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NUMERICAL METHOD FOR SYSTEM OF SPACE-FRACTIONAL EQUATIONS OF SUPERDIFFUSION TYPE WITH DELAY AND NEUMANN BOUNDARY CONDITIONS

We consider a system of two space-fractional superdiffusion equations with functional general delay and Neumann boundary conditions. For this problem, an analogue of the Crank–Nicolson method is constructed, based on the shifted Grünwald–Letnikov formulas for approximating fractional Riesz derivatives with respect to a spatial variable and using piecewise linear interpolation of discrete prehistory with extrapolation by continuation to take into account the delay effect. With the help of the Gershgorin theorem, the solvability of the difference scheme and its stability are proved. The order of convergence of the method is obtained. The results of numerical experiments are presented.

Keywords: superdiffusion equations, Neumann conditions, functional delay, Riesz derivatives, Grünwald–Letnikov approximation, Crank–Nicholson method, order of convergence.

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Introduction

Systems of ordinary differential equations with delay effect, including distributed ones, occur in many mathematical models, for example, predator–prey [1]. Modern mathematical models use, among other things, systems of partial differential equations with delay, including those with Neumann boundary conditions [2]. In recent years systems of fractional differential equations [3, 4] have become more and more popular. These papers also use the Neumann boundary conditions.

Due to the complexity of analytical studies of fractional equations, the development of adequate numerical methods becomes relevant. Numerical algorithms for solving equations, both time-fractional and space-fractional, are the subject of a huge number of works, we note some [5–14]. But they consider the Dirichlet boundary conditions.

In this work, we use numerical algorithms developed and studied earlier for the Dirichlet problem for partial differential equations with delay [15] and for diffusion equations with a fractional derivative with respect to space without delay [17].

§ 1. Problem definition

Let us consider a system of equations of the superdiffusion type with fractional Riesz derivatives with respect to space and with a functional delay of the form

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = D_u \dfrac{\partial^\alpha u}{\partial |x|^\alpha} + \tilde{f}(x,t,u(x,t),u_t(x,\cdot),v(x,t),v_t(x,\cdot)), \\ \dfrac{\partial v(x,t)}{\partial t} = D_v \dfrac{\partial^\alpha v}{\partial |x|^\alpha} + \check{f}(x,t,u(x,t),u_t(x,\cdot),v(x,t),v_t(x,\cdot)), \end{cases} \tag{1.1}$$

where $0 \leqslant t \leqslant T$, $0 \leqslant x \leqslant X$ are independent variables, $u(x,t)$, $v(x,t)$ are the required functions, $u_t(x,\cdot) = \{u(x,t+s),\ \tau \leqslant s < 0\}$ and $v_t(x,\cdot) = \{v(x,t+s),\ \tau \leqslant s < 0\}$ are prehistories of desired functions by the time $t, \tau > 0$ is the value of delay, $1 < \alpha < 2$, $D_u > 0$, $D_v > 0$.

Riesz derivatives of order α are defined by the relations [16, p. 3]

$$\frac{\partial^\alpha u}{\partial |x|^\alpha} = K\left(\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + \frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha}\right), \quad K = -\frac{1}{2\cos(\alpha\pi/2)},$$

where the left and right Riemann–Liouville partial derivatives of order α are defined respectively as

$$\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_0^x \frac{u(\xi,t)}{(x-\xi)^{\alpha-1}}d\xi,$$
$$\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_x^X \frac{u(\xi,t)}{(x-\xi)^{\alpha-1}}d\xi.$$

Derivatives $\frac{\partial^\alpha v}{\partial |x|^\alpha}$, $\frac{\partial^\alpha v(x,t)}{\partial_+ x^\alpha}$ and $\frac{\partial^\alpha v(x,t)}{\partial_- x^\alpha}$ are defined similarly.

Initial conditions are given

$$\begin{aligned} u(x,t) &= \varphi(x,t), \quad 0 \leqslant x \leqslant X, \quad -\tau \leqslant t \leqslant 0,\\ v(x,t) &= \psi(x,t), \quad 0 \leqslant x \leqslant X, \quad -\tau \leqslant t \leqslant 0. \end{aligned} \tag{1.2}$$

Homogeneous boundary conditions of the second type (Neumann conditions) are also set

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial x}|_{x=0} = 0, \quad \frac{\partial u(x,t)}{\partial x}|_{x=X} = 0, \quad 0 \leqslant t \leqslant T,\\ &\frac{\partial v(x,t)}{\partial x}|_{x=0} = 0, \quad \frac{\partial v(x,t)}{\partial x}|_{x=X} = 0, \quad 0 < t \leqslant T. \end{aligned} \tag{1.3}$$

We assume that the solution to the problem (1.1)–(1.3) exists and is unique. Moreover, when proving the convergence of numerical algorithms, we will assume the necessary smoothness of the solution $u(x,t)$, $v(x,t)$.

Denote by $Q = Q[-\tau, 0)$ the set of functions $w(s)$, piecewise continuous to $[-\tau, 0)$ with a finite number of discontinuity points of the first kind, at discontinuity points continuous on the right. We define the norm of functions on Q $\|w(\cdot)\|_Q = \sup_{s\in[-\tau,0)} |w(s)|$. Additionally, we will assume that the functionals $\tilde{f}(x,t,w,v(\cdot))$ and $\check{f}(x,t,w,v(\cdot))$ are defined on $\Omega \times [0,T] \times R \times Q$ and are Lipschitz in the last two arguments, i. e., there exists a constant L_f, such that for all $x \in \Omega$, $t \in [0,T]$, $w^1 \in R$, $w^2 \in R$, $w^1(\cdot) \in Q$, $w^2(\cdot) \in Q$ the following inequalities are satisfied:

$$|\tilde{f}(x,t,w^1,w^1(\cdot)) - \tilde{f}(x,t,w^2,w^2(\cdot))| \leqslant L_f(|w^1 - w^2| + \|w^1(\cdot) - w^2(\cdot)\|_Q),$$
$$|\check{f}(x,t,w^1,w^1(\cdot)) - \check{f}(x,t,w^2,w^2(\cdot))| \leqslant L_f(|w^1 - w^2| + \|w^1(\cdot) - w^2(\cdot)\|_Q).$$

We introduce the vector notation

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad F = \begin{pmatrix} \tilde{f} \\ \check{f} \end{pmatrix}, \quad D = \begin{pmatrix} D_u \\ D_v \end{pmatrix},$$

then the system (1.1) can be written as

$$\frac{\partial U(x,t)}{\partial t} = D \diamond \frac{\partial^\alpha U}{\partial |x|^\alpha} + F(x,t,U(x,t),U_t(x,\cdot)), \tag{1.4}$$

where $A \diamond U$ denotes the vector with coordinates $\begin{pmatrix} a_1 u \\ a_2 v \end{pmatrix}$, if $A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

§ 2. Difference scheme

Introduce the time step $\Delta = \frac{\tau}{M_0}$, where M_0 is a natural number and let $M = [\frac{T}{\Delta}]$. Introduce points $t_j = j\Delta$, $j = -M_0, \dots, M$. Let us divide the segment $[0, X]$ into parts with a step $h = X/N$, N is an integer, $N \geq 2$, by introducing the points $x_i = ih$, $i = 0, \dots, N$.

The approximation of the vector function $U(x_i, t_j)$ at the grid nodes will be denoted by the vector V_i^j with coordinates $\begin{pmatrix} u_i^j \\ v_i^j \end{pmatrix}$.

For every fixed $i = 0, \dots, N$ we introduce a discrete prehistory to the moment t_m, $m = 0, \dots, M$: $\{V_i^j\}_m = \{V_i^j, m - M_0 \leqslant j \leqslant m\}$.

The operator interpolation (with extrapolation by half a step) of a discrete prehistory is a mapping I: that associates a discrete prehistory $\{V_i^j\}_m$ with a vector function $V^m(t)_i$ defined on $[t_m - \tau, t_m + \frac{\Delta}{2}]$.

We will say that the interpolation operator has the order of error p on the exact solution $U(x, t)$, if there are constants C_1 and C_2 such that for all i, m and $t \in [t_m - \tau, t_m + \frac{\Delta}{2}]$ the following inequality holds:

$$\|V^m(t)_i - U(x_i, t)\| \leqslant C_1 \max_{m - M_0 \leqslant j \leqslant m} \|V_i^j - U(x_i, t_j)\| + C_2 \Delta^p.$$

Here and in what follows, the norm in two-dimensional space is determined by the relation $\|U\| = \max\{|u|, |v|\}$.

In what follows, for the methods under consideration, we will use piecewise linear interpolation

$$V^m(t)_i = \frac{1}{\Delta}((t_j - t)V_i^{j-1} + (t - t_{j-1})V_i^j), \quad t_{j-1} \leqslant t \leqslant t_j, \quad j \leqslant m, \tag{2.1}$$

with extrapolation by continuation

$$V^m(t)_i = \frac{1}{\Delta}((t_m - t)V_i^{m-1} + (t - t_{m-1})V_i^m), \quad t_m \leqslant t \leqslant t_m + \frac{\Delta}{2}. \tag{2.2}$$

This interpolation operator is of second order if the exact solution $U(x, t)$ is twice continuously differentiable with respect to t [18, p. 98, 102].

To approximate the left and right fractional Riemann–Liouville derivatives, we will use the shifted Grünwald formulas

$$\frac{\partial^\alpha U(x_i, t_j)}{\partial_+ x^\alpha} = \frac{1}{h^\alpha} \sum_{k=0}^{i+1} g_{\alpha,k} U(x_{i-k+1}, t_j) + R_i^j, \tag{2.3}$$

$$\frac{\partial^\alpha U(x_i, t_j)}{\partial_- x^\alpha} = \frac{1}{h^\alpha} \sum_{k=0}^{N-i+1} g_{\alpha,k} U(x_{i+k-1}, t_j) + Q_i^j, \tag{2.4}$$

where the normalized weights are defined as

$$g_{\alpha,0} = 1, \quad g_{\alpha,k} = (-1)^k \frac{(\alpha)(\alpha - 1) \dots (\alpha - k + 1)}{k!}, \quad k = 1, 2, 3, \dots.$$

Note some properties of normalized weights [17]

$$g_{\alpha,1} = -\alpha; \quad 1 > g_{\alpha,2} \geq g_{\alpha,3} \geq \dots > 0; \qquad \sum_{k=0}^{\infty} g_{\alpha,k} = 0; \qquad 0 < \sum_{k=0}^{m} g_{\alpha,k} < 1, \quad m \geq 1.$$

If the exact solution $U(x, t)$ is four times continuously differentiable with respect to x, then [16, p. 56]

$$\|R_i^j\| \leqslant Ch^\alpha, \quad \|Q_i^j\| \leqslant Ch.$$

From (2.3)–(2.4) for the Riesz derivative we get the representation

$$\frac{\partial^{\alpha}U(x_i,t_j)}{\partial|x|^{\alpha}}=\frac{K}{h^{\alpha}}\big(\sum_{k=0}^{i+1}g_{\alpha,k}U(x_{i-k+1},t_j)+\sum_{k=0}^{N-i+1}g_{\alpha,k}U(x_{i+k-1},t_j)\big)+P_i^j. \tag{2.5}$$

If the exact solution $U(x,t)$ is four times continuously differentiable with respect to x, then

$$\|P_i^j\|\leqslant Ch. \tag{2.6}$$

Let us discretize (1.4) at the nodes $(x_i,t_{m+1/2})$, applying a two-site approximation to the middle for the time derivative, using the shifted formulas (2.5) for the Riesz derivative with respect to space on the m-th and $m+1$-th time layers and using piecewise linear interpolation (with extrapolation by half a step) of the prehistory of the discrete model, we obtain an analogue of the Crank–Nicolson scheme

$$\begin{gathered}\frac{V_i^{m+1}-V_i^m}{\Delta}=D\diamond\frac{K}{2h^{\alpha}}\Big(\sum_{s=0}^{i+1}g_{\alpha,s}V_{i-s+1}^{m}+\sum_{s=0}^{i+1}g_{\alpha,s}V_{i-s+1}^{m+1}+\\+\sum_{s=0}^{N-i+1}g_{\alpha,s}V_{i+s-1}^{m}+\sum_{s=0}^{N-i+1}g_{\alpha,s}V_{i+s-1}^{m+1}\Big)+F_i^{m+\frac{1}{2}},\\F_i^{m+\frac{1}{2}}=F(x_i,t_m+\frac{\Delta}{2},V^m(t_{m+\frac{1}{2}})_i,V_{t_{m+\frac{1}{2}}}^m(\cdot)_i),\quad i=1,\dots,N-1,\quad m=0,\dots,M-1,\end{gathered} \tag{2.7}$$

where $V^m(t_{m+\frac{1}{2}})_i$ is the result of piecewise linear interpolation (2.1) with extrapolation by continuation (2.2) at the point $t_m+\frac{\Delta}{2}$, $V_{t_{m+\frac{1}{2}}}^m(\cdot)_i$ is the history of interpolation with extrapolation at this point.

Using the formulas for numerical differentiation by the boundary and the Neumann boundary conditions (1.3), we supplement the scheme (2.7) with the equalities

$$V_0^{m+1}=V_1^{m+1},\quad V_N^{m+1}=V_{N-1}^{m+1},\quad V_0^m=V_1^m,\quad V_N^m=V_{N-1}^m, \tag{2.8}$$

then the scheme (2.7) will take the form

$$\begin{gathered}\frac{V_i^{m+1}-V_i^m}{\Delta}=D\diamond\frac{K}{2h^{\alpha}}\Big(\sum_{s=0}^{i}g_{\alpha,s}V_{i-s+1}^{m}+g_{\alpha,i+1}V_1^m+\sum_{s=0}^{i}g_{\alpha,s}V_{i-s+1}^{m+1}+\\+g_{\alpha,i+1}V_1^{m+1}+\sum_{s=0}^{N-i}g_{\alpha,s}V_{i+s-1}^{m}+g_{\alpha,N-i+1}V_{N-1}^m+\sum_{s=0}^{N-i}g_{\alpha,s}V_{i+s-1}^{m+1}+\\+g_{\alpha,N-i+1}V_{N-1}^{m+1}\Big)+F_i^{m+\frac{1}{2}},\quad i=1,\dots,N-1,\quad m=0,\dots,M-1.\end{gathered} \tag{2.9}$$

Scheme (2.9) is completed with initial conditions from (1.2)

$$V_i^j=\begin{pmatrix}u_i^j=\varphi(x_i,t_j)\\v_i^j=\psi(x_i,t_j)\end{pmatrix},\quad i=0,\dots,N,\quad j=-M_0,\dots,0. \tag{2.10}$$

The scheme (2.9–2.10) represents, for each fixed m, two systems of linear algebraic equations of order $N-1$. Let us consider the question of their solvability and stability of the method.

§ 3. Solvability and stability of the method

We introduce vectors of size $2(N-1)$

$$V^m = \begin{pmatrix} V_1^m \\ \vdots \\ V_{N-1}^m \end{pmatrix}, \qquad F^m = \begin{pmatrix} F_1^{m+\frac{1}{2}} \\ \vdots \\ F_{N-1}^{m+\frac{1}{2}} \end{pmatrix},$$

and matrices $\hat{A}$, $\tilde{A}$, $A = \hat{A} + \tilde{A}$ of size $(N-1)\times(N-1)$, whose elements satisfy the relations

$$\hat{A}_{i,s} = \begin{cases} -2g_{\alpha,1}, & s = i, \\ -(g_{\alpha,2} + g_{\alpha,0}), & s = i-1, \\ -(g_{\alpha,0} + g_{\alpha,2}), & s = i+1, \\ -g_{\alpha,i-s+1}, & s < i-1, \\ -g_{\alpha,s-i+1}, & s > i+1, \end{cases}$$

$$\tilde{A}_{i,s} = \begin{cases} -g_{\alpha,i+1}, & s = 1, \\ -g_{\alpha,N-i+1}, & s = N-1, \\ 0, & 1 < s < N-1, \end{cases}$$

then (2.9) can be rewritten as

$$\frac{V^{m+1} - V^m}{\Delta} = D \diamond \frac{K}{2h^\alpha}\Big(- AV^m - AV^{m+1}\Big) + F^m$$

or

$$V^{m+1} + D \diamond \frac{K\Delta}{2h^\alpha} AV^{m+1} = V^m - D \diamond \frac{K\Delta}{2h^\alpha} AV^m + \Delta F^m. \tag{3.1}$$

L e m m a 3.1. *The eigenvalues of the matrix A have positive real parts.*

P r o o f. Let us show the diagonal dominance of the matrix $A = \hat{A} + \tilde{A}$. Its elements $A_{i,s}$ satisfy the relations

$$|A_{1,1}| - \sum_{s=2}^{N-1} |A_{1,s}| = |-2g_{\alpha,1} - g_{\alpha,2}| - |-(g_{\alpha,0} + g_{\alpha,2})| - \sum_{s=3}^{N-2} |-g_{\alpha,s}| - |-g_{\alpha,N-1} - g_{\alpha,N}| =$$

$$= -2g_{\alpha,1} - g_{\alpha,2} - g_{\alpha,0} - g_{\alpha,2} - \sum_{s=3}^{N-2} g_{\alpha,s} - g_{\alpha,N-1} - g_{\alpha,N} = -g_{\alpha,1} - g_{\alpha,2} - \sum_{s=0}^{N} g_{\alpha,s} < 0;$$

$$|A_{2,2}| - |A_{2,1}| - \sum_{s=3}^{N-1} |A_{2,s}| = -2g_{\alpha,1} - 2g_{\alpha,0} - 2g_{\alpha,2} - g_{\alpha,3} - \sum_{s=4}^{N-1} g_{\alpha,s-1} -$$

$$- g_{\alpha,N-1} = -\sum_{s=0}^{3} g_{\alpha,s} - \sum_{s=0}^{N-1} g_{\alpha,s} < 0;$$

$$|A_{N-2,N-2}| - \sum_{s=1}^{N-3} |A_{N-2,s}| - |A_{N-2,N-1}| = -2g_{\alpha,1} - g_{\alpha,N-1} - \sum_{s=1}^{N-4} g_{\alpha,N-1-s} -$$

$$- 2g_{\alpha,0} - 2g_{\alpha,2} - g_{\alpha,3} = -\sum_{s=0}^{3} g_{\alpha,s} - \sum_{s=0}^{N-1} g_{\alpha,s} < 0;$$

$$|A_{N-1,N-1}| - \sum_{s=1}^{N-2} |A_{N-1,s}| = -2g_{\alpha,1} - g_{\alpha,2} - (g_{\alpha,0} - g_{\alpha,2}) - \sum_{s=1}^{N-3} |-g_{\alpha,N-s}| - g_{\alpha,N}| =$$

$$= -g_{\alpha,1} - g_{\alpha,2} - \sum_{s=0}^{N} g_{\alpha,s} < 0;$$

If $i \neq 1, i \neq 2, i \neq N-2, i \neq N-1$, then

$$|A_{i,1}| - \sum_{s=2}^{N-1} |A_{1,s}| = -2g_{\alpha,1} - 2(g_{\alpha,0} - 2g_{\alpha,2}) - \sum_{s=1}^{i-2} g_{\alpha,i-s+1} - g_{\alpha,i+1} -$$

$$- \sum_{s=i+2}^{N-1} g_{\alpha,s-i+1} - g_{\alpha,N-i+1} = -\sum_{s=0}^{i+1} g_{\alpha,s} - \sum_{s=0}^{N-i+1} g_{\alpha,s} < 0.$$

Since there is diagonal dominance and all elements of the principal matrix A are positive, then by the Gershgorin theorem, see for example [19, p. 78], the real parts of the matrix eigenvalues are positive. $\square$

For further study of the solvability and stability of the method (3.1), we rewrite it in the form of two systems of dimensions $N-1$, typing vector

$$u^m = \begin{pmatrix} u_1^m \\ \vdots \\ u_{N-1}^m \end{pmatrix}, \quad v^m = \begin{pmatrix} v_1^m \\ \vdots \\ v_{N-1}^m \end{pmatrix}, \quad \tilde{f}^m = \begin{pmatrix} \tilde{f}_1^{m+\frac{1}{2}} \\ \vdots \\ \tilde{f}_{N-1}^{m+\frac{1}{2}}, \end{pmatrix}, \quad \check{f}^m = \begin{pmatrix} \check{f}_1^{m+\frac{1}{2}} \\ \vdots \\ \check{f}_{N-1}^{m+\frac{1}{2}}, \end{pmatrix},$$

where

$$\tilde{f}_i^{m+\frac{1}{2}} = \tilde{f}(x_i, t_m + \frac{\Delta}{2}, V^m(t_{m+\frac{1}{2}})_i, V^m_{t_{m+\frac{1}{2}}}(\cdot)_i),$$

$$\check{f}_i^{m+\frac{1}{2}} = \check{f}(x_i, t_m + \frac{\Delta}{2}, V^m(t_{m+\frac{1}{2}})_i, V^m_{t_{m+\frac{1}{2}}}(\cdot)_i), \quad i = 1, \dots, N-1, \quad m = 0, \dots, M-1.$$

Then the system (3.1) can be written as

$$u^{m+1} + D_u \frac{K\Delta}{2h^\alpha} A u^{m+1} = u^m - D_u \frac{K\Delta}{2h^\alpha} A u^m + \Delta \tilde{f}^m, \tag{3.2}$$

$$v^{m+1} + D_v \frac{K\Delta}{2h^\alpha} A v^{m+1} = v^m - D_v \frac{K\Delta}{2h^\alpha} A v^m + \Delta \check{f}^m. \tag{3.3}$$

T h e o r e m 3.1. *System* (3.2)–(3.3) *has a unique solution.*

P r o o f. By Lemma 3.1, the real parts of the eigenvalues of the matrix A are positive, then the real parts of the matrix $D_u \frac{K\Delta}{2h^\alpha} A$ are also positive, hence the real parts of the eigenvalues of the matrix $E + D_u \frac{K\Delta}{2h^\alpha} A$ is greater than one, where E is the identity matrix.

But then the absolute values of all eigenvalues of the matrix $E + D_u \frac{K\Delta}{2h^\alpha} A$ are greater than one, which implies that the matrix $E + D_u \frac{K\Delta}{2h^\alpha} A$ is nondegenerate, i.e., the system (3.2) is uniquely resolvable.

The unique solvability of the system (3.3) is shown similarly. $\square$

By virtue of the theorem (3.1), the system (3.2)–(3.3) can be rewritten as

$$u^{m+1} = S_1 u^m + \Delta \hat{S}_1 f1^m,$$

$$v^{m+1} = S_2 v^m + \Delta \hat{S}_2 f2^m,$$

$$S_1 = (E + D_u \frac{K\Delta}{2h^\alpha} A)^{-1} (E - D_u \frac{K\Delta}{2h^\alpha} A), \qquad \hat{S}_1 = (E + D_u \frac{K\Delta}{2h^\alpha} A)^{-1},$$

$$S_2 = (E + D_v \frac{K\Delta}{2h^\alpha} A)^{-1} (E - D_v \frac{K\Delta}{2h^\alpha} A), \qquad \hat{S}_2 = (E + D_v \frac{K\Delta}{2h^\alpha} A)^{-1}.$$

T h e o r e m 3.2 (stability of the method). *The eigenvalues of the matrices S_1 and S_2 are less than* 1 *in absolute value.*

P r o o f. Let $\lambda = \lambda(A) = \mathrm{Re}\,\lambda + i\mathrm{Im}\,\lambda$ be an eigenvalue of the matrix A and $k > 0$. Then

$$\lambda\left(\frac{E - kA}{E + kA}\right) = \frac{1 - k\lambda(A)}{1 + k\lambda(A)},$$

$$|1 - k\lambda|^2 = (\mathrm{Re}\,(1 - k\lambda))^2 + (\mathrm{Im}\,(1 - k\lambda))^2 = 1 - 2k\mathrm{Re}\,\lambda + (k\mathrm{Re}\,\lambda)^2 + (k\mathrm{Im}\,\lambda)^2,$$

$$|1 + k\lambda|^2 = (\mathrm{Re}\,(1 + k\lambda))^2 + (\mathrm{Im}\,(1 + k\lambda))^2 = 1 + 2k\mathrm{Re}\,\lambda + (k\mathrm{Re}\,\lambda)^2 + (k\mathrm{Im}\,\lambda)^2,$$

whence, by virtue of the lemma 3.1, it follows $|1 - k\lambda|^2 < |1 + k\lambda|^2$, or $|1 - k\lambda| < |1 + k\lambda|$. From here, by the definition of the matrices S_1 and S_2,, the conclusion of the theorem follows.□

§ 4. Error analysis

Denote the vector value of the error at the nodes $E_i^j = U(x_i, t_j) - V_i^j = (\epsilon_i^j, \varepsilon_i^j)$, $i = 0, \dots, N$, $j = -M_0, \dots, M$. By definition $\epsilon_i^j = \varepsilon_i^j = O$ for all $i = 0, \dots, N$, $j = -M_0, \dots, 0$.

We say that the method converges with order $h^p + \Delta^q$, if there exists a constant C such that $\|E_i^j\| \leqslant C(h^p + \Delta^q)$ for all $i = 0, \dots, N$, $j = 1, \dots, M$.

For any $m = 0, 1, \dots, M - 1$ and $i = 1, \dots, N - 1$, the residual of the method (2.9) is the value

$$\Psi_i^m = \frac{U(x_i, t_{m+1}) - U(x_i, t_m)}{\Delta} -$$

$$- D \diamond \frac{K}{2h^\alpha}\Big(\sum_{s=0}^{i} g_{\alpha,s} U(x_{i-s+1}, t_m) + g_{\alpha,i+1} U(x_1, t_m) +$$

$$+ \sum_{s=0}^{i} g_{\alpha,s} V(x_{i-s+1}, t_{m+1}) + g_{\alpha,i+1} U(x_1, t_{m+1}) + \sum_{s=0}^{N-i} g_{\alpha,s} U(x_{i+s-1}, t_m) +$$

$$+ g_{\alpha,N-i+1} U(x_{N-1}, t_m) + \sum_{s=0}^{N-i} g_{\alpha,s} U(x_{i+s-1}, t_{m+1}) + g_{\alpha,N-i+1} U(x_{N-1}, t_{m+1})\Big) - \hat{F}_i^{m+\frac{1}{2}},$$

$$\hat{F}_i^{m+\frac{1}{2}} = F(x_i, t_m + \frac{\Delta}{2}, U(x_i, t_{m+\frac{1}{2}}), U_{t_{m+\frac{1}{2}}}(x_i, \cdot)).$$

L e m m a 4.1. *If the exact solution $u(x,t)$, $v(x,t)$ of the problem* (1.1)–(1.3) *is four times continuously differentiable with respect to x and twice continuously differentiable with respect to t, then for the residual of the method the following is true:*

$$\|\Psi_i^m\| \leqslant C_3(h + \Delta^2), \quad i = 0, \dots, N, \quad m = 0, \dots, M - 1.$$

The proof is carried out using the Taylor decomposition of the residual at the nodes $(x_i, t_{m+1/2})$ using (2.6) and the fact that the piecewise linear interpolation is of the second order. Approximation (2.8) of homogeneous Neumann boundary conditions is also used.

T h e o r e m 4.1. *Let the conditions for the smoothness of the solution formulated in the Lemma* 4.1 *be satisfied, then the error of the method* (2.9) *has order $h + \Delta^2$.*

The assertion of the theorem is verified by embedding method (2.9) into the general difference scheme with aftereffect [15] using the Lemma 4.1 and the Theorem 3.2.

§ 5. Numerical experiments

E x a m p l e 5.1. Consider the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^\alpha u}{\partial |x|^\alpha} - v(x,t) + u(x,t)(|\cos(x)| - \sqrt{u^2(x,t)+v^2(x,t)}) + u(x,t-\tau) \\ \qquad - \dfrac{\sin(x-\pi/2+\alpha\pi/2)}{2\cos(\alpha\pi/2)}\cos(t) - \cos(x)\cos(t-\tau), \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial^\alpha v}{\partial |x|^\alpha} + u(x,t) + v(x,t)(|\cos(x)| - \sqrt{u^2(x,t)+v^2(x,t)}) + v(x,t-\tau) \\ \qquad - \dfrac{\sin(x-\pi/2+\alpha\pi/2)}{2\cos(\alpha\pi/2)}\sin(t) - \cos(x)\sin(t-\tau), \end{cases} \tag{5.1}$$

where $x \in [0,\pi]$, $t \in [0, 4\pi]$, $\alpha = 1.8$, $\tau = 1$. The Neumann boundary conditions are set

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=X} = 0, \quad 0 \le t \le T,$$

and initial conditions

$$u(x,t) = \cos(x)\cos(t), \quad 0 \le x \le \pi, \quad -\tau \le t \le 0,$$

$$v(x,t) = \cos(x)\sin(t), \quad 0 \le x \le \pi, \quad -\tau \le t \le 0.$$

The exact solution is $u = \cos(x)\cos(t)$, $v = \cos(x)\sin(t)$.

$M = N$	ϵ	ε
2^3	1.615026	1.413223
2^4	1.302408	7.381870×10^{-1}
2^5	1.109274	4.852900×10^{-1}
2^6	1.035347	3.171708×10^{-1}
2^7	1.010894	1.935689×10^{-1}
2^8	1.003349	1.116905×10^{-1}
2^9	1.001074	6.225966×10^{-2}
2^{10}	1.000374	3.405036×10^{-2}

Table 1. Error of the numerical solution of system (5.1) based on the proposed scheme when $M = N$.

M	N	ϵ	ε
2^4	2^5	1.012499	4.972836×10^{-1}
2^5	2^6	1.004089	3.191082×10^{-1}
2^6	2^7	1.001382	1.938703×10^{-1}
2^7	2^8	1.000607	1.117390×10^{-1}
2^8	2^9	1.000310	6.226816×10^{-2}
2^9	2^{10}	1.000166	3.405202×10^{-2}

Table 2. Error of the numerical solution of system (5.1) based on the proposed scheme for different values of M and N.

M	N	ϵ	ε
2^5	2^4	1.440611	6.650061×10^{-1}
2^6	2^5	1.156315	4.722424×10^{-1}
2^7	2^6	1.050867	3.150544×10^{-1}
2^8	2^7	1.015641	1.932433×10^{-1}
2^9	2^8	1.004720	1.116410×10^{-1}
2^{10}	2^9	1.001456	6.225190×10^{-2}

Table 3. Error of the numerical solution of system (5.1) based on the proposed scheme for different values of M and N.

E x a m p l e 5.2. Consider the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^\alpha u}{\partial |x|^\alpha} + \tilde{f}, \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial^\alpha v}{\partial |x|^\alpha} + \check{f}, \end{cases} \tag{5.2}$$

where $x \in [0, 1]$, $t \in [0, 1]$, $\alpha = 1.2$, $\tau = 0.5$. The Neumann boundary conditions are set

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=X} = 0, \quad 0 \le t \le T,$$

and initial conditions

$$\begin{aligned} u(x,t) &= e^{-t}(x^3)(1-x)^3, \quad 0 \le x \le \pi, \quad -\tau \le t \le 0, \\ v(x,t) &= -e^{-t}(x^3)(1-x)^3, \quad 0 \le x \le \pi, \quad -\tau \le t \le 0, \end{aligned}$$

The source term is given by:

$$\begin{aligned} \tilde{f} &= -e^{-t}(x^3)(1-x)^3 + \frac{e^{-t}}{2\cos(\alpha\pi/2)}\left(\frac{\Gamma(4)}{\Gamma(2.8)}(x^{1.8} + (1-x)^{1.8}) - \frac{3\Gamma(5)}{\Gamma(3.8)}(x^{2.8} + (1-x)^{2.8}) + \right. \\ &\left. + \frac{3\Gamma(6)}{\Gamma(4.8)}(x^{3.8} + (1-x)^{3.8}) - \frac{\Gamma(7)}{\Gamma(5.8)}(x^{4.8} + (1-x)^{4.8})\right) \frac{u(x,t-\tau)}{e^{-(t+\tau)}(x^3)(1-x)^3} + v(x,t) + u(x,t), \\ \check{f} &= e^{-t}(x^3)(1-x)^3 - \frac{e^{-t}}{2\cos(\alpha\pi/2)}\left(\frac{\Gamma(4)}{\Gamma(2.8)}(x^{1.8} + (1-x)^{1.8}) - \frac{3\Gamma(5)}{\Gamma(3.8)}(x^{2.8} + (1-x)^{2.8}) + \right. \\ &\left. + \frac{3\Gamma(6)}{\Gamma(4.8)}(x^{3.8} + (1-x)^{3.8}) - \frac{\Gamma(7)}{\Gamma(5.8)}(x^{4.8} + (1-x)^{4.8})\right) \frac{v(x,t-\tau)}{e^{-(t+\tau)}(x^3)(1-x)^3} + v(x,t) + u(x,t). \end{aligned}$$

The exact solution is: $u = e^{-t}(x^3)(1-x)^3$, $v = -e^{-t}(x^3)(1-x)^3$.

$M = N$	ϵ	ε
2^3	4.788462×10^{-04}	4.806325×10^{-04}
2^4	2.665764×10^{-04}	2.672903×10^{-04}
2^5	1.390116×10^{-04}	1.393004×10^{-04}
2^6	7.288451×10^{-05}	7.301299×10^{-05}
2^7	3.856764×10^{-05}	3.862954×10^{-05}
2^8	1.099096×10^{-05}	1.100758×10^{-05}
2^9	6.395759×10^{-06}	6.409708×10^{-06}
2^{10}	2.374209×10^{-06}	2.381952×10^{-06}

Table 4. Error of the numerical solution of system (5.2) based on the proposed scheme when $M = N$.

M	N	ϵ	ε
2^3	2^4	3.185572×10^{-04}	3.194584×10^{-04}
2^4	2^5	1.457558×10^{-04}	1.461059×10^{-04}
2^5	2^6	7.436044×10^{-05}	7.450573×10^{-05}
2^6	2^7	3.894708×10^{-05}	3.901345×10^{-05}
2^7	2^8	2.063072×10^{-05}	2.066324×10^{-05}
2^8	2^9	1.099096×10^{-05}	1.100758×10^{-05}
2^9	2^{10}	6.100022×10^{-06}	6.111265×10^{-06}

Table 5. Error of the numerical solution of system (5.2) based on the proposed scheme for different values of M and N.

M	N	ϵ	ε
2^4	2^3	4.773200×10^{-04}	4.788415×10^{-04}
2^5	2^4	2.594554×10^{-04}	2.600584×10^{-04}
2^6	2^5	1.366533×10^{-04}	1.369104×10^{-04}
2^7	2^6	7.221225×10^{-05}	7.233228×10^{-05}
2^8	2^7	3.838352×10^{-05}	3.844318×10^{-05}
2^9	2^8	1.587867×10^{-05}	1.605439×10^{-05}
2^{10}	2^9	8.174771×10^{-06}	8.194037×10^{-06}

Table 6. Error of the numerical solution of system (5.2) based on the proposed scheme for different values of M and N.

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М. Ибрагим, В. Г. Пименов

Численный метод для системы дробных по пространству уравнений супердиффузионного типа с запаздыванием и граничными условиями Неймана

Ключевые слова: супердиффузионные уравнения, условия Неймана, функциональное запаздывание, производные Рисса, аппроксимация Грюнвальда–Летникова, метод Кранка–Никольсон, порядок сходимости.

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Рассматривается система двух дробных по пространству уравнений супердиффузии с функциональным запаздыванием общего вида и краевыми условиями Неймана. Для этой задачи конструируется аналог метода Кранка–Никольсон, основанный на сдвинутых формулах Грюнвальда–Летникова для аппроксимации дробных производных Рисса по пространственной переменной и применении кусочно-линейной интерполяции дискретной предыстории с экстраполяцией продолжением для учета эффекта запаздывания. С помощью теоремы Гершгорина доказана разрешимость разностной схемы и ее устойчивость. Получен порядок сходимости метода. Представлены результаты численных экспериментов.

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