}(\operatorname{co}\{f(0,u_1), \dots, f(0,u_m)\})$ hold. Then there exists an $\varepsilon_0 > 0$ such that for each point $x_0 \in O_{\varepsilon_0}(0)$ an ε-capture occurs in the game $\Gamma(x_0)$.*

Remark 1. According to the proof of Theorem 1, the motion generated by the winning strategy of the pursuer resides inside the ball $D_{x_0}(0)$. Therefore, it suffices to have functions $f(\,\cdot\,,\cdot\,)$ and $g(\,\cdot\,,\cdot\,)$ defined in some neighborhood of zero in the state space. In this case, these functions can be locally Lipschitz in the above-indicated sense.

Remark 2. Without loss of generality, we can assume that $U = \{1, \dots, m\}$, because the pursuer's control is constant on the partition intervals, i.e., the function f has the form $f(x,j) = f_j(x)$, where $f_j : \mathbb{R}^k \to \mathbb{R}^k$ is a function Lipschitz in x. Moreover, the set U can be an arbitrary nonempty subset in $\mathbb{R}^l$ under the condition that for each $u \in U$ the function f is Lipschitz in x. In this case, if there exists a finite tuple of numbers $\{u_1, \dots, u_m\} \subset U$ satisfying the assumptions of Theorem 1, then an ε-capture occurs.

2. CAPTURE STRATEGY CONSTRUCTED IN [22]

Let us present the winning strategy of the pursuer found in [22], the accompanying notation, and some of the results established in the proof of Theorem 1. We assume that the conditions of this theorem are satisfied. The existence of the parameters indicated in this section was established in [22] in the proof of Theorem 1.

There exist $\alpha > 0$ and $\varepsilon_0 > 0$ such that for each point $x \in D_{\varepsilon_0}(0)$ and each vector $p \in \mathbb{R}^k$, $\|p\| = 1$, there exists an $i \in \{1, \dots, m\}$ such that for any $v \in V$ one has the inequality

$$\big\langle f(x, u_i) + g(x, v), p \big\rangle \geq \alpha,$$

where

$$\alpha = \min_{x \in D_{\varepsilon_0}(0)} \min_{\|p\|=1} \min_{v \in V} \max_{i=1,\dots,m} \big\langle f(x, u_i) + g(x, v), p \big\rangle.$$

There exists a number $h > 0$ such that for each $x_0 \in D_{\varepsilon_0}(0) \setminus \{0\}$ and each $v \in V$ the inequality

$$\big\langle f(x, \overline{u}_0) + g(x, v), -x_0/\|x_0\| \big\rangle \geq \alpha/2 = \overline{\alpha} \tag{3}$$

holds for all $x \in D_h(x_0)$. Here $\overline{u}_0$ is found from the following maximum:

$$\max_{u \in U} \big\langle f(x_0, u), -x_0/\|x_0\| \big\rangle = \big\langle f(x_0, \overline{u}_0), -x_0/\|x_0\| \big\rangle. \tag{4}$$

In this case, it suffices to take $h = \alpha/(2L_1 + 2L_2)$.

Let D be a number for which the inequality $\|f(x, u_i) + g(x, v)\| \leq D$ holds for all $x \in D_{\varepsilon_0}(0)$, any $v \in V$, and each $i \in \{1, \dots, m\}$.

Denote

$$\Delta(\xi) = \min\big\{\overline{\alpha}\|\xi\|/D^2, h/D\big\}. \tag{5}$$

It was shown in the paper [22] that when realizing the pursuer's strategy, the interval partition length $[\tau_j, \tau_{j+1})$, $j = 0, 1, \dots$, is selected using the function $\Delta(\cdot)$ defined by relation (5) and is given by the relation $\tau_{j+1} - \tau_j = \Delta(x(\tau_j))$. The control $\overline{u}_j$ is found from the following maximum:

$$\max_{u \in U} \Big\langle f\big(x(\tau_j), u\big), -x(\tau_j)/\big\|x(\tau_j)\big\| \Big\rangle = \Big\langle f\big(x(\tau_j), \overline{u}_j\big), -x(\tau_j)/\big\|x(\tau_j)\big\| \Big\rangle. \tag{6}$$

For each $j = 0, 1, \dots$, one has the estimate

$$\begin{aligned} \big\|x(\tau_{j+1})\big\|^2 &= \big\|x(\tau_j)\big\|^2 + \left\| \int\limits_{\tau_j}^{\tau_j + \Delta\left(x(\tau_j)\right)} \Big(f\big(x(s), \overline{u}_j\big) + g\big(x(s), v(s)\big)\Big)\, ds \right\|^2 \\ &\quad + 2 \int\limits_{\tau_j}^{\tau_j + \Delta(x(\tau_j))} \Big\langle f\big(x(s), \overline{u}_j\big) + g\big(x(s), v(s)\big), x(\tau_j) \Big\rangle\, ds \\ &\leq \big\|x(\tau_j)\big\|^2 + D^2 \Big(\Delta\big(x(\tau_j)\big)\Big)^2 - 2\Delta\big(x(\tau_j)\big)\overline{\alpha}\big\|x(\tau_j)\big\| \\ &\leq \big\|x(\tau_j)\big\|^2 - \Delta\big(x(\tau_j)\big)\overline{\alpha}\big\|x(\tau_j)\big\|. \end{aligned} \tag{7}$$

Moreover, $\|x(t)\| < \|x(\tau_j)\|$, $t \in (\tau_j, \tau_{j+1})$.

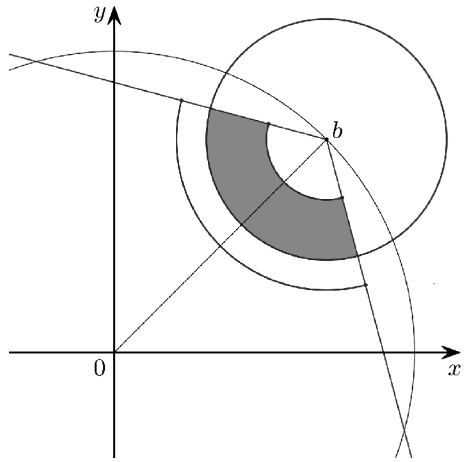

Fig. 1. Geometric meaning of the parameter choice.

An ε-capture occurs when using this strategy. The proof of Theorem 1 (see [22]) gives a general upper bound (i.e., an upper bound valid for all $x_0 \in D_{\varepsilon_0}(0) \setminus \{0\}$) of the ε-capture time.

The geometric meaning of this choice of the parameters for the above-described strategy is as follows.

Let the initial position be at a point $b \in D_{\varepsilon_0}(0)$ (Fig. 1). The pursuer's control is chosen according to the maximum (4), where $x_0 = b$. Then for all $t \in [0, \Delta(b)]$ the inclusion $x(t) \in D_h(b)$ holds (Fig. 1, small circle). Therefore, up to the time $\Delta(b)$, inequality (3) for the velocity will hold; i.e., the velocity vector will be in the convex cone defined by the positive number $\overline{\alpha}$. Thus, the trajectory will also be contained in the convex cone defined by the number $\overline{\alpha}$ but with a vertex at the point b (Fig. 1, rays issuing from the point b). By virtue of the definition of the function $\Delta(\cdot)$, by the time $\Delta(b)$ the trajectory in the cone will go no further than a certain distance (Fig. 1, large arc). Moreover, this distance is equal to half the length of an arbitrary chord drawn from the point b along the cone boundary. Since, by virtue of (3), the inequality $\|\dot{x}(t)\| \geq \overline{\alpha}$ is true for all $t \in [0, \Delta(b)]$, at the moment $\Delta(b)$ the point of the trajectory will be located in the cone at a distance from the point b no closer than $\overline{\alpha}\Delta(b)$ (Fig. 1, small arc). It follows from the above that by the time $\Delta(b)$ the trajectory of the system will be in the shaded domain (Fig. 1).

3. GUARANTEED CAPTURE TIME

By $\mathfrak{S}$ we denote the set of systems satisfying the statement of the problem and Theorem 1. In other words, by an element $\mathfrak{s} \in \mathfrak{S}$ we will mean a tuple $(f(\,\cdot\,,\cdot\,), g(\,\cdot\,,\cdot\,), U, V)$ for which the following conditions are satisfied:

1. $k, l, s, m \in \mathbb{N}$, $k \geq 2$.
2. The set $U = \{u_1, \ldots, u_m\}$ is finite, $u_i \in \mathbb{R}^l$, $i = 1, \ldots, m$.
3. The set $V \subset \mathbb{R}^s$ is a compact set.
4. The function $f : \mathbb{R}^k \times U \to \mathbb{R}^k$ is Lipschitz in x for each $u \in U$, and the function $g : \mathbb{R}^k \times V \to \mathbb{R}^k$ is jointly Lipschitz in all variables; i.e., there exist positive numbers L_1 and L_2 such that the estimates (2) hold true.
5. The vectors $f(0, u_1), \ldots, f(0, u_m)$ form a positive basis in $\mathbb{R}^k$, and one has the inclusion $-g(0, V) \subset \operatorname{Int}(\operatorname{co}\{f(0, u_1), \ldots, f(0, u_m)\})$.

The differential game (1) corresponding to the quadruple $\mathfrak{s} \in \mathfrak{S}$ and the initial position x_0 will be denoted by $\Gamma(\mathfrak{s}, x_0)$.

Let $\mathfrak{s} \in \mathfrak{S}$. Define a number $\varepsilon_0(\mathfrak{s})$ by the condition

$$\varepsilon_0(\mathfrak{s}) = \sup\{r \geq 0 : -g(x, V) \subset \operatorname{Int}(\operatorname{co}\{f(x, u_1), \ldots, f(x, u_m)\}), \quad x \in D_r(0)\}.$$

Further, we define the set $O(\mathfrak{s}) \subset \mathbb{R}^k$ by the equality

$$O(\mathfrak{s}) = \begin{cases} O_{\varepsilon_0(\mathfrak{s})}(0), & \varepsilon_0(\mathfrak{s}) < +\infty \\ \mathbb{R}^k, & \varepsilon_0(\mathfrak{s}) = +\infty. \end{cases}$$

Note that, according to Theorem 1, for each $\mathfrak{s} \in \mathfrak{S}$ and any $x_0 \in O(\mathfrak{s})$ an ε-capture occurs in the game $\Gamma(\mathfrak{s}, x_0)$.

Set

$$\begin{aligned} D(\mathfrak{s}, r) &= \max\Big\{\big\|f(x,u) + g(x,v)\big\| : x \in D_r(0), u \in U, v \in V\Big\}, \\ \alpha(\mathfrak{s}, x_0) &= \min_{x \in D_{\|x_0\|}(0)} \min_{\|p\|=1} \min_{v \in V} \max_{i=1,\ldots,m} \big\langle f(x,u_i) + g(x,v), p\big\rangle. \end{aligned} \tag{8}$$

For $\mu \in [0,1]$, we define the functions

$$h(\mathfrak{s}, \mu) = \frac{\mu \alpha(\mathfrak{s}, x_0)}{2(L_1 + L_2)} \tag{9}$$

and

$$\overline{\alpha}(\mathfrak{s}, \mu) = \alpha(\mathfrak{s}, x_0)\left(1 - \frac{\mu}{2}\right). \tag{10}$$

Theorem 2. *Let $\mathfrak{s} \in \mathfrak{S}$ and $x_0 \in O(\mathfrak{s})$, $x_0 \neq 0$, and let L_1, and L_2 be the Lipschitz constants corresponding to the quadruple $\mathfrak{s}$. Then in the game $\Gamma(\mathfrak{s}, x_0)$ for each $\delta > 0$, $\delta < \|x_0\|$, the system trajectory can be transferred to the ball $D_\delta(0)$ using a piecewise constant pursuer's strategy with the fixed partition step $\Delta = h(\mathfrak{s}, \mu_0)/D(\mathfrak{s}, \|x_0\|)$ in time*

$$T_\delta \leq \frac{\|x_0\|^2 - \delta^2}{2\overline{\alpha}(\mathfrak{s}, \mu_0)\delta - \Big(D\big(\mathfrak{s}, \|x_0\|\big)\Big)^2 \Delta} + \Delta,$$

where

$$\mu_0 = \min\Big\{1, (L_1 + L_2)\delta / D\big(\mathfrak{s}, \|x_0\|\big)\Big\}. \tag{11}$$

Proof. Since the quadruple $\mathfrak{s}$ is fixed, we denote $D(\mathfrak{s}, \|x_0\|) = D$, $\alpha(\mathfrak{s}, x_0) = \alpha(x_0)$, $h(\mathfrak{s}, \mu) = h(\mu)$, and $\overline{\alpha}(\mathfrak{s}, \mu) = \overline{\alpha}(\mu)$ to simplify the notation.

1⁰. In this part of the proof, we derive the estimates corresponding to the functions $\alpha(x_0)$, $h(\mu)$, and $\overline{\alpha}(\mu)$.

Let $p \in \mathbb{R}^k$, $\|p\| = 1$, $\mu \in (0,1]$, $\xi \in D_{\|x_0\|}(0)$, and $\overline{x} \in D_{h(\mu)}(\xi)$. Choose a value $\overline{u} \in U$ on which the following maximum is achieved:

$$\max_{u \in U} \langle f(\xi, u), p\rangle = \langle f(\xi, \overline{u}), p\rangle.$$

Let us prove that for any $v \in V$ one has the inequality

$$\langle f(\overline{x}, \overline{u}) + g(\overline{x}, v), p\rangle \geq \overline{\alpha}(\mu). \tag{12}$$

Using definitions (8), we estimate the inner product in (12),

$$\begin{aligned} \big\langle f(\overline{x}, \overline{u}) + g(\overline{x}, v), p\big\rangle &= \big\langle f(\overline{x}, \overline{u}) - f(\xi, \overline{u}) + f(\xi, \overline{u}) + g(\overline{x}, v) - g(\xi, v) + g(\xi, v), p\big\rangle \\ &= \big\langle f(\xi, \overline{u}) + g(\xi, v), p\big\rangle + \big\langle f(\overline{x}, \overline{u}) - f(\xi, \overline{u}), p\big\rangle + \big\langle g(\overline{x}, v) - g(\xi, v), p\big\rangle \\ &\geq \alpha(x_0) - \big\|f(\overline{x}, \overline{u}) - f(\xi, \overline{u})\big\| - \big\|g(\overline{x}, v) - g(\xi, v)\big\| \\ &\geq \alpha(x_0) - L_1\|\overline{x} - \xi\| - L_2\|\overline{x} - \xi\| \geq \alpha(x_0) - (L_1 + L_2)h(\mu) \\ &= \alpha(x_0) - (L_1 + L_2)\frac{\mu\alpha(x_0)}{2(L_1 + L_2)} = \overline{\alpha}(\mu). \end{aligned}$$

Thus, inequality (12) has been proved.

Note that the function $h(\mu)$ is strictly increasing, the function $\overline{\alpha}(\mu)$ is strictly decreasing, and the following double inequalities hold:

$$0 \le h(\mu) \le \frac{\alpha(x_0)}{2(L_1+L_2)} \quad \text{and} \quad \frac{\alpha(x_0)}{2} \le \overline{\alpha}(\mu) \le \alpha(x_0), \quad \mu \in [0,1]. \tag{13}$$

2⁰. In this part of the proof, we construct a strategy that sends the system trajectory into the ball $D_\delta(0)$.

By virtue of definitions (9) and (11) and inequalities (13), we have the estimate

$$h(\mu_0) \le \overline{\alpha}(\mu_0)\delta/D. \tag{14}$$

Define a fixed partition step $\Delta = h(\mu_0)/D$. We choose a fixed pursuer's control $\overline{u}_j \in U$ on the interval $[\tau_j, \tau_{j+1})$, $j = 0, \dots, \eta$, from condition (6).

Let us estimate the squared norm using inequalities (12), (14), and (8). For all $t \in (0, \tau_1]$, we have

$$\begin{aligned} \|x(t)\|^2 &= \left\| x_0 + \int\limits_0^t \Big(f\big(x(s), \overline{u}_0\big) + g\big(x(s), v(s)\big) \Big)\, ds \right\|^2 \\ &= \|x_0\|^2 + \left\| \int\limits_0^t \Big(f\big(x(s), \overline{u}_0\big) + g\big(x(s), v(s)\big) \Big)\, ds \right\|^2 \\ &\quad + 2\int\limits_0^t \Big\langle f\big(x(s), \overline{u}_0\big) + g\big(x(s), v(s)\big), x_0 \Big\rangle\, ds \\ &\le \|x_0\|^2 + D^2t^2 - 2t\overline{\alpha}(\mu_0)\|x_0\| \le \|x_0\|^2 + D^2 t\Delta - 2tDh(\mu_0) \\ &= \|x_0\|^2 + Dth(\mu_0) - 2tDh(\mu_0) < \|x_0\|^2. \end{aligned} \tag{15}$$

Note that under the conditions in question, by virtue of the choice (6) of the pursuer's control, the system trajectory never leaves the ball $D_{\|x_0\|}(0)$. Therefore, when deriving inequalities (15), it is correct to estimate the norm of the system velocity by the quantity $D = D(\mathfrak{s}, \|x_0\|)$.

The following inequality holds by virtue of (15):

$$\|x(\tau_1)\|^2 \le \|x_0\|^2 + D^2\Delta^2 - 2\Delta\overline{\alpha}(\mu_0)\delta < \|x_0\|^2. \tag{16}$$

Let $x(\tau_1), \dots, x(\tau_{\eta-1}) \notin D_\delta(0)$. Then, by analogy with (16), for all $j = 1, \dots, \eta$ the inequalities $\|x(\tau_j)\|^2 \le \|x(\tau_{j-1})\|^2 + D^2\Delta^2 - 2\Delta\overline{\alpha}(\mu_0)\delta < \|x(\tau_{j-1})\|^2$ hold. Consequently,

$$\|x(\tau_\eta)\|^2 \le \|x_0\|^2 + \eta D^2\Delta^2 - 2\eta\Delta\overline{\alpha}(\mu_0)\delta.$$

Based on this, if $\|x_0\|^2 + \eta D^2\Delta^2 - 2\eta\Delta\overline{\alpha}(\mu_0)\delta \le \delta^2$, then $x(\tau_\eta) \in D_\delta(0)$. Thus,

$$\eta \le \left[\frac{\|x_0\|^2 - \delta^2}{2\Delta\overline{\alpha}(\mu_0)\delta - D^2\Delta^2} \right] + 1. \tag{17}$$

Here $[\cdot]$ stands for the integer part of a number. If η is strictly greater than the right-hand side of inequality (17), then $x(\tau_{\eta-1}) \in D_\delta(0)$; this contradicts the assumption $x(\tau_{\eta-1}) \notin D_\delta(0)$.

Let us estimate τ_η,

$$\tau_\eta = \eta\Delta \le \left(\left[\frac{\|x_0\|^2 - \delta^2}{2\Delta\overline{\alpha}(\mu_0)\delta - D^2\Delta^2} \right] + 1 \right)\Delta \le \left(\frac{\|x_0\|^2 - \delta^2}{2\Delta\overline{\alpha}(\mu_0)\delta - D^2\Delta^2} + 1 \right)\Delta = \frac{\|x_0\|^2 - \delta^2}{2\overline{\alpha}(\mu_0)\delta - D^2\Delta} + \Delta.$$

Thus,

$$T_\delta \le \frac{\|x_0\|^2 - \delta^2}{2\overline{\alpha}(\mu_0)\delta - D^2\Delta} + \Delta,$$

where $\mu_0 = \min\{1, (L_1 + L_2)\delta/D\}$ and $\Delta = h(\mu_0)/D$. The proof of the theorem is complete.

Denote

$$T(\mathfrak{s}, x_0) = \|x_0\|/\alpha(\mathfrak{s}, x_0).$$

Theorem 3. *The set $\mathfrak{S}$ possesses the following properties*:

1. *For each $\mathfrak{s} \in \mathfrak{S}$ and any $x_0 \in O(\mathfrak{s})$, an ε-capture occurs in the game $\Gamma(\mathfrak{s}, x_0)$ in time $T(\mathfrak{s}, x_0)$.*
2. *There exist $\mathfrak{c} \in \mathfrak{S}$ and $x_0 \in O(\mathfrak{c})$ for which an ε-capture does not happen within any time $\overline{T} < T(\mathfrak{c}, x_0)$ in the game $\Gamma(\mathfrak{c}, x_0)$.*

Proof. Since the quadruple $\mathfrak{s}$ is fixed, to simplify the notation, we denote $D(\mathfrak{s}, \|x_0\|) = D$, $\alpha(\mathfrak{s}, x_0) = \alpha(x_0)$, $h(\mathfrak{s}, \mu) = h(\mu)$, $\overline{\alpha}(\mathfrak{s}, \mu) = \overline{\alpha}(\mu)$, and $T(x_0) = T(\mathfrak{s}, x_0)$.

1⁰. In this part of the proof, we construct a capture strategy using Theorem 2.

Fix an arbitrary number $\omega \in (0, 1)$. Choose a μ_1 such that

$$0 < \mu_1 \le \min\bigl\{1, (L_1 + L_2)\omega\|x_0\|/D\bigr\}. \tag{18}$$

Then, by analogy with (14), we have the estimate

$$h(\mu_1) \le \overline{\alpha}(\mu_1)\omega\|x_0\|/D. \tag{19}$$

Denote $\Delta_1 = h(\mu_1)/D$. By analogy with **2⁰** in the proof of Theorem 2, using a fixed partition step Δ_1, it can be shown that the system trajectory is transferred into the ball $D_{\omega\|x_0\|}(0)$ in time T_1, where

$$T_1 \le \frac{\|x_0\|^2(1 - \omega^2)}{2\overline{\alpha}(\mu_1)\omega\|x_0\| - D^2\Delta_1} + \Delta_1.$$

Further, let $\mu_2 = \omega\mu_1$. Then $h(\mu_2) = \omega h(\mu_1)$. Consequently, since $\overline{\alpha}(\cdot)$ is a strictly decreasing function, by virtue of (19), we have the estimate

$$h(\mu_2) \le \overline{\alpha}(\mu_1)\omega^2\|x_0\|/D \le \overline{\alpha}(\mu_2)\omega^2\|x_0\|/D. \tag{20}$$

Set $\Delta_2 = h(\mu_2)/D$. Thus, using a fixed partition step Δ_2, it can be seen that the system trajectory is sent into the ball $D_{\omega^2\|x_0\|}(0)$ from the position $x(T_1)$ in time T_2. Since $\|x(T_1)\| \le \omega\|x_0\|$, we have

$$T_2 \le \frac{\|x(T_1)\|^2 - \omega^4\|x_0\|^2}{2\overline{\alpha}(\mu_2)\omega^2\|x_0\| - D^2\Delta_2} + \Delta_2 \le \frac{\omega^2\|x_0\|^2(1 - \omega^2)}{2\overline{\alpha}(\mu_2)\omega^2\|x_0\| - D^2\Delta_2} + \Delta_2. \tag{21}$$

Then we repeat this procedure. Set $\mu_3 = \omega\mu_2$; then $h(\mu_3) = \omega h(\mu_2)$ and the inequality

$$h(\mu_3) \le \overline{\alpha}(\mu_2)\omega^3\|x_0\|/D \le \overline{\alpha}(\mu_3)\omega^3\|x_0\|/D$$

holds by virtue of (20). The fixed step is $\Delta_3 = h(\mu_3)/D$. By analogy with (21), we estimate T_3 as

$$T_3 \le \frac{\omega^4\|x_0\|^2(1 - \omega^2)}{2\overline{\alpha}(\mu_3)\omega^3\|x_0\| - D^2\Delta_3} + \Delta_3.$$

And so on for each $q \in \mathbb{N}$. As a result, we obtain

$$\begin{gathered} \mu_q = \omega^{q-1}\mu_1, \quad h(\mu_q) = \omega^{q-1}h(\mu_1), \quad \Delta_q = \omega^{q-1}\Delta_1, \\ T_q \le \frac{\omega^{2(q-1)}\|x_0\|^2(1 - \omega^2)}{2\overline{\alpha}(\mu_q)\omega^q\|x_0\| - D^2\Delta_q} + \Delta_q. \end{gathered} \tag{22}$$

By virtue of the construction of this procedure, we have the inequality

$$x(T_1 + \ldots + T_q) \le \omega^q \|x_0\|.$$

Thus, we can transfer the system trajectory into any predetermined neighborhood of zero. Consequently, for an ε-capture to occur when using this procedure, it remains to show that the quantity $\sum_{q=1}^{\infty} T_q$ is bounded above.

Using inequalities (22), let us transform the estimate for T_q as follows:

$$\begin{aligned} T_q &\le \frac{\omega^{2(q-1)}\|x_0\|^2(1-\omega^2)}{2\overline{\alpha}(\mu_q)\omega^q\|x_0\| - D^2\Delta_q} + \Delta_q = \frac{\omega^{2(q-1)}\|x_0\|^2(1-\omega^2)}{2\overline{\alpha}(\mu_q)\omega^q\|x_0\| - D^2\omega^{q-1}\Delta_1} + \omega^{q-1}\Delta_1 \\ &= \frac{\|x_0\|^2(1-\omega^2)}{2\overline{\alpha}(\mu_q)\omega\|x_0\| - Dh(\mu_1)}\omega^{q-1} + \omega^{q-1}\Delta_1 \le \frac{\|x_0\|^2(1-\omega^2)}{2\overline{\alpha}(\mu_1)\omega\|x_0\| - Dh(\mu_1)}\omega^{q-1} + \omega^{q-1}\Delta_1. \end{aligned} \tag{23}$$

Now, using (23), we estimate the sum

$$\begin{aligned} \sum_{q=1}^{\infty} T_q &\le \sum_{q=1}^{\infty} \frac{\|x_0\|^2(1-\omega^2)}{2\overline{\alpha}(\mu_1)\omega\|x_0\| - Dh(\mu_1)}\omega^{q-1} + \sum_{q=1}^{\infty}\omega^{q-1}\Delta_1 \\ &= \frac{\|x_0\|^2(1-\omega^2)}{2\overline{\alpha}(\mu_1)\omega\|x_0\| - Dh(\mu_1)}\frac{1}{1-\omega} + \frac{1}{1-\omega}\Delta_1 = \frac{\|x_0\|^2(1+\omega)}{2\overline{\alpha}(\mu_1)\omega\|x_0\| - Dh(\mu_1)} + \frac{h(\mu_1)}{D(1-\omega)}. \end{aligned}$$

Thus, an ε-capture from the initial position x_0 occurs in finite time $T(\omega, \mu_1)$ determined by the relation

$$T(\omega, \mu_1) = \frac{\|x_0\|^2(1+\omega)}{2\overline{\alpha}(\mu_1)\omega\|x_0\| - Dh(\mu_1)} + \frac{h(\mu_1)}{D(1-\omega)}. \tag{24}$$

2^0. In this part of the proof, we estimate the capture time using relation (24). Based on the construction of this estimate, let us show that the properties in the condition of the theorem are satisfied.

Set

$$\overline{\mu} = \min\{1, (L_1 + L_2)\omega\|x_0\|/D\}.$$

Note that the number $T(\omega, \mu_1)$ is defined for each $\mu_1 \in (0, \overline{\mu}]$ by virtue of (18). Since the function $h(\cdot)$ is strictly increasing and $\overline{\alpha}(\cdot)$ is strictly decreasing, we have

$$\inf_{\mu_1 \in (0,\overline{\mu}]} T(\omega, \mu_1) = \lim_{\mu_1 \to 0+} T(\omega, \mu_1).$$

Using definitions (9) and (10), let us find the value of the last limit, which we will denote by $T(\omega, 0)$; i.e.,

$$\lim_{\mu_1 \to 0+} T(\omega, \mu_1) = \lim_{\mu_1 \to 0+} \left(\frac{\|x_0\|^2(1+\omega)}{2\overline{\alpha}(\mu_1)\omega\|x_0\| - Dh(\mu_1)} + \frac{h(\mu_1)}{D(1-\omega)} \right) = \frac{\|x_0\|^2(1+\omega)}{2\alpha(x_0)\omega\|x_0\|}.$$

Thus, for each $\omega \in (0,1)$ an ε-capture occurs in any time $T > T(\omega, 0)$.

The function $T(\omega, 0)$ is strictly decreasing for $\omega \in (0,1)$. Therefore,

$$\inf_{\omega \in (0,1)} T(\omega, 0) = \lim_{\omega \to 1-} T(\omega, 0) = \lim_{\omega \to 1-} \frac{\|x_0\|^2(1+\omega)}{2\alpha(x_0)\omega\|x_0\|} = \frac{\|x_0\|}{\alpha(x_0)} = T(x_0).$$

Let us show that an ε-capture occurs in time $T(x_0)$. By construction, an ε-capture occurs in any time $T > T(x_0)$. Let $\delta > 0$ and $\overline{T} = T(x_0) + \delta/(2D)$. Then, in time $\overline{T}$, an ε-capture occurs. Consequently, there exists a piecewise constant pursuer strategy such that $\|x(\tau)\| < \delta/2$ for some $\tau \in [0, \overline{T}]$. By virtue of definitions (8), the inequality

$$\Big| \|x(\tau)\| - \big\|x\big(\tau - \delta/(2D)\big)\big\| \Big| \le D\delta/(2D) = \delta/2$$

holds. Therefore, $\|x(\tau - \delta/(2D))\| \le \|x(\tau)\| + \delta/2 < \delta$. Moreover, $\tau - \delta/(2D) \le T(x_0)$. Thus, it has been proved that an ε-capture occurs in time $T(x_0)$. Consequently, property 1 holds in the statement of the theorem.

Let us provide an example for which property 2 in the statement of the theorem holds true. In the case of $k = s = 2$, consider the system

$$\dot{x}_1 = u_1 + v_1, \quad \dot{x}_2 = u_2 + v_2,$$
$$f(x,u) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad g(x,v) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$U = \left\{ \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}, \begin{pmatrix} -1.5 \\ 1.5 \end{pmatrix}, \begin{pmatrix} -1.5 \\ -1.5 \end{pmatrix}, \begin{pmatrix} 1.5 \\ -1.5 \end{pmatrix} \right\}, \quad V = [-1,1] \times [-1,1], \quad v(t) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad t \ge 0.$$

Note that here, according to Theorem 1, an ε-capture will occur from any initial position $x_0 \in \mathbb{R}^2$. For this example, $\alpha(x_0) = 0.5$ and $T(x_0) = 2$.

Let us estimate $x_2(t)$,

$$x_2(t) = 1 + \int\limits_0^t (u_2(s) + 1)\, ds \ge 1 + \int\limits_0^t (-1.5 + 1)\, ds = 1 - 0.5t.$$

Based on this, if $T \in [0, 2)$, then $x(T) \notin O_{1-0.5T}(0)$. Consequently, an ε-capture does not occur in time $T < 2 = T(x_0)$. Thus, property 2 in the statement of the theorem does not occur. The proof of the theorem is complete.

4. COMPUTER MODELING

Consider a differential game in $\mathbb{R}^2$. The system (2) of differential equations has the form

$$\dot{x}_1 = u_1 \cos\big(|x_1| + |x_2|\big) - u_2 \sin\big(|x_1| + |x_2|\big) + v_1 \cos\left(\frac{\pi}{2} - |x_1| - |x_2|\right) - v_2 \sin\left(\frac{\pi}{2} - |x_1| - |x_2|\right),$$
$$\dot{x}_2 = u_1 \sin\big(|x_1| + |x_2|\big) + u_2 \cos\big(|x_1| + |x_2|\big) + v_1 \sin\left(\frac{\pi}{2} - |x_1| - |x_2|\right) + v_2 \cos\left(\frac{\pi}{2} - |x_1| - |x_2|\right),$$
$$u(t) = \big(u_1(t), u_2(t)\big) \in U = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\},$$
$$v(t) = \big(v_1(t), v_2(t)\big) \in V = \operatorname{co} \left\{ \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} \right\}$$

with the initial condition

$$x_0 = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} = \begin{pmatrix} 0.2 \\ 3 \end{pmatrix}.$$

Thus,

$$f(x,u) = A\big(|x_1| + |x_2|\big)u, \quad g(x,v) = A\left(\frac{\pi}{2} - |x_1| - |x_2|\right)v,$$

where $A(\cdot)$ is the rotation matrix, $u = (u_1, u_2)^{\mathrm{T}}$, and $v = (v_1, v_2)^{\mathrm{T}}$.

This system satisfies the assumptions of Theorem 1 with an ε-capture occurring for each $x_0 \in \mathbb{R}^2$. The system has the following parameters:

$$\alpha(x_0) = 1 - 0.5\sqrt{2}, \quad L_1 = 2\sqrt{2}, \quad L_2 = \sqrt{2}, \quad D = 1.5\sqrt{2}.$$

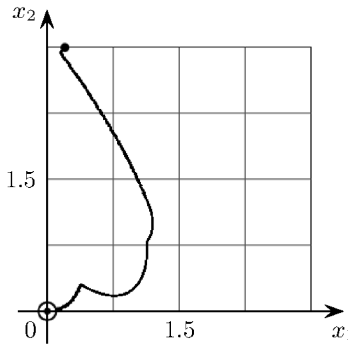

Fig. 2. Resulting trajectory $(x_1(t), x_2(t))$.

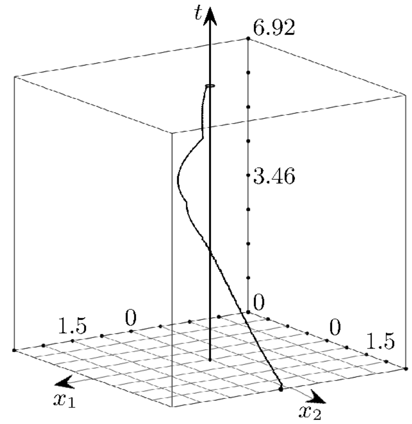

Fig. 3. Resulting solution $(x_1(t), x_2(t), t)$.

Take $\delta = 0.1$. Then, according to Theorem 2, to transfer the system trajectory into the ball $D_\delta(0)$, it suffices to use a fixed partition step $\Delta \le (1 - 0.5\sqrt{2})/18$. Choose $\Delta = 0.0162$.

An approximate solution of this system will be found by the third-order Runge–Kutta method with step 10^{-4}. The evader's control at each step of the method is constant; it is chosen based on the following maximum: $\max\limits_{v\in V}\langle g(\hat{x}, v), \hat{x}/\|\hat{x}\|\rangle = \langle g(\hat{x}, \hat{v}), \hat{x}/\|\hat{x}\|\rangle$, where $\hat{x}$ is the position at the beginning of a method step.

The simulation result is as follows: the time of reaching the ball $D_\delta(0)$ is equal to $T_\delta = 6.9187$; the system trajectory and the resulting solution are presented in Figs. 2 and 3. Note that $T_\delta < T((1 - \delta/\|x_0\|)x_0) \approx 9.9239$.

CONCLUSIONS

For one class of nonlinear differential pursuit games, it is shown that it is possible to use the pursuer's strategy with a constant step of partitioning the time interval. An estimate of the capture time from a given initial position is obtained, which is sharp in a certain sense.

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