Izvestiya Instituta Matematiki i Informatiki Udmurtskogo Gosudarstvennogo Universiteta
2022. Volume 59. Pp. 15–24

MSC2020: 35D30, 35J67, 46E35, 47H11

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EXISTENCE OF WEAK SOLUTIONS FOR A $P(X)$-LAPLACIAN EQUATION VIA TOPOLOGICAL DEGREE

We consider the $p(x)$-Laplacian equation with a Dirichlet boundary value condition

$$\begin{cases} -\Delta_{p(x)}(u) + |u|^{p(x)-2}u = g(x, u, \nabla u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

Using the topological degree constructed by Berkovits, we prove, under appropriate assumptions, the existence of weak solutions for this equation.

Keywords: weak solution, Dirichlet boundary condition, variable exponent Sobolev space, topological degree, $p(x)$-Laplacian.

DOI: 10.35634/2226-3594-2022-59-02

Introduction

The problem studied in this paper concerns the $p(x)$-Laplacian operator and the variable exponent $p(x)$. The study of various mathematical problems with variable exponents has received considerable attention in recent years, as these problems model several physics concerning thermorheological fluids [7], image restoration [9], electrorheological fluids [21,22] and elastic materials [27]. The $p(x)$-Laplacian is a generalization of the p-Laplacian, and it has more complicated nonlinearities than the p-Laplacian. Due to its inhomogeneous characteristic, it is reasonable to expect that the $p(x)$-Laplacian operator is suitable for modelling inhomogeneous materials. Recently, several works devoted to problems involving the p-Laplacian operator have been extended to the case of the $p(x)$-Laplacian operator. We can cite in this context the papers [4, 6, 19, 20, 24] and the references therein.

Consider the following problem with a Dirichlet boundary condition

$$\begin{cases} -\Delta_{p(x)}(u) + |u|^{p(x)-2}u = g(x, u, \nabla u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{0.1}$$

where $-\Delta_{p(x)}(u) = -\mathrm{div}\,(|\nabla u|^{p(x)-2}\nabla u)$, $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $p(\cdot)$ is a variable exponent satisfying some conditions to be seen in the paper suite and g is a Carathéodory function satisfying a growth condition with a variable exponent that is suitably controlled by $p(\cdot)$.

For g independent of ∇u, the authors in [18] have shown the existence of infinitely many pairs of solutions for this problem by applying the Fountain theorem and the dual Fountain theorem respectively. When $g(x, u, \nabla u) = |u|^{q(x)-2}u$, Alsaedi [5] studied this problem as a perturbed non-homogeneous Dirichlet problem. Several others have studied the problem (0.1) without the term $|u|^{p(x)-2}u$ with different methods in both cases where g is dependent or not on ∇u (see for example [3, 13, 16, 23]). Note that, by passing the term $|u|^{p(x)-2}u$ to the right in (0.1) and posing $f(x, u, \nabla u) = g(x, u, \nabla u) - |u|^{p(x)-2}u$, we find the problem (1) of [3] and the problem (1) of [23]. But the growth conditions (f_2) in [3] and (H_f) in [23] will no longer be satisfied because of the presence of an exponent $p(\cdot)$, although we will adopt this condition for g in our paper. For example, for $g \equiv 0$, we will have that $|f(x, u, \nabla u)| = |u(x)|^{p(x)-1}$ does not satisfy (f_2) of [3] or (H_f) of [23]: here, the exponent $q(\cdot)$ of these assumptions attains $p(\cdot)$ and we no longer have $q^+ < p^-$. In this paper we prove the existence of weak solutions for problem (0.1) with a growth condition similar to (f_2) in [3] and (H_f) in [23] but only satisfied by g, as part of f and not by the entire f, despite the appearance of the exponent $p(\cdot)$.

Fan and Han [12] discussed the existence and multiplicity of solutions of the following $p(x)$-Laplacian equation in $\mathbb{R}^N$:

$$\begin{cases} -\Delta_{p(x)}(u) + |u|^{p(x)-2}u = f(x,u), \quad x \in \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N). \end{cases}$$

This problem was later studied by Ge and Lv [15] by adding a potential term and using the mountain pass theorem and vanishing lemma. They obtained a weak solution u_λ of the perturbation equations. They proved that u_λ tends to u, a nontrivial solution of the original problem, when $\lambda \to 0$.

In this paper, motivated by the above work, we study the problem (0.1) using another approach based on the topological degree method constructed by Berkovits [8] for some classes of operators in Banach reflexive spaces. The reader can refer to [1–3, 8] and the references therein for more details about this method.

This paper is organized as follows. Section 1 is reserved for some mathematical preliminaries. In Section 2, we give our basic assumptions, some technical lemmas, and also give and prove our results of existence.

§ 1. Mathematical Preliminaries

§ 1.1. Definitions and proposition

Let us start with a short reminder of the classes of operators mentioned in the introduction and of an important proposition which will be the key to proving the existence of at least one weak solution of the problem (0.1).

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle \cdot, \cdot \rangle$ and let Ω be a nonempty subset of X. The symbol $\to$ $(\rightharpoonup)$ stands for strong (weak) convergence; (u_n) denotes a sequence $(n \in \mathbb{N})$ and $\limsup$ denotes the superior limit given by, for a sequence (v_n),

$$\limsup v_n := \lim_{n\to\infty} (\sup_{m\geq n} v_m).$$

Let Y be a real Banach space. We recall that a mapping $F\colon \Omega \subset X \to Y$ is *bounded*, if it takes any bounded set into a bounded set; F is said to be *demicontinuous*, if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$; F is said to be *compact*, if it is continuous and the image of any bounded set is relatively compact. A mapping $F\colon \Omega \subset X \to X^*$ is said to be *of class* (S_+), if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup\langle Fu_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$; F is said to be *quasimonotone*, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, it follows that $\limsup\langle Fu_n, u_n - u\rangle \geq 0$.

For any operator $F\colon \Omega \subset X \to X$ and any bounded operator $T : \Omega_1 \subset X \to X^*$ such that $\Omega \subset \Omega_1$, we say that F satisfies condition $(S_+)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup\langle Fu_n, y_n - y\rangle \leq 0$, we have $u_n \to u$. For any $\Omega \subset X$, we consider the following classes of operators:

$$\mathcal{F}_1(\Omega) := \{F\colon \Omega \to X^* \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+)\},$$
$$\mathcal{F}_{T,B}(\Omega) := \{F\colon \Omega \to X \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+)_T\}$$
$$\mathcal{F}_T(\Omega) := \{F\colon \Omega \to X \mid F \text{ is demicontinuous and satisfies condition } (S_+)_T\}.$$

Proposition 1.1. *Let $S\colon X \to X^*$ and $T\colon X^* \to X$ be two operators bounded and continuous such that S is quasimonotone and T is a homeomorphism, strictly monotone and of class (S_+). If*

$$\Lambda := \{v \in X^* \mid v + tS \circ Tv = 0 \text{ for some } t \in [0,1]\}$$

is bounded in X^*, then the equation

$$v + S \circ Tv = 0$$

admits at least one solution in X^*.

Proof. Since Λ is bounded in X^*, there exists $R > 0$ such that

$$\|v\|_{X^*} < R \text{ for all } v \in \Lambda.$$

This means that $v + tS \circ Tv \neq 0$ for all $v \in \partial B_R(0)$ and all $t \in [0,1]$, where $B_R(0)$ is the ball of center 0 and radius R in X^*. Thanks to the Minty–Browder Theorem [26, Theorem 26A], the inverse operator $L := T^{-1}$ is bounded, continuous and of type (S_+). From [8, Lemma 2.2 and 2.4] it follows that

$$I + S \circ T \in \mathcal{F}_T(\overline{B_R(0)}) \text{ and } I = L \circ T \in \mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators I, S and T are bounded, $I + S \circ T$ is also bounded. We conclude that

$$I + S \circ T \in \mathcal{F}_{T,B}(\overline{B_R(0)}) \text{ and } I \in \mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Consider a homotopy $H\colon [0,1] \times \overline{B_R(0)} \to X^*$ given by

$$H(t,v) := v + tS \circ Tv \text{ for } (t,v) \in [0,1] \times \overline{B_R(0)}.$$

Let us apply the homotopy invariance and normalization property of the Berkovits degree (which we denote by d) introduced in [8], we get

$$d(I + S \circ T, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

and hence there exists a point $v \in B_R(0)$ such that

$$v + S \circ Tv = 0.$$

§ 1.2. Functional framework

In the sequel, Ω is an open bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz boundary $\partial\Omega$ (that is $\partial\Omega$ is "sufficiently regular" in the sense that it can be thought of as locally being the graph of a Lipschitz continuous function).

In order to discuss the problem (0.1), we start with the definition of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and the variable exponent Sobolev spaces $W_0^{1,p(\cdot)}(\Omega)$, and some properties of them; for more details, see [14, 17].

Let us denote

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega})\colon h(x) > 1 \text{ for every } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we write

$$h^- := \min_{x \in \overline{\Omega}} h(x), \quad h^+ := \max_{x \in \overline{\Omega}} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space by

$$L^{p(\cdot)}(\Omega) = \{u \mid u\colon \Omega \to \mathbb{R} \text{ is measurable and } \rho_{p(\cdot)}(u) < \infty\},$$

where

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx.$$

We consider this space to be endowed with the so-called *Luxemburg norm*:

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0\colon \rho_{p(\cdot)}\Big(\frac{u}{\lambda}\Big) \le 1\}.$$

We define the variable exponent Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega)\colon |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $W_0^{1,p(\cdot)}(\Omega)$ is defined by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. With these norms, the spaces $L^{p(\cdot)}(\Omega), W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable reflexive Banach spaces.

The conjugate space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the Hölder inequality holds [17, Theorem 2.1]:

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \le 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}. \tag{1.1}$$

If $p(\cdot),\ q(\cdot) \in C_+(\overline{\Omega}),\ q(\cdot) \le p(\cdot)$ a. e. in Ω then there exists a continuous embedding $L^{p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Omega)$.

In this paper, we suppose that $p(\cdot)$ satisfies the log-Hölder continuity condition, i. e., there exists $C > 0$ such that for all $x, y \in \Omega$, $x \ne y$, one has

$$|p(x) - p(y)| \log\left(e + \frac{1}{|x-y|}\right) \le C. \tag{1.2}$$

An interesting feature of generalized variable exponent Sobolev space is that smooth functions are not dense in it without additional assumptions on the exponent $p(\cdot)$. However, when the exponent satisfies the log-Hölder condition (1.2), we recall the Poincaré inequality (see [11, Theorem 8.2.4] and [14, Theorem 2.7]): there exists a constant $C > 0$ depending only on Ω and the function p such that

$$\|u\|_{p(\cdot)} \le C\|\nabla u\|_{p(\cdot)}, \quad \forall u \in W_0^{1,p(\cdot)}(\Omega). \tag{1.3}$$

In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has a norm given by

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)},$$

which is equivalent to the norm $\|\cdot\|_{W^{1,p(\cdot)}}$. Moreover, the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact (see [17]). The space $(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is also a separable and reflexive Banach space.

The dual space of $W_0^{1,p(\cdot)}(\Omega)$, denoted $W^{-1,p'(\cdot)}(\Omega)$, is equipped with the norm

$$\|v\|_{-1,p'(\cdot)} = \inf\{\|v_0\|_{p'(\cdot)} + \sum_{i=1}^N \|v_i\|_{p'(\cdot)}\},$$

where the infimum is taken on all possible decompositions $v = v_0 - \operatorname{div} F$ with $v_0 \in L^{p'(\cdot)}(\Omega)$ and $F = (v_1, \dots, v_N) \in (L^{p'(\cdot)}(\Omega))^N$.

P r o p o s i t i o n 1.2 (see [14]). *Let* $(u_n) \subset L^{p(\cdot)}(\Omega)$ *and* $u \in L^{p(\cdot)}(\Omega)$. *Then we have*

1) $\|u\|_{p(\cdot)} \geq 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}$;

2) $\|u\|_{p(\cdot)} \leq 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}$;

3) $\lim_{n\to\infty} \|u_n - u\|_{p(\cdot)} = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$;

4) $\|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u) + 1$;

5) $\rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-} + \|u\|_{p(\cdot)}^{p^+}$.

In this paper, we will use also the following equivalent norm on $W^{1,p(\cdot)}(\Omega)$:

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 \colon \rho_{p(\cdot)}\Big(\frac{\nabla u}{\lambda}\Big) + \rho_{p(\cdot)}\Big(\frac{u}{\lambda}\Big) \leq 1\}.$$

If we denote $I(u) = \rho_{p(\cdot)}(\nabla u) + \rho_{p(\cdot)}(u)$, then, similar to Proposition 1.2, we have

P r o p o s i t i o n 1.3 (see [10]). *Let* $(u_n) \subset W^{1,p(\cdot)}(\Omega)$ *and* $u \in W^{1,p(\cdot)}(\Omega)$. *Then we have*

1) $\|u\|_{p(\cdot)} \geq 1 \quad \Rightarrow \quad \|u\|^{p^-} \leq I(u) \leq \|u\|^{p^+}$;

2) $\|u\|_{p(\cdot)} \leq 1 \quad \Rightarrow \quad \|u\|^{p^+} \leq I(u) \leq \|u\|^{p^-}$;

3) $\lim_{n\to\infty} \|u_n - u\| = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} I(u_n - u) = 0$;

4) $\|u\| \leq I(u) + 1$;

5) $I(u) \leq \|u\|^{p^-} + \|u\|^{p^+}$.

§ 2. Basic assumptions and main results

In this section, we study the strongly nonlinear problem (0.1) based on the Berkovits degree, where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is an open bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\overline{\Omega})$ satisfies the log-Hölder continuity condition (1.2) such that $1 < p(x)$ and $g \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a real-valued function such that:

(g_1) g satisfies the Carathéodory condition, that is, $g(.,\eta,\zeta)$ is measurable on Ω for all $(\eta,\zeta) \in \mathbb{R} \times \mathbb{R}^N$ and $g(x,.,.)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for a. e. $x \in \Omega$;

(g_2) g has the growth condition

$$|g(x,\eta,\zeta)| \leq c(k(x) + |\eta|^{q(x)-1} + |\zeta|^{q(x)-1})$$

for a. e. $x \in \Omega$ and all $(\eta,\zeta) \in \mathbb{R} \times \mathbb{R}^N$, where c is a positive constant, $k \in L^{p'(x)}(\Omega)$ and $1 < q^- \leq q(x) \leq q^+ < p^-$.

L e m m a 2.1 (see [3, Lemma 2]). *Suppose that assumptions* (g_1) *and* (g_2) *hold. Then the operator* $S \colon W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *defined by*

$$\langle Su, v\rangle = -\int_\Omega (g(x,u,\nabla u))v\,dx, \quad u,v \in W_0^{1,p(x)}(\Omega)$$

is compact.

Let $A\colon W_0^{1,p(\cdot)}(\Omega)\to W^{-1,p'(\cdot)}(\Omega)$ be the operator defined by

$$\langle A(u),v\rangle=\int_\Omega(|\nabla u|^{p(x)-2}\nabla u\cdot\nabla v+|u|^{p(x)-2}uv)\,dx,\quad u,v\in W_0^{1,p(\cdot)}(\Omega). \tag{2.1}$$

L e m m a 2.2 (see [12, Lemma 3.1]). *A is strictly monotone, bounded homeomorphism and is of type (S_+).*

Let us first define a weak solution of the problem (0.1).

D e f i n i t i o n 2.1. We say that $u\in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution of (0.1) if

$$\int_\Omega(|\nabla u|^{p(x)-2}\nabla u\cdot\nabla v+|u|^{p(x)-2}uv)\,dx=\int_\Omega(g(x,u,\nabla u))v\,dx\quad\forall v\in W_0^{1,p(\cdot)}(\Omega).$$

T h e o r e m 2.1. *Suppose that the assumptions (g_1) and (g_2) hold true. Then there exists at least one weak solution of the problem* (0.1) *in* $W_0^{1,p(\cdot)}(\Omega)$.

P r o o f. Let A and $S\colon W_0^{1,p(\cdot)}(\Omega)\to W^{-1,p'(\cdot)}(\Omega)$ be as in (2.1) and Lemma 2.1 respectively. Then $u\in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution of (0.1) if and only if

$$Au=-Su. \tag{2.2}$$

Thanks to the properties of the operator A seen in Lemma 2.2 and in view of Minty–Browder Theorem [26, Theorem 26A], the inverse operator $T:=A^{-1}\colon W^{-1,p'(\cdot)}(\Omega)\to W_0^{1,p(\cdot)}(\Omega)$ is bounded, continuous and of type (S_+). Moreover, note from Lemma 2.1 that the operator S is bounded, continuous and quasimonotone. Therefore, equation (2.2) is equivalent to

$$u=Tv \text{ and } v+S\circ Tv=0. \tag{2.3}$$

To solve equation (2.3), we will apply the Proposition 1.1. It is sufficient to show that the set

$$\Lambda:=\{v\in W^{-1,p'(\cdot)}(\Omega)\mid v+tS\circ Tv=0 \text{ for some } t\in[0,1]\}$$

is bounded.

Indeed, let $v\in\Lambda$ and set $u:=Tv$, then, by the equivalence of the norms $\|\cdot\|_{1,p(\cdot)}$ and $\|\cdot\|$, there exists $\alpha>0$ such that $\|Tv\|_{1,p(\cdot)}=\|u\|_{1,p(\cdot)}\le\alpha\|u\|$.

If $\|u\|\le1$, then $\|Tv\|_{1,p(\cdot)}$ is bounded. If $\|u\|>1$, then we have by Proposition 1.3

$$\|Tv\|_{1,p(\cdot)}^{p^-}\le\alpha^{p^-}\|u\|^{p-}\le\alpha^{p^-}I(u).$$

We get by the growth condition (g_2), the Hölder inequality (1.1), the inequality (5) of Proposition 1.2 and the Young inequality the estimate

$$\begin{aligned}
\|Tv\|_{1,p(\cdot)}^{p^-} &\le \alpha^{p^-}I(u)\\
&= \alpha^{p^-}\langle Au,u\rangle\\
&= \alpha^{p^-}\langle v,Tv\rangle\\
&= -t\alpha^{p^-}\langle S\circ Tv,Tv\rangle\\
&= t\alpha^{p^-}\int_\Omega g(x,u,\nabla u)u\,dx\\
&\le \text{const}\,(\int_\Omega|k(x)u(x)|dx+\rho_{q(\cdot)}(u)+\int_\Omega|\nabla u|^{q(x)-1}|u|\,dx)\\
&\le \text{const}\,(2\|k\|_{p'(\cdot)}\|u\|_{p(\cdot)}+\|u\|_{q(\cdot)}^{q^+}+\|u\|_{q(\cdot)}^{q^-}+\frac{1}{q'^-}\rho_{q(\cdot)}(\nabla u)+\frac{1}{q^-}\rho_{q(\cdot)}(u))\\
&\le \text{const}\,(\|u\|_{p(\cdot)}+\|u\|_{q(\cdot)}^{q^+}+\|u\|_{q(\cdot)}^{q^-}+\|\nabla u\|_{q(\cdot)}^{q^+}).
\end{aligned}$$

From the Poincaré inequality (1.3) and the continuous embedding $L^{p(\cdot)} \hookrightarrow L^{q(\cdot)}$, we can deduct the estimate

$$\|Tv\|_{1,p(\cdot)}^{p^-} \leq \text{const}\,(\|Tv\|_{1,p(\cdot)} + \|Tv\|_{1,p(\cdot)}^{q^+}).$$

It follows that $\{Tv|v \in B\}$ is bounded. Since the operator S is bounded, it is obvious from (2.3) that the set Λ is bounded in $W^{-1,p'(\cdot)}(\Omega)$. Hence, in virtue of Proposition 1.1, the equation $v + S \circ Tv$ has at least one non trivial solution $\bar{v}$ in $W^{-1,p'(\cdot)}(\Omega)$. So, $\bar{u} = T\bar{v}$ is a weak solution of (0.1).

E x a m p l e 2.1. As examples of functions g satisfying the assumptions (g_1) and (g_2), we can take:

- $g(x,\eta,\zeta) = g(\eta) = c|\eta|^{q-2}\eta$ where c is a positive constant and $1 < q < p^-$.
- $g(x,\eta,\zeta) = g(x,\eta) = |\eta|^{q(x)-2}\eta\log(1+|\eta|)$ where $q \in C_+(\bar{\Omega})$ with $q^+ < p^-$.
- $g(x,\eta,\zeta) = |\eta|^{q(x)-2}\eta + |\zeta|^{q(x)-1}$ or $g(x,\eta,\zeta) = k(x) + |\eta|^{q(x)-2}\eta + |\zeta|^{q(x)-1}$ where $k \in L^{p'(x)}(\Omega)$ is positive and $q \in C_+(\bar{\Omega})$ with $q^+ < p^-$.

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Received 28.12.2021

Accepted 26.04.2022

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Citation: M. Ait Hammou, E. H. Rami. Existence of weak solutions for a $p(x)$-Laplacian equation via topological degree, *Izvestiya Instituta Matematiki i Informatiki Udmurtskogo Gosudarstvennogo Universiteta*, 2022, vol. 59, pp. 15–24.

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Существование слабых решений для $p(x)$-уравнения Лапласа через топологическую степень

Ключевые слова: слабое решение, граничные условия Дирихле, пространство Соболева с переменной экспонентой, топологическая степень, $p(x)$-лапласиан.

УДК: 517.95

DOI: 10.35634/2226-3594-2022-59-02

Мы рассматриваем уравнение Лапласа с $p(x)$-лапласианом с граничным условием Дирихле

$$\begin{cases} -\Delta_{p(x)}(u) + |u|^{p(x)-2}u = g(x, u, \nabla u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Используя топологическую степень, предложенную Берковицем, мы доказываем, при соответствующих предположениях, существование слабых решений для этого уравнения.

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Поступила в редакцию 28.12.2021
Принята в печать 26.04.2022

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Цитирование: М. Аит Хамму, Э. Х. Рами. Существование слабых решений для $p(x)$-уравнения Лапласа через топологическую степень // Известия Института математики и информатики Удмуртского государственного университета. 2022. Т. 59. С. 15–24.