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PURSUIT–EVASION DIFFERENTIAL GAMES WITH GR-CONSTRAINTS ON CONTROLS

In the paper, a pursuit–evasion differential game is considered when controls of the players are subject to differential constraints in the form of Grönwall's integral inequality. The strategy of parallel pursuit (briefly, Π-strategy) was introduced and used by L. A. Petrosyan to solve simple pursuit problems under phase constraints on the states of the players in the case when control functions of both players are chosen from the class L_∞. In the present work, the Π-strategy is constructed for a simple pursuit problem in the cases when control functions of both players are chosen from different classes of the Grönwall type constraints, and sufficient conditions of capture and optimal capture time are found in these cases. To solve the evasion problem, we suggest a control function for the Evader and find sufficient conditions of evasion. In addition, an attainability domain of the players is constructed and its conditions of embedding in respect to time are given. Results of this work continue and extend the works of R. Isaacs, L. A. Petrosyan, B. N. Pshenichnyi, A. A. Chirii, A. A. Azamov and other researchers, including the authors.

Keywords: differential game, Grönwall's inequality, pursuit, evasion, optimal strategy, capture time.

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Introduction

Advanced differential games constitute the theory of mathematical methods of control processes, including dynamism, control, awareness, fighting and other particular properties, and describe one of the most complex mathematical models of real processes having great practical significance. At present, there are hundreds of monographs on the theory. Nonetheless, completely solved problems of differential games are quite few. In the theory of differential games, problems of pursuit–evasion occupy a special place due to a number of specific qualities. The works of L. S. Pontryagin [1] and N. N. Krasovskii [2] have a great importance in the construction of the fundamental theory of differential games. The book of R. Isaacs [3] includes certain game problems which were investigated comprehensively and devoted for further research. Some game problems in [3] have been studied in details by L. A. Petrosyan [4]. In the book [4], the concept of strategy of parallel convergence (briefly, Π-strategy) was first given and applied to solve the quality problems for the differential game. The Π-strategy suggested in the book [4] served as the starting point for the progress of the pursuit method in games with multiple pursuers (see e. g. B. N. Pshenichnyi [5], B. N. Pshenichnyi, A. A. Chikrii, and I. S. Rappoport [6], A. A. Azamov [7], A. A. Azamov and B. T. Samatov [8], N. N Petrov [9, 10], N. N Petrov and A. Ya. Narmanov [11], A. I. Blagodatskikh and N. N Petrov [12], A. A. Chikrii [13], N. L. Grigorenko [14], L. A. Petrosyan [15, 16], L. A. Petrosyan and Yu. G. Dutkevich [17], L. A. Petrosyan and B. B. Rikhsiev [18], L. A. Petrosyan and V. V. Mazalov [19], B. T. Samatov [20–24]).

In the theory of differential games, the problems are mainly considered for the cases when geometric, integral or mixed constraints are imposed on control functions (N. N Petrov [10], A. N. Dar'in and A. B. Kurzhanskii [25], D. V. Kornev and N. Yu. Lukoyanov [26], N. Yu. Satimov [27], G. I. Ibragimov [28, 29]). Yet, differential type constraints on controls have been triggering a substantial interest in many applied problems such as technical, economical and ecological problems (J. P. Aubin and A. Cellina [30], J. S. Pang and D. E. Stewart [31]).

At the present time, there exists a large interest from the point of practical and theoretical view to investigate different type problems of optimal control theory for more complicated dynamical systems where the controls and system are subject to various kinds of stationary and non-stationary constraints (for instance, [28, 32, 33]). Differential games with phase constraints can be involved in such problems. In solving such problems, an attainability domain of players takes up a great importance. Constructing the attainability domain of players is considered as the significant result in the problems of avoidance of meeting [4] and in the conflict problems too [34–37]. A great deal of optimal control problems including linear and nonlinear dynamical systems were considered for the cases where stationary and non-stationary constraints are applied to controls. Furthermore, their applications were provided in details (for instance, [38, 39]).

In the work of B. T. Samatov, G. I. Ibragimov and I. V. Hodjibayeva [40], the notion of Gr-constraint (Grönwall type constraint) on controls of players, which expresses more general form of geometric constraints, is first introduced. In the present work, a pursuit–evasion differential game with Gr-constraints is considered for more generalized cases. In the paper, we will rely on Pontryagin's formalization [1] and use the method of resolving functions (see [5,6,13,14,27,40]), which allows using a much smaller amount of current information for the construction of the pursuer strategy. In order to solve the pursuit problem, the Π-strategy of the Pursuer is defined and sufficient conditions of pursuit are shown and also, an attainability domain of the players, which contains the meeting points of the players, is constructed. Besides, in the evasion problem, a special strategy is suggested to the Evader and sufficient conditions of evasion are obtained.

§ 1. Formulation of the problems

Consider two objects, the Pursuer P and the Evader E, moving in the space $\mathbb{R}^n$. Suppose that x and y are the locations of the objects respectively.

Let the movements of the Pursuer P and the Evader E be based on the following differential equations

$$P\colon \dot{x} = u, \quad x(0) = x_0, \tag{1.1}$$
$$E\colon \dot{y} = v, \quad y(0) = y_0, \tag{1.2}$$

where x, y, x_0, y_0, u, $v \in \mathbb{R}^n$, $n \geq 2$, $x_0 \neq y_0$. In the present paper, we propose a new set of controls of the Pursuer and the Evader described by the following Grönwall type constraints [40]

$$|u(t)|^2 \leq \rho^2 + 2k_1 \int_0^t |u(s)|^2 ds, \quad t \geq 0, \tag{1.3}$$
$$|v(t)|^2 \leq \sigma^2 + 2k_2 \int_0^t |v(s)|^2 ds, \quad t \geq 0, \tag{1.4}$$

respectively, where ρ, σ are given positive numbers and k_1, k_2 are given non-negative numbers.

Here u is the velocity vector of the Pursuer and the temporal variation of u must be a measurable function $u(\cdot)\colon [0, \infty) \to \mathbb{R}^n$. We denote by $\mathbb{U}_{Gr}$ the set of all measurable functions $u(\cdot)$ that satisfy the Grönwall type constraint (briefly, Gr-constraint) (1.3).

Similarly, v is the velocity vector of the Evader and here the temporal variation of v must be a measurable function $v(\cdot)\colon [0, \infty) \to \mathbb{R}^n$. We denote by $\mathbb{V}_{Gr}$ the set of all measurable functions $v(\cdot)$ that satisfy Gr-constraint (1.4).

Since the problem (1.1)–(1.4) for the case $k_1 = k_2$ was studied enough completely (Samatov et al., [40]). In the present work, we will assume $k_1 \neq k_2$.

D e f i n i t i o n 1.1. The measurable functions $u(\cdot) \in \mathbb{U}_{Gr}$ and $v(\cdot) \in \mathbb{V}_{Gr}$ are called *controls of the Pursuer and Evader* respectively.

D e f i n i t i o n 1.2. For the pairs $(x_0, u(\cdot))$, $u(\cdot) \in \mathbb{U}_{Gr}$ and $(y_0, v(\cdot))$, $v(\cdot) \in \mathbb{V}_{Gr}$ the solutions of the equations (1.1), (1.2), that is, $x(t) = x_0 + \int_0^t u(s)\,ds$ and $y(t) = y_0 + \int_0^t v(s)\,ds$ are respectively called the *trajectories of the Pursuer and the Evader* on the interval $t \geq 0$.

The goal of the Pursuer is to capture the Evader, i. e., achievement of the equality $x(t) = y(t)$ (Pursuit problem) and the Evader E strives to avoid an encounter (Evasion problem), i. e., to achieve the inequality $x(t) \neq y(t)$ for all $t \geq 0$, and in the opposite case, to delay the instant of the encounter as long as possible.

We use the following statement.

L e m m a 1.1 (Grönwall's inequality, see [41]). *If* $|\gamma(t)|^2 \leq \alpha^2 + 2k \int_0^t |\gamma(s)|^2 ds$, *then* $|\gamma(t)| \leq \leq \alpha e^{kt}$, *where* $\gamma(t)$, $t \geq 0$, *is a measurable function and* α, k *are non-negative numbers.*

By Lemma 1.1, if $u(\cdot) \in \mathbb{U}_{Gr}$ and $v(\cdot) \in \mathbb{V}_{Gr}$, then

$$|u(t)| \leq \rho e^{k_1 t}, \quad |v(t)| \leq \sigma e^{k_2 t}, \quad t \geq 0. \tag{1.5}$$

It can be easily presented that the converse is not satisfied, that is, the inequalities (1.5) do not mean the inequalities (1.3) and (1.4). To define the notions of optimal strategies of the players and optimal pursuit time, we will consider two games.

D e f i n i t i o n 1.3. Let $B(c; r)$ denote the ball of radius r and centered at the point $c \in \mathbb{R}^n$. Then we say that the function $\mathbf{u}(t, v) \colon \mathbb{R}_+ \times B(0; \sigma e^{k_2 t}) \to B(0; \rho e^{k_1 t})$ is *a strategy of the Pursuer* if $\mathbf{u}(t, v)$ is *Lebesgue measurable* in t for each fixed v, *Borel measurable* in v for each fixed t.

Let $z = x - y$, $z_0 = x_0 - y_0$. Then from (1.1), (1.2) we have the equation in the form

$$\dot{z} = u - v, \quad z(0) = z_0. \tag{1.6}$$

For solving the pursuit–evasion problems, the following definitions are important.

D e f i n i t i o n 1.4. The strategy $\mathbf{u}(t, v)$ is called *winning* for the Pursuer P on the interval $[0, T_{Gr}]$ in the simple game (1.1)–(1.4) if for every $v(\cdot) \in \mathbb{V}_{Gr}$ there exists some moment $t^* \in [0, T_{Gr}]$ that the solution $z(t)$ of the following Cauchy problem, which follows from (1.6),

$$\dot{z} = \mathbf{u}(t, v(t)) - v(t), \quad z(0) = z_0, \tag{1.7}$$

is equal to zero at this moment, i. e., $z(t^*) = 0$, and this implies that $x(t^*) = y(t^*)$. Here the number T_{Gr} is called *a guaranteed capture time*.

D e f i n i t i o n 1.5. The strategy $\mathbf{u}(t, v)$ is called *a parallel pursuit strategy*, or Π-*strategy*, if for every $v(\cdot) \in \mathbb{V}_{Gr}$ the solution $z(t)$ of the Cauchy problem (1.7) can be represented as

$$z(t) = \Lambda_{Gr}(t, v(\cdot)) z_0, \quad \Lambda_{Gr}(0, v(\cdot)) = 1,$$

where $\Lambda_{Gr}(t, v(\cdot))$ is some scalar continuous function of t, $t \geq 0$, that can be called *a convergence function* in the pursuit problem.

D e f i n i t i o n 1.6. A control function $\mathbf{v}^*(\cdot) \in \mathbb{V}_{Gr}$ for the player E is called *winning* in the simple game (1.1)–(1.4) if for every $u(\cdot) \in \mathbb{U}_{Gr}$ the solution $z(t)$ of the Cauchy problem

$$\dot{z} = u(t) - \mathbf{v}^*(t), \quad z(0) = z_0, \tag{1.8}$$

is different from zero for all $t \geq 0$, i. e., $z(t) \neq 0$, $t \geq 0$.

This paper is devoted to solving the following problems for Grönwall type constraints on controls.

1. Solve pursuit problem in the simple game (1.1)–(1.4) with the Grönwall type constraints (briefly, Gr-Game of Pursuit).
2. Solve evasion problem in the simple game (1.1)–(1.4) with the Grönwall type constraints (briefly, Gr-Game of Evasion).
3. Construction of an attainability domain of the Pursuer.

§ 2. A solution of the pursuit problem in the case when $\rho \geq \sigma$

In this section, we will construct optimal strategies of players and give a formula for optimal pursuit time.

D e f i n i t i o n 2.1. If a) $\rho \geq \sigma$ and $k_1 > k_2$, or b) $\rho > \sigma$ and $k_2 > k_1$, then the function

$$\mathbf{u}_{Gr}(t, v) = v - \lambda^*_{Gr}(t, v)\xi_0, \quad \lambda^*_{Gr}(t, v) = \langle v, \xi_0 \rangle + \sqrt{\langle v, \xi_0 \rangle^2 + \rho^2 e^{2k_1 t} - |v|^2}, \tag{2.1}$$

is called *the Π_{Gr}-strategy of the Pursuer* in the simple game (1.1)–(1.4), where $\xi_0 = z_0/|z_0|$, and $\langle v, \xi_0 \rangle$ is the scalar product of the vectors v and ξ_0 in the space $\mathbb{R}^n$.

P r o p o s i t i o n 2.1. *If* a) $\rho \geq \sigma$ *and* $k_1 > k_2$, *or* b) $\rho > \sigma$ *and* $k_2 > k_1$ *when* $t \in [0, \theta]$, $\theta = \frac{1}{k_2 - k_1} \ln \frac{\rho}{\sigma}$, *then, for all* v *such that* $|v| \leq \sigma e^{k_2 t}$, *the function* $\lambda^*_{Gr}(t, v)$ *is defined and non-negative and besides, for* $\boldsymbol{u}_{Gr}(t, v)$ *in* (2.1) *the following holds*

$$|\boldsymbol{u}_{Gr}(t, v)| = \rho e^{k_1 t}. \tag{2.2}$$

L e m m a 2.1. *Let one of the following conditions*

(1) $\rho \geq \sigma$, $k_1 > k_2$;

(2) $\rho > \sigma$, $k_1 < k_2$, $|z_0| \leq A$;

be valid. Then the following equation

$$|z_0| - \frac{\rho}{k_1} e^{k_1 t} + \frac{\sigma}{k_2} e^{k_2 t} - \frac{\sigma}{k_2} + \frac{\rho}{k_1} = 0 \tag{2.3}$$

with respect to t, has at least one positive root and we denote it by T_{Gr}, *where*

$$A = \rho^{\frac{k_2}{k_2 - k_1}} \sigma^{\frac{k_1}{k_1 - k_2}} \left(\frac{1}{k_1} - \frac{1}{k_2} \right) + \frac{\sigma}{k_2} - \frac{\rho}{k_1}. \tag{2.4}$$

P r o o f. **1)** Let $\rho \geq \sigma$, $k_1 > k_2$. Using the equation (2.3) we will form the function

$$F_1(t) = |z_0| - \frac{\rho}{k_1} e^{k_1 t} + \frac{\sigma}{k_2} e^{k_2 t} - \frac{\sigma}{k_2} + \frac{\rho}{k_1}$$

and analyze it as follows:

a) $F_1(0) = |z_0| > 0$;

b) the derivative of the function $F_1(t)$ equals $F_1'(t) = \sigma e^{k_2 t} - \rho e^{k_1 t} \leq 0$;

c) the function $F_1(t)$ is decreasing when $t \geq \theta$;

d) take the limit of the function $F_1(t)$ as $t \to \infty$

$$\lim_{t\to\infty} F_1(t) = \lim_{t\to\infty} e^{k_1 t}\left(\frac{\sigma}{k_2}e^{(k_2-k_1)t} - \frac{\rho}{k_1} - \frac{|z_0| - \frac{\sigma}{k_2} + \frac{\rho}{k_1}}{e^{k_1 t}}\right) = -\infty.$$

2) Let $\rho > \sigma$, $k_1 < k_2$. Then the Pursuer should complete the pursuit on the interval $[0, \theta]$ depending on the part b) of Proposition 2.1.

Then we will investigate the function $F_1(t)$ as follows:

a) $F_1(0) = |z_0|$;

b) the function $F_1(t)$ is decreasing on the interval $[0, \theta]$. Therefore, $F_1(t)$ at $t = \theta$ must be negative for a positive root of (2.3) to exist. From this, we get $|z_0| \leq A$ (see (2.4)). □

Theorem 2.1. *If one of the conditions of Lemma* 2.1 *holds, then the* Π_{Gr}*-strategy* (2.1) *is winning in the Gr-game of pursuit on time interval* $[0, T_{Gr}]$.

Proof. Assume that the Pursuer applies the Π_{Gr}-strategy for any control function of the Evader $v(\cdot) \in \mathbb{V}_{Gr}$. Then in accordance with (1.6) we have the Cauchy problem

$$\dot{z} = \dot{x} - \dot{y} = -\lambda^*_{Gr}(t, v(t))\xi_0, \quad z(0) = z_0, \quad z_0 = x_0 - y_0,$$

and write its solution as follows:

$$z(t) = \Lambda_{Gr}(t, v(\cdot))z_0, \qquad \Lambda_{Gr}(t, v(\cdot)) = 1 - \frac{1}{|z_0|}\int_0^t \lambda^*_{Gr}(s, v(s))ds. \tag{2.5}$$

Now we will study the behavior of the function $\Lambda_{Gr}(t, v(\cdot))$ of t. Using the definition of the function $\lambda^*_{Gr}(t, v(t))$, we have

$$\Lambda_{Gr}(t, v(\cdot)) = 1 - \frac{1}{|z_0|}\int_0^t \left(\langle v(s), \xi_0\rangle + \sqrt{\langle v(s), \xi_0\rangle^2 + \rho^2 e^{2k_1 s} - |v(s)|^2}\right) ds.$$

Since the function $f(t, w) = |v(t)|w + \sqrt{|v(t)|^2 w^2 + \rho^2 e^{2k_1 t} - |v(t)|^2}$, $w \in [-1, 1]$, is monotone increasing with respect to w for every $t \geq 0$. Therefore, from the second inequality in (1.5) we obtain

$$\Lambda_{Gr}(t, v(\cdot)) \leq 1 - \frac{1}{|z_0|}\int_0^t (\rho e^{k_1 s} - \sigma e^{k_2 s})ds = 1 - \frac{1}{|z_0|}\left(\frac{\rho}{k_1}(e^{k_1 t} - 1) - \frac{\sigma}{k_2}(e^{k_2 t} - 1)\right) = \Lambda_1(t).$$

Clearly, the function $\Lambda_1(t)$ is monotone decreasing on $[0, T_{Gr}]$ and $\Lambda_1(T_{Gr}) = 0$ (see Lemma 2.1). Consequently, there exists some time t^*, $0 \leq t^* \leq T_{Gr}$, such that $\Lambda_{Gr}(t^*, v(\cdot)) = 0$, and hence by (2.5), we obtain that $z(t^*) = 0$. □

§ 3. A solution of the pursuit problem in the case when $\rho < \sigma$

Let $\rho < \sigma$, $k_2 < k_1$ be satisfied. Then the inequality $\rho e^{k_1 t} \geq \sigma e^{k_2 t}$ doesn't hold on the interval $[0, \theta)$, where $\theta = \frac{1}{k_2 - k_1}\ln\frac{\rho}{\sigma}$. Here we assume that the Pursuer begins the pursuit from the moment $t = \theta$ and the Evader moves with some control function $v(\cdot) \in \mathbb{V}_{Gr}$ from the initial state y_0 to the state $y(\theta)$ by $t = \theta$, where $y(\theta) = y_0 + \displaystyle\int_0^\theta v(s)\, ds$.

Lemma 3.1. *For the Evader, the inclusion* $y(t) \in S_{R_0}(y_0)$ *holds at every moment* t, $t \in [0, \theta]$, *where* $S_{R_0}(y_0)$ *is a ball whose center is on the point* y_0 *and whose radius is* $R_0 = \sigma(e^{k_2\theta} - 1)/k_2$.

P r o o f. Let $v(\cdot) \in \mathbb{V}_{Gr}$. Then from (1.2) and (1.4) it proceeds that

$$|y(t) - y_0| \leq \int_0^\theta |v(s)|\, ds \leq \int_0^\theta \sigma e^{k_2 s} ds = \sigma(e^{k_2\theta} - 1)/k_2 = R_0$$

for all $t \in [0, \theta]$. $\square$

C o r o l l a r y 3.1. *Lemma* 3.1 *means that for the function* $z(t)$*, the relation*

$$\max_{0 \leq t \leq \theta} |z(t)| \leq |z_0| + R_0 = |z_0^*|$$

is satisfied on time interval $[0, \theta]$.

D e f i n i t i o n 3.1. We call *the* Π^*_{Gr}*-strategy of the Pursuer* the function

$$\mathbf{u}^*_{Gr}(t, v) = \begin{cases} 0, & \text{if } 0 \leq t < \theta; \\ v - \lambda^{**}_{Gr}(t, v)\xi_0^*, & \text{if } \theta \leq t; \end{cases} \tag{3.1}$$

on time interval $[\theta, T^*_{Gr}]$, where

$$\lambda^{**}_{Gr}(t, v) = \langle v, \xi_0^* \rangle + \sqrt{\langle v, \xi_0^* \rangle^2 + \rho^2 e^{2k_1 t} - |v|^2}, \qquad \xi_0^* = \frac{z_0^*}{|z_0^*|}, \qquad z_0^* = x_0 - y(\theta),$$

and T^*_{Gr} is the first positive root of the equation

$$\frac{\rho}{k_1} e^{k_1 t} - \frac{\sigma}{k_2} e^{k_2 t} + \frac{\sigma}{k_2} e^{k_2 \theta} - \frac{\rho}{k_1} e^{k_1 \theta} - |z_0^*| = 0$$

with respect to t.

Note that the equality (2.2) holds for the strategy (3.1) when $t \geq \theta$.

T h e o r e m 3.1. *Let* $\rho < \sigma$ *and* $k_1 > k_2$ *be valid. Then in the simple game* (1.1)–(1.4)*, the* Π^*_{Gr}*-strategy* (3.1) *is winning on time interval* $(0, T^{**}_{Gr}]$*, where* $T^{**}_{Gr} = \theta + T^*_{Gr}$.

P r o o f. Suppose that the Pursuer applies the Π^*_{Gr}-strategy (3.1) from the moment θ when the Evader implements any control function $v(\cdot) \in \mathbb{V}_{Gr}$. Then using the equations (1.1), (1.2) we obtain the equation

$$\dot{z} = -\lambda^{**}_{Gr}(t, v(t))\xi_0^*, \quad z(\theta) = z_0^*.$$

Integrate both sides of this equation over $[\theta, t]$

$$z(t) = \Lambda^*_{Gr}(t, v(\cdot)) z_0^*, \tag{3.2}$$

where

$$\Lambda^*_{Gr}(t, v(\cdot)) = 1 - \frac{1}{|z_0^*|} \int_\theta^t \lambda^{**}_{Gr}(s, v(s))\, ds.$$

Now we will study the behavior of the function $\Lambda^*_{Gr}(t, v(\cdot))$ of t. Using the definition of the function $\lambda^{**}_{Gr}(t, v(t))$, we have

$$\Lambda^*_{Gr}(t, v(\cdot)) = 1 - \frac{1}{|z_0^*|} \int_\theta^t \left(\langle v(s), \xi_0^* \rangle + \sqrt{\langle v(s), \xi_0^* \rangle^2 + \rho^2 e^{2k_1 s} - |v(s)|^2} \right) ds.$$

Since the function $f(t,\mu) = |v(t)|\mu + \sqrt{|v(t)|^2\mu^2 + \rho^2 e^{2k_1 t} - |v(t)|^2}$, $\mu \in [-1,1]$ is monotone increasing with respect to μ for every $t \geq \theta$. Therefore, from the second inequality in (1.5) we have

$$\Lambda^*_{Gr}(t, v(\cdot)) \leq 1 - \frac{1}{|z^*_0|}\int_\theta^t (\rho e^{k_1 s} - \sigma e^{k_2 s})\, ds = F_2(t)/|z^*_0| = \Lambda_2(t),$$

where $F_2(t) = |z^*_0| - \frac{\rho}{k_1}e^{k_1 t} + \frac{\sigma}{k_2}e^{k_2 t} - \frac{\sigma}{k_2}e^{k_2\theta} + \frac{\rho}{k_1}e^{k_1\theta}$.

Now we will show that there exists a positive solution for the equation

$$\Lambda_2(t) = 0 \tag{3.3}$$

on the interval $[\theta, T^*_{Gr}]$. For this purpose, we will analyze the function $F_2(t) = |z^*_0|\Lambda_2(t)$, that is, the function has the following properties.

a) $F_2(\theta) = |z^*_0| > 0$.

b) It is obvious that $\frac{dF_2(t)}{dt} \leq 0$ for every $t > \theta$, i. e., the function $F_2(t)$ is decreasing. Moreover, the limit of this function as $t \to \infty$ is equal to

$$\lim_{t\to\infty} F_2(t) = -\lim_{t\to\infty} e^{k_1 t}\left(\frac{\rho k_2}{\sigma k_1} + \frac{\sigma k_1 - \rho k_2 e^{k_1\theta} - |z_0|k_1k_2}{\sigma k_1 e^{k_1 t}} - e^{(k_2-k_1)t}\right) = -\infty.$$

Hence, we can conclude that the equation (3.3) has at least one positive root on the interval $[\theta, +\infty)$.

Clearly, the function $\Lambda_2(t)$ is monotone decreasing on $[0, T^*_{Gr}]$ and $\Lambda_2(T^*_{Gr}) = 0$. Consequently, there exists some time t^*_1, $\theta < t^*_1 \leq T^*_{Gr}$, such that $\Lambda^*_{Gr}(t^*_1, v(\cdot)) = 0$, and hence, by (3.2) it follows that $z(t^*_1) = 0$.

Next, we will prove the admissibility of the strategies (2.1), (3.1) for all $t \geq 0$. Let $v(\cdot) \in \mathbb{V}_{Gr}$ be an arbitrary control of the Evader. According to (1.3) and (2.2), we obtain:

a)

$$\begin{aligned}|\mathbf{u}_{Gr}(t, v(t))|^2 &= \rho^2 + \rho^2(e^{2k_1 t} - 1) = \rho^2 + 2k_1\int_0^t \rho^2 e^{2k_1 s} ds = \\ &= \rho^2 + 2k_1\int_0^t |\mathbf{u}_{Gr}(s, v(s))|^2 ds,\end{aligned}$$

b)

$$\begin{aligned}|\mathbf{u}^*_{Gr}(t, v(t))|^2 &= \rho^2 + \rho^2(e^{2k_1 t} - 1) = \rho^2 + 2k_1\int_\theta^t \rho^2 e^{2k_1 s} ds = \\ &= \rho^2 + 2k_1\int_\theta^t |\mathbf{u}^*_{Gr}(s, v(s))|^2 ds.\end{aligned}$$

□

§ 4. A solution of the evasion problem

D e f i n i t i o n 4.1. In the simple game (1.1)–(1.4), we call *the strategy of the Evader* the function

$$\mathbf{v}^*_{Gr}(t) = -\sigma e^{k_2 t}\xi_0, \quad t \geq 0. \tag{4.1}$$

T h e o r e m 4.1. *Let one of the following conditions holds:* 1) $\rho \le \sigma$, $k_2 \ge k_1$; 2) $\rho > \sigma$, $k_2 > k_1$, $|z_0| > A$ (*see* (2.4)). *Then in the simple game* (1.1)–(1.4), *the strategy* (4.1) *is winning and the distance between the players changes according to the function*

$$F_1(t) = |z_0| - \frac{\rho}{k_1}e^{k_1 t} + \frac{\sigma}{k_2}e^{k_2 t} - \frac{\sigma}{k_2} + \frac{\rho}{k_1}.$$

P r o o f. Let us assume that the Evader implements the strategy (4.1) for any control function $u(\cdot) \in \mathbb{U}_{Gr}$ of the Pursuer. Obviously, $\mathbf{v}^*{}_{Gr}(t) \in \mathbb{V}_{Gr}$. Then for any $u(t)$, it follows the Cauchy problem (1.8). For the solution $z(t)$ of this problem, we can write the following estimations:

$$|z(t)| = \left| z_0 - \int_0^t \mathbf{v}^*{}_{Gr}(s)\, ds \right| - \int_0^t |u(s)|\, ds \ge |z_0| + \sigma \int_0^t e^{k_2 s}\xi_0\, ds - \int_0^t |u(s)|\, ds.$$

Using the first inequality in (1.5) we obtain $|z(t)| \ge F_1(t)$, where

$$F_1(t) = |z_0| - \frac{\rho}{k_1}e^{k_1 t} + \frac{\sigma}{k_2}e^{k_2 t} - \frac{\sigma}{k_2} + \frac{\rho}{k_1}.$$

1. Let $\rho > \sigma$ and $k_2 > k_1$.

a) The function $F_1(t)$ is increasing on the interval $[\theta, \infty)$, $\theta = \frac{1}{k_2 - k_1} \ln \frac{\rho}{\sigma}$.
b) $F_1(0) = |z_0| > 0$.

If the function $F_1(t)$ is positive at value $t = \theta$, i. e., $f(\theta) > 0$ or $|z_0| > A$ (see (2.4)), then this function is positive for all t, $t \in [\theta, \infty)$.

2. Let $\rho = \sigma$ and $k_2 > k_1$.

a) $F_1(0) = |z_0| > 0$.
b) The function $F_1(t)$ is increasing on the interval $[0, \infty)$.

Thus the function $F_1(t)$ is different from zero and positive for all t, $t \in [\theta, \infty)$.

3. Let $\rho < \sigma$ and $k_2 > k_1$.

a) $F_1(0) = |z_0| > 0$.
b) The function $f(t)$ is increasing on the interval $[\theta, \infty)$, $\theta = \frac{1}{k_2 - k_1} \ln \frac{\rho}{\sigma} < 0$.

Thereby the function $F_1(t)$ is positive for all t, $t \in [0, \infty)$.

4. Let $\rho \le \sigma$ and $k_1 = k_2$. This condition was studied in [40].

□

§ 5. Dynamics of the attainability domain

Suppose that $\rho > \sigma$, $k_1 > k_2$ are valid and the Pursuer applies the strategy (2.1) when the Evader implements any control function $v(\cdot) \in V_{Gr}$. In relation to the equations (1.1), (1.2), it follows that the trajectories of the players are defined, respectively, as

$$x(t) = x_0 + \int_0^t \mathbf{u}_{Gr}(s, v(s))\, ds, \quad y(t) = y_0 + \int_0^t v(s)\, ds.$$

For the pair of $(x(t), y(t))$ we construct the following sets

$$W(t) = \{w \colon \sigma e^{k_2 t}|w - x(t)| \ge \rho e^{k_1 t}|w - y(t)|\}, \tag{5.1}$$
$$W(0) = \{w \colon \sigma|w - x_0| \ge \rho|w - y_0|\}.$$

We can see that the inclusion $y(t) \in W(t)$ is valid for all t, $t \in [0, T_{Gr}]$.

L e m m a 5.1. *If $\rho > \sigma$, $k_1 > k_2$, then the set* (5.1) *can be written as follows:*

$$W(t) = x(t) + \Lambda_{Gr}(t, v(\cdot))(R(t, z_0)S + C(t, z_0)),$$

where

$$R(t, z_0) = \frac{\rho\sigma e^{(k_1+k_2)t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}|z_0|, \qquad C(t, z_0) = -\frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}} z_0,$$

and S is the unit sphere whose center is at the zero point in $\mathbb{R}^n$.

P r o o f. We introduce the denotation $w - x(t) = \bar{w}$. Substitute $\bar{w} + x(t)$ for w and the set (5.1) will take the form

$$\begin{aligned} W(t) = \{\bar{w} + x(t) \colon \sigma e^{k_2 t}|\bar{w}| \geq \rho e^{k_1 t}|\bar{w} + z(t)|\} = \\ = x(t) + \{\bar{w} \colon \sigma e^{k_2 t}|\bar{w}| \geq \rho e^{k_1 t}|\bar{w} + z(t)|\}. \end{aligned}$$

Hence,

$$W(t) = x(t) + W^*(t), \tag{5.2}$$

where $W^*(t) = \{w \colon \sigma e^{k_2 t}|w| \geq \rho e^{k_1 t}|w + z(t)|\}$.

Here we square the both sides of the inequality in $W^*(t)$

$$\begin{gathered} \sigma^2 e^{2k_2 t}|w|^2 \geq \rho^2 e^{2k_1 t}(|w|^2 + 2\langle w, z(t)\rangle + |z(t)|^2) \Longrightarrow \\ |w|^2 + \frac{2\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\langle w, z(t)\rangle + \frac{\rho^2 e^{2k_1 t}|z(t)|^2}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}} \leq 0 \Longrightarrow \\ |w|^2 + 2\left\langle w, \frac{\rho^2 e^{2k_1 t} z(t)}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right\rangle + \left(\frac{\rho^2 e^{2k_1 t} z(t)}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)^2 \leq \\ \leq \left(\frac{\rho^2 e^{2k_1 t} z(t)}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)^2 - \frac{\rho^2 e^{2k_1 t}|z(t)|^2}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}. \end{gathered}$$

Reduce the right-hand side of the last inequality to the canonical form

$$\left|w + \frac{\rho^2 e^{2k_1 t} z(t)}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right| \leq \frac{\rho\sigma e^{(k_1+k_2)t}|z(t)|}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}.$$

We denote as

$$R(t, z(t)) = \frac{\rho\sigma e^{(k_1+k_2)t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}|z(t)|, \qquad C(t, z(t)) = -\frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}} z(t).$$

Consequently, using (2.5) we can write the set (5.2) in the form

$$\begin{gathered} W(t) = x(t) + C(t, z(t)) + R(t, z(t))S = \\ = x(t) + \Lambda_{Gr}(t, v(\cdot))\left(\frac{\rho\sigma e^{(k_1+k_2)t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}|z_0|S - \frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}} z_0\right). \end{gathered}$$

This finishes the proof. $\square$

L e m m a 5.2. *If the conditions of Lemma* 5.1 *are valid for all $t \in [0, T_{Gr}]$, then the set $W(t)$ is monotone decreasing with respect to t, i. e., $W_1(t) \supset W_2(t)$ for $t_1 < t_2$, t_1, $t_2 \in [0, T_{Gr}]$.*

P r o o f. To prove Lemma 5.2 we construct the support function (see Blagodatskikh [42]) $F(W(t),\psi)$ of the set $W(t)$, where $\psi \in \mathbb{R}^n$ and $|\psi| = 1$.

Now take the derivative of $F(W(t),\psi)$ in terms of t

$$\frac{d}{dt}F(W(t),\psi) = \frac{d}{dt}F(x(t) + \Lambda(t,v(\cdot))[R(t,z_0)S + C(t,z_0),\psi]) =$$
$$= (\langle \dot{x}(t),\psi\rangle + \dot{\Lambda}(t,v(\cdot))[R(t,z_0) + \langle C(t,z_0),\psi\rangle] +$$
$$+ \Lambda(t,v(\cdot))[\dot{R}(t,z_0) + \langle \dot{C}(t,z_0),\psi\rangle] = \Phi_1(t,\psi) + \Phi_2(t,\psi),$$

where

$$\Phi_1(t,\psi) = \langle \dot{x}(t),\psi\rangle + \dot{\Lambda}(t,v(\cdot))[R(t,z_0) + \langle C(t,z_0),\psi\rangle],$$
$$\Phi_2(t,\psi) = \Lambda(t,v(\cdot))[\dot{R}(t,z_0) + \langle \dot{C}(t,z_0),\psi\rangle].$$

Now we prove that the inequality

$$\frac{d}{dt}F(W(t),\psi) = \Phi_1(t,\psi) + \Phi_2(t,\psi) \le 0$$

is true on $t \in (0, T_{Gr}]$. To do this, we first show that $\Phi_1(t,\mu) \le 0$. Multiply the expression $(\rho^2 e^{2k_1 t})/(\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t})$ to the both sides of the inequality $|v(t)|^2 \le \sigma^2 e^{2k_2 t}$ and reduce to the simple form

$$|v(t)|^2 \le \frac{\sigma^2 e^{2k_2 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}(\rho^2 e^{2k_1 t} - |v(t)|^2). \tag{5.3}$$

Considering the equality $\lambda(t,v(t))(\lambda(t,v(t)) - 2\langle v(t),\xi_0\rangle) = \rho^2 e^{2k_1 t} - |v(t)|^2$, the inequality (5.3) can be transformed into the form

$$|v(t)|^2 \le \frac{\sigma^2 e^{2k_2 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\lambda(t,v(t))(\lambda(t,v(t)) - 2\langle v(t),\xi_0\rangle) \Longrightarrow$$
$$|v(t)|^2 + 2\lambda(t,v(t))\frac{\sigma^2 e^{2k_2 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\langle v(t),\xi_0\rangle \le \lambda^2(t,v(t))\frac{\sigma^2 e^{2k_2 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}.$$

From this, we get

$$|v(t)|^2 + 2\lambda(t,v(t))\frac{\sigma^2 e^{2k_2 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\langle v(t),\xi_0\rangle + \lambda^2(t,v(t))\frac{\sigma^4 e^{4k_2 t}}{(\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t})^2} \le$$
$$\le \lambda^2(t,v(t))\frac{\rho^2\sigma^2 e^{2(k_1+k_2)t}}{(\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t})^2}.$$

Hence, we have

$$\left| v(t) + \frac{\sigma^2 e^{2k_2 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\xi_0\lambda(t,v(t))\right| \le \lambda(t,v(t))\frac{\rho\sigma e^{(k_1+k_2)t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}.$$

According to the property of the support function, for any vector $\psi \in \mathbb{R}^n$, $|\psi| = 1$, the inequality

$$\left\langle v(t) + \lambda(t,v(t))\frac{\sigma^2 e^{2k_2 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\xi_0, \psi\right\rangle \le \left| v(t) + \lambda(t,v(t))\frac{\sigma^2 e^{2k_2 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\xi_0\right|$$

is valid. Using this we can write the following

$$\left\langle v(t) + \lambda(t, v(t)) \frac{\sigma^2 e^{2k_2 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}} \xi_0, \psi \right\rangle \le \lambda(t, v(t)) \frac{\rho\sigma e^{(k_1+k_2)t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}} \Longrightarrow$$

$$\langle v(t), \psi\rangle - \lambda(t, v(t)) \left(1 - \frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right) \langle \xi_0, \psi\rangle \le \lambda(t, v(t)) \frac{\rho\sigma e^{(k_1+k_2)t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}} \Longrightarrow$$

$$\langle v(t) - \lambda(t, v(t))\xi_0, \psi\rangle + \lambda(t, v(t)) \frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}} \langle \xi_0, \psi\rangle - \lambda(t, v(t)) \frac{\rho\sigma e^{(k_1+k_2)t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}} \le 0 \Longrightarrow$$

$$\langle \dot{x}(t), \psi\rangle - \frac{\lambda(t, v(t))}{|z_0|} \left[\frac{\rho\sigma e^{(k_1+k_2)t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}} |z_0| - \frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}} \langle z_0, \psi\rangle\right] \le 0 \Longrightarrow$$

$$\langle \dot{x}(t), \psi\rangle + \dot{\Lambda}(t, v(\cdot))[R(t, z_0) + \langle C(t, z_0), \psi\rangle] \le 0.$$

This inequality means that $\Phi_1(t, \psi) \le 0$.

Now we will show $\Phi_2(t, \psi) \le 0$. Firstly, the function $\Phi_2(t, \psi)$ is rewritten in the following form

$$\Phi_2(t, \psi) = \Lambda(t, v(\cdot))[\dot{R}(t, z_0) + \langle \dot{C}(t, z_0), \psi\rangle] = \Lambda(t, v(\cdot))\Phi_3(t, \psi),$$

where

$$\Phi_3(t, \psi) = \left(\frac{\rho\sigma e^{(k_1+k_2)t}|z_0|}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)' - \left\langle \left(\frac{\rho^2 e^{2k_1 t} z_0}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)', \psi \right\rangle.$$

Because of $\Lambda(t, v(\cdot)) \ge 0$ on $t \in (0, T_{Gr}]$, it is enough to the prove $\Phi_3(t, \psi) \le 0$.

First, calculate the second derivative in $\Phi_3(t, \psi)$ and according to the conditions of Lemma 5.1, we have

$$\left(\frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)' = \frac{2\rho^2\sigma^2 e^{2(k_1+k_2)t}(k_2 - k_1)}{(\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t})^2} \le 0$$

for all $t \in [0, T_{Gr}]$. From this

$$-\left(\frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)' \ge 0.$$

According to the definition of the support function, the inequality $\langle \xi_0, \psi\rangle \le 1$ holds, and multiply the expression $\left(-(\rho^2 e^{2k_1 t})/(\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t})\right)'$ to the both sides of this inequality

$$-\left(\frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)' \langle \xi_0, \psi\rangle \le -\left(\frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)'.$$

Add the expression $\left((\rho\sigma e^{(k_1+k_2)t})/(\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t})\right)'$ to the last inequality

$$\begin{aligned} &\left(\frac{\rho\sigma e^{(k_1+k_2)t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)' - \left(\frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)' \langle \xi_0, \psi\rangle \le \\ &\le \left(\frac{\rho\sigma e^{(k_1+k_2)t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)' - \left(\frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)' = -\frac{\rho\sigma e^{(k_1+k_2)t}(k_1 - k_2)}{(\rho e^{k_1 t} + \sigma e^{k_2 t})^2}. \end{aligned} \tag{5.4}$$

This is negative for all t, $t \ge 0$, in accordance with the conditions of Lemma 5.1. Therefore the left-hand side of the inequality (5.4) is negative too, i. e.,

$$\left(\frac{\rho\sigma e^{(k_1+k_2)t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)' - \left(\frac{\rho^2 e^{2k_1 t}}{\rho^2 e^{2k_1 t} - \sigma^2 e^{2k_2 t}}\right)' \langle \xi_0, \psi\rangle \le 0. \tag{5.5}$$

Multiply $|z_0|$ to the both sides of the inequality (5.5)

$$\Phi_3(t,\psi) = \left(\frac{\rho\sigma e^{(k_1+k_2)t}|z_0|}{\rho^2 e^{2k_1t} - \sigma^2 e^{2k_2t}}\right)' - \left\langle \left(\frac{\rho^2 e^{2k_1t} z_0}{\rho^2 e^{2k_1t} - \sigma^2 e^{2k_2t}}\right)', \psi \right\rangle \leq 0.$$

So we can write the relation

$$\Phi_2(t,\psi) = \Lambda(t, v(\cdot))[\dot{R}(t,z_0) + \langle \dot{C}(t,z_0), \psi\rangle] = \Lambda(t, v(\cdot))\Phi_3(t,\psi) \leq 0$$

for the function $\Phi_2(t,\psi)$. Hence, we have shown that the support function $F(W(t),\psi)$ is decreasing for all $\psi \in \mathbb{R}^n$, $|\psi| = 1$, i. e.,

$$\frac{d}{dt}F(W(t),\psi) \leq 0.$$

On the other hand, the set $W(t)$ is monotone decreasing in $t \in [0, T_{Gr}]$. This completes the proof. □

§ 6. Individual cases for the pursuit-evasion games with Gr-constraints

<table>
<tr><th colspan="7">Pursuit game</th></tr>
<tr><th>№</th><th colspan="2">Capture conditions</th><th>Resolving function</th><th>Duration of the strategy</th><th>Guaranteed capture time</th></tr>
<tr><td>1</td><td>$k_1 = k_2$</td><td>$\rho > \sigma$</td><td>$\lambda_{Gr}(t,v)$</td><td>$[0,\infty)$</td><td>$\frac{1}{k}\ln\left(1 + \frac{k|z_0|}{\rho-\sigma}\right)$</td></tr>
<tr><td>2</td><td>$k_1 > k_2$</td><td>$\rho > \sigma$</td><td rowspan="3">$\lambda^*_{Gr}(t,v)$</td><td>$[0,\infty)$</td><td rowspan="3">the first positive root of $F_1(t) = 0$</td></tr>
<tr><td>3</td><td>$k_1 < k_2$</td><td>$\rho > \sigma$, $|z_0| \leq A$</td><td>$[0,\theta]$</td></tr>
<tr><td>4</td><td>$k_1 > k_2$</td><td>$\rho = \sigma$</td><td>$[0,\infty)$</td></tr>
<tr><td>5</td><td>$k_1 > k_2$</td><td>$\rho < \sigma$</td><td>$\lambda^{**}_{Gr}(t,v)$</td><td>$[\theta,\infty)$</td><td>the first positive root of $F_2(t) = 0$</td></tr>
</table>

where

$$\lambda_{Gr}(t,v) = \langle v, \xi_0\rangle + \sqrt{\langle v, \xi_0\rangle^2 + \delta e^{2kt}},\ \delta = \rho^2 - \sigma^2,$$
$$\lambda^*_{Gr}(t,v) = \langle v, \xi_0\rangle + \sqrt{\langle v, \xi_0\rangle^2 + \rho^2 e^{2k_1t} - |v|^2},$$
$$\lambda^{**}_{Gr}(t,v) = \langle v, \xi^*_0\rangle + \sqrt{\langle v, \xi^*_0\rangle^2 + \rho^2 e^{2k_1t} - |v|^2}.$$

<table>
<tr><th colspan="6">Evasion game</th></tr>
<tr><th>№</th><th colspan="2">Evasion conditions</th><th>Strategy of the Evader</th><th>Duration of the strategy</th><th>The distance function between the players</th></tr>
<tr><td>1</td><td>$k_1 = k_2$</td><td>$\rho = \sigma$</td><td rowspan="5">$\mathbf{v}^*(t) = -\sigma e^{k_2t}\xi_0$</td><td rowspan="5">$t \in [0,\infty)$</td><td rowspan="2">$|z(t)| \geq |z_0| + (\sigma-\rho)(e^{kt}-1)/k$</td></tr>
<tr><td>2</td><td>$k_1 = k_2$</td><td>$\rho < \sigma$</td></tr>
<tr><td>3</td><td>$k_1 < k_2$</td><td>$\rho > \sigma$, $|z_0| > A$</td><td>$|z(t)| \geq |z_0| - A$</td></tr>
<tr><td>4</td><td>$k_1 < k_2$</td><td>$\rho = \sigma$</td><td rowspan="2">$|z(t)| \geq |z_0| - \frac{\rho}{k_1}e^{k_1t} + \frac{\sigma}{k_2}e^{k_2t} - \frac{\sigma}{k_2} + \frac{\rho}{k_1}$</td></tr>
<tr><td>5</td><td>$k_1 < k_2$</td><td>$\rho < \sigma$</td></tr>
</table>

where

$$A = \rho^{\frac{k_2}{k_2-k_1}}\sigma^{\frac{k_1}{k_1-k_2}}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) + \frac{\sigma}{k_2} - \frac{\rho}{k_1}.$$

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Б. Т. Саматов, А. Х. Акбаров, Б. И. Жураев
Дифференциальные игры преследования–убегания при Gr–ограничениях на управления

Ключевые слова: дифференциальная игра, неравенство Гронуолла, преследование, убегание, оптимальная стратегия, время поимки.

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В этой статье исследуется дифференциальная игра преследования–убегания, когда на управления игроков налагаются дифференциальные ограничения вида интегрального неравенства Гронуолла. Отметим, что стратегия параллельного преследования (короче, П-стратегия) была введена и использована Л. А. Петросяном для решения задач простого преследования при фазовых ограничениях на состояний игроков для случая, когда функции управления обоих игроков выбираются из класса L_∞. В настоящей работе для решения задачи простого преследования построена П-стратегия, когда функции управления обоих игроков выбираются из различных классов с ограничениями типа Гронуолла и для этого случая найдены достаточные условия поимки и оптимальное время поимки. Для решения задачи убегания предлагается функция управления для убегающего и находятся достаточные условия убегания. Кроме того, построена область достижимости игроков и даны условия вложения ее по времени. Полученные результаты являются развитием и продолжением работ Р. Айзекса, Л. А. Петросяна, Б. Н. Пшеничного, А. А. Чикрия, А. А. Азамова и других исследователей, включая авторов этой работы.

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