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ANDREEV STATES IN A QUASI-ONE-DIMENSIONAL SUPERCONDUCTOR ON THE SURFACE OF A TOPOLOGICAL INSULATOR

Yu. P. Chuburin* **and T. S. Tinyukova†**

We study bound states in an s-wave superconducting strip on the surface of a topological superconductor with the perpendicular Zeeman field. We prove analytically that an arbitrarily small local perturbation of the Zeeman field generates Andreev bound states with energies near the superconducting gap edges, while the (nonmagnetic) impurity potential does not produce such an effect. Rather large perturbations of the Zeeman field can lead to the appearance of Andreev bound states with energies near zero. We analytically find wave functions of the Andreev bound states under consideration. In contrast to the one-dimensional case, the wave functions do not satisfy the conjugation conditions that are characteristic of Majorana states because of the influence of neighboring subbands.

Keywords: topological insulator, superconducting gap, Andreev bound state, subband, Zeeman field

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1. Introduction

Active studies of topological superconductors are continued at present, which is related to the appearance of Majorana bound states (MBSs) in the topologically nontrivial phase at their boundaries; they are zero-energy quasiparticles of the "electron–hole" form and obey the non-Abelian quantum statistics. There exists a high probability of using MBSs in quantum calculations (see [1]–[3]).

In what follows, by the Andreev bound states (ABSs), we mean quasiparticles generated by the impurity or a local change in the constant magnetic field with the energy inside the superconducting gap (cf. an analogous understanding of ABSs, e.g., in [4], [5]). The difference between the MBSs and ABSs is a serious problem in experiments [4], [6], [7]; therefore, studies of the origin and behavior of the ABSs are important. This problem has usually been studied within the framework of the one-dimensional Bogoliubov–de Gennes equation (BdG), and hence the influence of neighboring subbands was not taken into account. However, it became difficult to identify quasiparticles in the case of a significant filling of subbands [8].

*Udmurt Federal Research Center, Ural Branch, Russian Academy of Sciences, Izhevsk, Russia, e-mail: chuburin@ftiudm.ru (corresponding author).

†Udmurt State University, Izhevsk, Russia, e-mail: ttinyukova@mail.ru.

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In this paper, we study the ABSs in a one-dimensional strip on the surface of a three-dimensional topological insulator (TI) in the presence of a constant perpendicular Zeeman field. We analytically prove that for an arbitrarily small local perturbation of the Zeeman field, energy levels appear near the boundaries of the superconducting gap, as in the one-dimensional model [9]. However, in the one-dimensional case, the ABSs have (separately for particles with spin up and spin down for a spinor in Eq. (4) below) the "particle–hole" symmetry (see (29) below), which is not the case for the strip (cf. the numerical results in [10], [11]).

In contrast to [12], where the authors studied a spinless p-wave model, the nonmagnetic impurity does not generate bound states in a TI. In addition, in the case of a TI, the wave functions for the topological and trivial phases are symmetric, but the wave functions differ strongly from each other for the p-wave model and the symmetry appears only if resonant states are taken into account. We note that in the one-dimensional superconducting structure, at the boundary of a two-dimensional TI [9], the nonmagnetic impurity can generate bound states in the trivial phase.

2. Hamiltonian and basic equations

In the presence of a perpendicular Zeeman field, the two-dimensional boundary of a three-dimensional TI with superconducting order induced due to the proximity effect (see the discussion of this effect for the three-dimensional TI boundary in [1], [13]) is described by the BdG Hamiltonian of the form (see [14]–[16])

$$H=\begin{pmatrix} M & -i\partial_x-\partial_y & 0 & \Delta \\ -i\partial_x+\partial_y & -M & -\Delta & 0 \\ 0 & -\Delta & -M & -i\partial_x+\partial_y \\ \Delta & 0 & -i\partial_x-\partial_y & M \end{pmatrix}, \tag{1}$$

where $\Delta=\mathrm{const}\neq 0$ is a real pairing potential and M is the parameter of the perpendicular Zeeman field.

The Hamiltonian H acts on spinors of the form

$$\Psi(x,y)=(\psi_{\mathrm{e}\uparrow}(x,y),\psi_{\mathrm{e}\downarrow}(x,y),\psi_{\mathrm{h}\uparrow}(x,y),\psi_{\mathrm{h}\downarrow}(x,y))^{\mathrm{T}}, \tag{2}$$

where the arrow indicates the spin direction; the first two components describe electrons, and last two, holes. We consider the solutions $\Psi(x,y)$ of the BdG equation in the strip $-\infty<x<\infty$, $0<y<W$, assuming that the solutions are periodic in y with the period W. In what follows, W is assumed to be large, which allows assuming that the errors caused by the choice of boundary conditions with respect to y are small; however, it should not significantly exceed the coherence length [17]. Thus,

$$\Psi(x,y)=\frac{1}{\sqrt{W}}\sum_{n=-\infty}^{\infty}\Psi^{(n)}(x)e^{-2\pi iny/W}, \tag{3}$$

where

$$\Psi^{(n)}(x)=(\psi^{(n)}_{\mathrm{e}\uparrow}(x),\psi^{(n)}_{\mathrm{e}\downarrow}(x),\psi^{(n)}_{\mathrm{h}\uparrow}(x),\psi^{(n)}_{\mathrm{h}\downarrow}(x))^{\mathrm{T}}=\frac{1}{\sqrt{W}}\int_0^W\Psi(x,y)e^{2\pi iny/W}\,dy. \tag{4}$$

Hamiltonian (1) acts on the nth term in (3) in accordance with the formula

$$H(\Psi^{(n)}(x)e^{-2\pi iny/W})=\begin{pmatrix} M & -i\partial_x+2\pi in/W & 0 & \Delta \\ -i\partial_x-2\pi in/W & -M & -\Delta & 0 \\ 0 & -\Delta & -M & -i\partial_x-2\pi in/W \\ \Delta & 0 & -i\partial_x+2\pi in/W & M \end{pmatrix}\times$$
$$\times(\psi^{(n)}_{\mathrm{e}\uparrow}(x),\psi^{(n)}_{\mathrm{e}\downarrow}(x),\psi^{(n)}_{\mathrm{h}\uparrow}(x),\psi^{(n)}_{\mathrm{h}\downarrow}(x))^{\mathrm{T}},\qquad n=0,\pm1,\dots. \tag{5}$$

We let $H^{(n)}$ denote the matrix in relation (5); it determines the Hamiltonian H in the nth band. In what follows, we need the Green's function of the Hamiltonian $H^{(n)}$; it is given in the appendix.

The spectrum of $H^{(n)}$ is described by the inequality (see (A.3))

$$|E| \geqslant \sqrt{(|M|-|\Delta|)^2 + \left(\frac{2\pi n}{W}\right)^2},$$

where E is the energy. The energy gap width increases with increasing $|n|$. The spectrum of H coincides with that of $H^{(0)}$ and is defined by the inequality $|E| \geqslant ||M|-|\Delta||$.

We consider the BdG equation

$$(H+V)\Psi = E\Psi \tag{6}$$

with the potential

$$V = V(x,y) = m \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (e^{-2\pi iy/W} + e^{2\pi iy/W})\delta(x), \tag{7}$$

where $\delta(x)$ is the Dirac delta function. The potential V describes the perturbation of a constant Zeeman field, which is local with respect to x, and generates a coupling of the subbands due to harmonics. We assume that the potential energy is distributed symmetrically with respect to zero, i.e., even approximations of the delta function $\delta(x)$ are used in calculations. Below, we are interested only in the bands with $n = 0, \pm 1$; they are the nearest neighbors to $E = 0$, and we can therefore approximately write Eq. (6) in the form

$$\begin{aligned}&(H^{(0)}-E)\Psi^{(0)}(x) + (H^{(1)}-E)\Psi^{(1)}(x)e^{-2\pi iy/W} + (H^{(-1)}-E)\Psi^{(-1)}(x)e^{2\pi iy/W} = \\ &= -m\left(\begin{pmatrix} \psi^{(0)}_{\mathrm{e}\uparrow}(x) \\ -\psi^{(0)}_{\mathrm{e}\downarrow}(x) \\ -\psi^{(0)}_{\mathrm{h}\uparrow}(x) \\ \psi^{(0)}_{\mathrm{h}\downarrow}(x) \end{pmatrix}(e^{-2\pi iy/W} + e^{2\pi iy/W}) + \begin{pmatrix} \psi^{(1)}_{\mathrm{e}\uparrow}(x) \\ -\psi^{(1)}_{\mathrm{e}\downarrow}(x) \\ -\psi^{(1)}_{\mathrm{h}\uparrow}(x) \\ \psi^{(1)}_{\mathrm{h}\downarrow}(x) \end{pmatrix} + \begin{pmatrix} \psi^{(-1)}_{\mathrm{e}\uparrow}(x) \\ -\psi^{(-1)}_{\mathrm{e}\downarrow}(x) \\ -\psi^{(-1)}_{\mathrm{h}\uparrow}(x) \\ \psi^{(-1)}_{\mathrm{h}\downarrow}(x) \end{pmatrix}\right)\delta(x).\end{aligned} \tag{8}$$

Hence we obtain

$$(H^{(0)}-E)\begin{pmatrix} \psi^{(0)}_{\mathrm{e}\uparrow}(x) \\ \psi^{(0)}_{\mathrm{e}\downarrow}(x) \\ \psi^{(0)}_{\mathrm{h}\uparrow}(x) \\ \psi^{(0)}_{\mathrm{h}\downarrow}(x) \end{pmatrix} = -m \begin{pmatrix} \psi^{(1)}_{\mathrm{e}\uparrow}(x) + \psi^{(-1)}_{\mathrm{e}\uparrow}(x) \\ -(\psi^{(1)}_{\mathrm{e}\downarrow}(x) + \psi^{(-1)}_{\mathrm{e}\downarrow}(x)) \\ -(\psi^{(1)}_{\mathrm{h}\uparrow}(x) + \psi^{(-1)}_{\mathrm{h}\uparrow}(x)) \\ \psi^{(1)}_{\mathrm{h}\downarrow}(x) + \psi^{(-1)}_{\mathrm{h}\downarrow}(x) \end{pmatrix}\delta(x), \tag{9}$$

$$(H^{(\pm1)}-E)\begin{pmatrix} \psi^{(\pm1)}_{\mathrm{e}\uparrow}(x) \\ \psi^{(\pm1)}_{\mathrm{e}\downarrow}(x) \\ \psi^{(\pm1)}_{\mathrm{h}\uparrow}(x) \\ \psi^{(\pm1)}_{\mathrm{h}\downarrow}(x) \end{pmatrix} = -m \begin{pmatrix} \psi^{(0)}_{\mathrm{e}\uparrow}(x) \\ -\psi^{(0)}_{\mathrm{e}\downarrow}(x) \\ -\psi^{(0)}_{\mathrm{h}\uparrow}(x) \\ \psi^{(0)}_{\mathrm{h}\downarrow}(x) \end{pmatrix}\delta(x). \tag{10}$$

We next assume that

$$M, \Delta > 0, \qquad |M-\Delta| \ll \min\{M, \Delta\}, \qquad |E| < |M-\Delta|. \tag{11}$$

In particular, the superconducting gap is small. We also assume that the quantities p, $1/W$, and $M-\Delta$ are approximately of the same order, in particular, $(1/W)^2 \ll |M-\Delta|$, and $|M-\Delta|^2 \ll 1/W$. We set

$$a_n^2 = E^2 - \left(\frac{2\pi n}{W}\right)^2 - (M-\Delta)^2. \tag{12}$$

Using form (A.7) of the Green's function $G^{(n)}(x-x',E)$ of the Hamiltonian $H^{(n)}$ at $n=0$, we pass from (9) to the equation

$$
\begin{aligned}
&(\psi^{(0)}_{\mathrm{e}\uparrow}(x),\psi^{(0)}_{\mathrm{e}\downarrow}(x),\psi^{(0)}_{\mathrm{h}\uparrow}(x),\psi^{(0)}_{\mathrm{h}\downarrow}(x))^{\mathrm{T}} = \\
&\quad= \frac{me^{ia_0|x|}}{4ia_0}\begin{pmatrix} M-\Delta+E & a_0\,\mathrm{sgn}(x) & -a_0\,\mathrm{sgn}(x) & -(M-\Delta+E) \\ a_0\,\mathrm{sgn}(x) & -(M-\Delta-E) & M-\Delta-E & -a_0\,\mathrm{sgn}(x) \\ -a_0\,\mathrm{sgn}(x) & M-\Delta-E & -(M-\Delta-E) & a_0\,\mathrm{sgn}(x) \\ -(M-\Delta+E) & -a_0\,\mathrm{sgn}(x) & a_0\,\mathrm{sgn}(x) & M-\Delta+E \end{pmatrix}\times \\
&\quad\times\begin{pmatrix} \psi^{(1)}_{\mathrm{e}\uparrow}(0)+\psi^{(-1)}_{\mathrm{e}\uparrow}(0) \\ -(\psi^{(1)}_{\mathrm{e}\downarrow}(0)+\psi^{(-1)}_{\mathrm{e}\downarrow}(0)) \\ -(\psi^{(1)}_{\mathrm{h}\uparrow}(0)+\psi^{(-1)}_{\mathrm{h}\uparrow}(0)) \\ \psi^{(1)}_{\mathrm{h}\downarrow}(0)+\psi^{(-1)}_{\mathrm{h}\downarrow}(0) \end{pmatrix}
\end{aligned}
\tag{13}
$$

(for brevity, we set $\psi(0)=(\psi(+0)+\psi(-0))/2$). Multiplying (13) by $\delta(x)$ and integrating, we obtain two equations

$$
\begin{aligned}
\begin{pmatrix}\psi^{(0)}_{\mathrm{e}\uparrow}(0)\\ \psi^{(0)}_{\mathrm{h}\downarrow}(0)\end{pmatrix} &= \frac{m(M-\Delta+E)}{4ia_0}(\psi^{(1)}_{\mathrm{e}\uparrow}(0)+\psi^{(-1)}_{\mathrm{e}\uparrow}(0)-\psi^{(1)}_{\mathrm{h}\downarrow}(0)-\psi^{(-1)}_{\mathrm{h}\downarrow}(0))\begin{pmatrix}1\\-1\end{pmatrix}, \\
\begin{pmatrix}\psi^{(0)}_{\mathrm{e}\downarrow}(0)\\ \psi^{(0)}_{\mathrm{h}\uparrow}(0)\end{pmatrix} &= \frac{m(M-\Delta-E)}{4ia_0}(\psi^{(1)}_{\mathrm{e}\downarrow}(0)+\psi^{(-1)}_{\mathrm{e}\downarrow}(0)-\psi^{(1)}_{\mathrm{h}\uparrow}(0)-\psi^{(-1)}_{\mathrm{h}\uparrow}(0))\begin{pmatrix}1\\-1\end{pmatrix}.
\end{aligned}
\tag{14}
$$

From (A.7) and (10), we similarly find

$$
\begin{aligned}
&(\psi^{(\pm1)}_{\mathrm{e}\uparrow}(0),\psi^{(\pm1)}_{\mathrm{e}\downarrow}(0),\psi^{(\pm1)}_{\mathrm{h}\uparrow}(0),\psi^{(\pm1)}_{\mathrm{h}\downarrow}(0)) = \\
&\quad= \frac{m}{4ia_1}\begin{pmatrix} M-\Delta+E & \pm2\pi i/W & \mp2\pi i/W & -(M-\Delta+E) \\ \mp2\pi i/W & -(M-\Delta-E) & M-\Delta-E & \pm2\pi i/W \\ \pm2\pi i/W & M-\Delta-E & -(M-\Delta-E) & \mp2\pi i/W \\ -(M-\Delta+E) & \mp2\pi i/W & \pm2\pi i/W & M-\Delta+E \end{pmatrix}\times \\
&\quad\times(\psi^{(0)}_{\mathrm{e}\uparrow}(0),-\psi^{(0)}_{\mathrm{e}\downarrow}(0),-\psi^{(0)}_{\mathrm{h}\uparrow}(0),\psi^{(0)}_{\mathrm{h}\downarrow}(0))^{\mathrm{T}}.
\end{aligned}
\tag{15}
$$

3. ABSs with energies near the superconducting gap edges

We first consider the case of the topologically trivial phase where $M-\Delta>0$ (see [1], [2]). We prove the existence of ABSs with the energies $E=M-\Delta-\varepsilon$, where $0<\varepsilon\ll|M-\Delta|$, i.e., near the upper superconducting gap edge. From (14), we approximately find

$$
\begin{aligned}
&\begin{pmatrix}\psi^{(0)}_{\mathrm{e}\uparrow}(0)\\ \psi^{(0)}_{\mathrm{h}\downarrow}(0)\end{pmatrix} = \frac{(M-\Delta)m}{2ia_0}(\psi^{(1)}_{\mathrm{e}\uparrow}(0)+\psi^{(-1)}_{\mathrm{e}\uparrow}(0)-\psi^{(1)}_{\mathrm{h}\downarrow}(0)-\psi^{(-1)}_{\mathrm{h}\downarrow}(0))\begin{pmatrix}1\\-1\end{pmatrix}, \\
&\psi^{(0)}_{\mathrm{e}\downarrow}(0)=\psi^{(0)}_{\mathrm{h}\uparrow}(0)=0.
\end{aligned}
\tag{16}
$$

Using (16) and the assumptions introduced above, we write (15) in the form

$$
\begin{aligned}
\begin{pmatrix}\psi^{(\pm1)}_{\mathrm{e}\uparrow}(0)\\ \psi^{(\pm1)}_{\mathrm{e}\downarrow}(0)\\ \psi^{(\pm1)}_{\mathrm{h}\uparrow}(0)\\ \psi^{(\pm1)}_{\mathrm{h}\downarrow}(0)\end{pmatrix} &= -\frac{(M-\Delta)m^2}{4a_0a_1}(\psi^{(1)}_{\mathrm{e}\uparrow}(0)+\psi^{(-1)}_{\mathrm{e}\uparrow}(0)-\psi^{(1)}_{\mathrm{h}\downarrow}(0)-\psi^{(-1)}_{\mathrm{h}\downarrow}(0))\times \\
&\quad\times\left(M-\Delta,\mp\frac{\pi i}{W},\pm\frac{\pi i}{W},-(M-\Delta)\right)^{\mathrm{T}}.
\end{aligned}
\tag{17}
$$

In accordance with (12), we approximately have

$$a_0 = \pm i\sqrt{2\varepsilon(M-\Delta)}, \tag{18}$$

where the signs + and − correspond to the respective bound and resonant states. The resonant states appear in the case where the energy ε measured from the gap boundary is transferred to the second sheet of the Riemann surface of the square root. In this case, in view of (13), the wave function increases at infinity because of the factor $e^{ia_0|x|}$. We note that the quantity $1/|a_0|$ characterizes the lifetime of the particle [18], as do the $1/|a_\pm|$, where

$$a_1 = a_{-1} = \pm i\sqrt{\left(\frac{2\pi}{W}\right)^2 + (M-\Delta)^2 - E^2} = \pm i\sqrt{\left(\frac{2\pi}{W}\right)^2 + 2\varepsilon(M-\Delta)} \approx \pm\frac{2\pi i}{W}. \tag{19}$$

In what follows, unless stipulated otherwise, we choose the sign +; it corresponds to bound sates in (18) and (19). We introduce the notation

$$\alpha = -\frac{m^2(M-\Delta)}{2a_0a_1} = \frac{m^2\sqrt{M-\Delta}}{4\sqrt{2\varepsilon}(\pi/W)}. \tag{20}$$

From (17), we obtain the equalities

$$\psi^{(\pm1)}_{\mathrm{h}\downarrow}(0) = -\psi^{(\pm1)}_{\mathrm{e}\uparrow}(0), \qquad \psi^{(\pm1)}_{\mathrm{h}\uparrow}(0) = -\psi^{(\pm1)}_{\mathrm{e}\downarrow}(0). \tag{21}$$

Using (20), (21), we write the system of equations for the unknowns $\psi^{(\pm1)}_{\mathrm{e}\uparrow}(0)$, $\psi^{(\pm1)}_{\mathrm{e}\downarrow}(0)$:

$$\begin{aligned}
\psi^{(\pm1)}_{\mathrm{e}\uparrow}(0) &= 2\alpha(M-\Delta)(\psi^{(1)}_{\mathrm{e}\uparrow}(0) + \psi^{(-1)}_{\mathrm{e}\uparrow}(0)),\\
\psi^{(\pm1)}_{\mathrm{e}\downarrow}(0) &= \mp 2\alpha\frac{\pi i}{W}(\psi^{(1)}_{\mathrm{e}\uparrow}(0) + \psi^{(-1)}_{\mathrm{e}\uparrow}(0)).
\end{aligned} \tag{22}$$

The determinant of system (22) is $d = 1 - 4\alpha(M-\Delta)$. Then the existence conditions for a nonzero solution of the system has the form

$$1 - 4\alpha(M-\Delta) = 0 \tag{23}$$

and, in view of (20),

$$\sqrt{\varepsilon} = \frac{m^2(M-\Delta)^{3/2}}{\sqrt{2}\pi/W}. \tag{24}$$

Condition (24) means that for all rather small m, there exists a stable energy level near the gap edge. In contrast to the one-dimensional case [9], this condition is independent of the sign of m.

We find the wave function corresponding to this level. Under condition (23), system (22) can be written in the form

$$\begin{pmatrix} 1/2 & -1/2 & 0 & 0\\ -1/2 & 1/2 & 0 & 0\\ i\pi/(2W(M-\Delta)) & i\pi/(2W(M-\Delta)) & 1 & 0\\ -i\pi/(2W(M-\Delta)) & -i\pi/(2W(M-\Delta)) & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi^{(1)}_{\mathrm{e}\uparrow}(0)\\ \psi^{(-1)}_{\mathrm{e}\uparrow}(0)\\ \psi^{(1)}_{\mathrm{e}\downarrow}(0)\\ \psi^{(-1)}_{\mathrm{e}\downarrow}(0) \end{pmatrix} = 0, \tag{25}$$

whence we have $\psi^{(1)}_{e\uparrow}(0) = \psi^{(-1)}_{e\uparrow}(0) = C = \text{const}$, $\psi^{(1)}_{e\downarrow}(0) = -\psi^{(-1)}_{e\downarrow}(0) = -\pi iC/(W(M-\Delta))$. Substituting these quantities in (13) and using (21), at $C = 1$, we have

$$
\begin{aligned}
&(\psi^{(0)}_{e\uparrow}(x), \psi^{(0)}_{e\downarrow}(x), \psi^{(0)}_{h\uparrow}(x), \psi^{(0)}_{h\downarrow}(x))^{\mathrm{T}} = \\
&\quad = \frac{me^{ia_0|x|}}{2ia_0}\begin{pmatrix} 2(M-\Delta) & a_0\,\mathrm{sgn}(x) & -a_0\,\mathrm{sgn}(x) & -2(M-\Delta) \\ a_0\,\mathrm{sgn}(x) & 0 & 0 & -a_0\,\mathrm{sgn}(x) \\ -a_0\,\mathrm{sgn}(x) & 0 & 0 & a_0\,\mathrm{sgn}(x) \\ -2(M-\Delta) & -a_0\,\mathrm{sgn}(x) & a_0\,\mathrm{sgn}(x) & 2(M-\Delta) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \\
&\quad = \frac{me^{ia_0|x|}}{ia_0}\,(2(M-\Delta), a_0\,\mathrm{sgn}(x), -a_0\,\mathrm{sgn}(x), -2(M-\Delta))^{\mathrm{T}}\,. \qquad (26)
\end{aligned}
$$

Similarly, using (10), (26), (A.7), we obtain

$$
\begin{aligned}
&(\psi^{(\pm1)}_{e\uparrow}(x), \psi^{(\pm1)}_{e\downarrow}(x), \psi^{(\pm1)}_{h\uparrow}(x), \psi^{(\pm1)}_{h\downarrow}(x))^{\mathrm{T}} = \\
&\quad = -\frac{(M-\Delta)m^2e^{ia_1|x|}}{2a_0a_1}\begin{pmatrix} 2(M-\Delta) & 0 & 0 & -2(M-\Delta) \\ a_1\,\mathrm{sgn}(x) \mp 2\pi i/W & 0 & 0 & -(a_1\,\mathrm{sgn}(x) \mp 2\pi i/W) \\ -(a_1\,\mathrm{sgn}(x) \mp 2\pi i/W) & 0 & 0 & a_1\,\mathrm{sgn}(x) \mp 2\pi i/W \\ -2(M-\Delta) & 0 & 0 & 2(M-\Delta) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \\
&\quad = -\frac{2(M-\Delta)m^2e^{ia_1|x|}}{a_0a_1}\begin{pmatrix} M-\Delta \\ (\pi i/W)(\mathrm{sgn}(x) \mp 1) \\ -(\pi i/W)(\mathrm{sgn}(x) \mp 1) \\ -(M-\Delta) \end{pmatrix}. \qquad (27)
\end{aligned}
$$

Substituting (26), (27) in (3) at $n = 0, \pm1$, using (18), (19), and omitting the common factor m/ia_0, we find the wave function of the bound state (see Fig. 1):

$$
\begin{aligned}
\Psi(x,y) = \frac{1}{\sqrt{W}}\Bigg(&e^{-\sqrt{2\varepsilon(M-\Delta)}|x|}(2(M-\Delta), i\sqrt{2\varepsilon(M-\Delta)}\,\mathrm{sgn}(x), -i\sqrt{2\varepsilon(M-\Delta)}\,\mathrm{sgn}(x), -2(M-\Delta)) - \\
&- \frac{(M-\Delta)me^{-(2\pi/W)|x|}}{\pi/W}\left(\left(M-\Delta, \frac{2\pi i}{W}\theta(x), -\frac{2\pi i}{W}\theta(x), -(M-\Delta)\right)e^{2\pi iy/W} + \right. \\
&\left. + \left(M-\Delta, -\frac{2\pi i}{W}\theta(-x), \frac{2\pi i}{W}\theta(-x), -(M-\Delta)\right)e^{-2\pi iy/W}\right)\Bigg)^{\mathrm{T}}. \qquad (28)
\end{aligned}
$$

The "valley" in Fig. 1 is due to the second term in the right-hand side of (28), i.e., due to the effect of neighboring subbands. The amplitude of the wave function of an ABS with the energy near the lower boundary has a similar form.

In contrast to the one-dimensional model [9], wave function (28) has no "particle–hole" symmetry, which is described by the conjugation conditions

$$
(\Psi_{e\uparrow})^*(x,y) = \Psi_{h\downarrow}(x,y), \qquad (\Psi_{e\downarrow})^*(x,y) = \Psi_{h\uparrow}(x,y) \qquad (29)
$$

(this symmetry is characteristic of MBSs). We note that as ε increases, the ABS localization increases.

We now consider the case $E = -(M-\Delta) + \varepsilon$, where $0 < \varepsilon \ll \Delta - M$. In view of (14), we have

$$
\begin{aligned}
&\psi^{(0)}_{e\uparrow}(0) = \psi^{(0)}_{h\downarrow}(0) = 0, \\
&\begin{pmatrix} \psi^{(0)}_{e\downarrow}(0) \\ \psi^{(0)}_{h\uparrow}(0) \end{pmatrix} = \frac{m(M-\Delta)}{2ia_0}(\psi^{(1)}_{e\downarrow}(0) + \psi^{(-1)}_{e\downarrow}(0) - \psi^{(1)}_{h\uparrow}(0) - \psi^{(-1)}_{h\uparrow}(0))\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \qquad (30)
\end{aligned}
$$

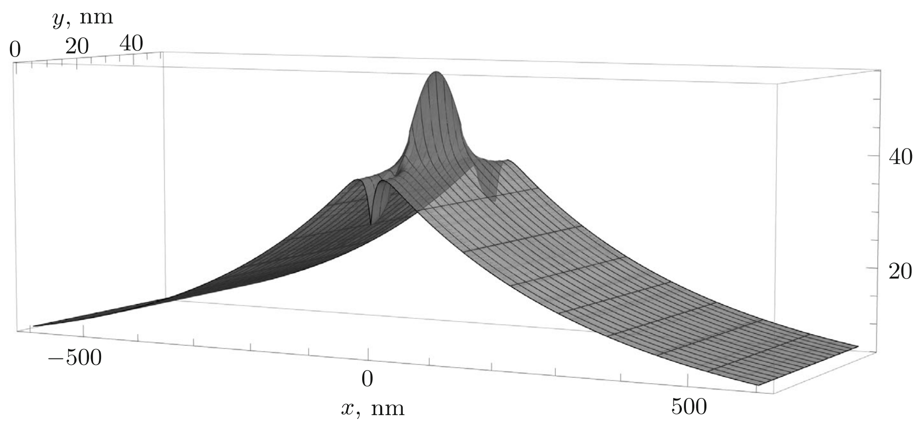

Fig. 1. Amplitude $|\Psi|^2$ of the ABS wave function with the energy near the upper gap edge. Here, $M-\Delta = 0{,}1$ meV and $m = 0{,}1$ meV.

In accordance with (15), as above, we obtain the equation

$$\begin{pmatrix} \psi^{(\pm1)}_{\mathrm{e}\uparrow}(0) \\ \psi^{(\pm1)}_{\mathrm{e}\downarrow}(0) \\ \psi^{(\pm1)}_{\mathrm{h}\uparrow}(0) \\ \psi^{(\pm1)}_{\mathrm{h}\downarrow}(0) \end{pmatrix} = \frac{(M-\Delta)m^2}{2a_0a_1}(\psi^{(1)}_{\mathrm{e}\downarrow}(0) + \psi^{(-1)}_{\mathrm{e}\downarrow}(0) - \psi^{(1)}_{\mathrm{h}\uparrow}(0) - \psi^{(-1)}_{\mathrm{h}\uparrow}(0)) \times$$
$$\times \left(\pm\frac{\pi i}{W}, -(M-\Delta), M-\Delta, \mp\frac{\pi i}{W}\right)^{\mathrm{T}}. \tag{31}$$

From (31), we obtain the equalities $\psi^{(\pm1)}_{\mathrm{h}\downarrow}(0) = -\psi^{(\pm1)}_{\mathrm{e}\uparrow}(0)$, $\psi^{(\pm1)}_{\mathrm{h}\uparrow}(0) = -\psi^{(\pm1)}_{\mathrm{e}\downarrow}(0)$ and the system

$$\psi^{(\pm1)}_{\mathrm{e}\uparrow}(0) = \mp 2\alpha\frac{\pi i}{W}(\psi^{(1)}_{\mathrm{e}\downarrow}(0) + \psi^{(-1)}_{\mathrm{e}\downarrow}(0)),$$

$$\psi^{(\pm1)}_{\mathrm{e}\downarrow}(0) = 2\alpha(M-\Delta)(\psi^{(1)}_{\mathrm{e}\downarrow}(0) + \psi^{(-1)}_{\mathrm{e}\downarrow}(0)).$$

The existence condition for a nonzero solution of this system coincides with (23). To find the wave function, we have the equation

$$\begin{pmatrix} 1 & 0 & i\pi/(2W(M-\Delta)) & i\pi/(2W(M-\Delta)) \\ 0 & 1 & -i\pi/(2W(M-\Delta)) & -i\pi/(2W(M-\Delta)) \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} \psi^{(1)}_{\mathrm{e}\uparrow}(0) \\ \psi^{(-1)}_{\mathrm{e}\uparrow}(0) \\ \psi^{(1)}_{\mathrm{e}\downarrow}(0) \\ \psi^{(-1)}_{\mathrm{e}\downarrow}(0) \end{pmatrix} = 0, \tag{32}$$

which is analogous to (25). The wave function has the form

$$\Psi(x,y) = \frac{1}{\sqrt{W}}\Bigg(e^{-\sqrt{2\varepsilon(M-\Delta)}|x|}(-i\sqrt{2\varepsilon(M-\Delta)}\,\mathrm{sgn}(x), 2(M-\Delta), -2(M-\Delta), i\sqrt{2\varepsilon(M-\Delta)}\,\mathrm{sgn}(x)) +$$
$$+ \frac{(M-\Delta)me^{-(2\pi/W)|x|}}{\pi/W}\Bigg(\left(-\frac{2\pi i}{W}\theta(-x), -(M-\Delta), M-\Delta, \frac{2\pi i}{W}\theta(-x)\right) \times$$
$$\times e^{2\pi i y/W} + \left(\frac{2\pi i}{W}\theta(x), -(M-\Delta), M-\Delta, -\frac{2\pi i}{W}\theta(x)\right)e^{-2\pi i y/W}\Bigg)\Bigg)^{\mathrm{T}}. \tag{33}$$

We note that the states with energies near the upper and lower gap edges have opposite spins (cf. [19], [20]).

If the signs of a_0 and a_1 in (20) change at the same time, which corresponds to the second sheet of the Riemann surface of the Green's function, then this equality remain unchanged, but the bound state becomes resonant with a finite lifetime (see the foregoing). But under the above assumptions, the continuous transformation of the bound state into a resonant one is impossible (cf. [21]) because the quantities ε and $M-\Delta$ must have the same order of smallness as the sign of a_1 changes, which contradicts the above conditions.

We now consider the case of the topologically nontrivial phase $\Delta - M > 0$. We set $E = \Delta - M - \varepsilon$, where $0 < \varepsilon \ll \Delta - M$. Then from (14), we obtain (30) and then (31), but with $a_0 = i\sqrt{2\varepsilon(\Delta - M)}$. The existence condition for an ABS has the form (cf. (24))

$$\sqrt{\varepsilon} = \frac{m^2(\Delta - M)^{3/2}}{\sqrt{2}\pi/W}. \tag{34}$$

The changes in the wave function are analogous to those when passing from the upper gap edge to the lower one in the trivial phase.

4. ABSs with the energy near zero

We consider states with the energies $E = \varepsilon \approx 0$. From (14), (15), we have

$$\begin{aligned}
\begin{pmatrix} \psi^{(0)}_{\mathrm{e}\uparrow}(0) \\ \psi^{(0)}_{\mathrm{h}\downarrow}(0) \end{pmatrix} &= \frac{m(M-\Delta)}{4ia_0}\left(\psi^{(1)}_{\mathrm{e}\uparrow}(0) + \psi^{(-1)}_{\mathrm{e}\uparrow}(0) - \psi^{(1)}_{\mathrm{h}\downarrow}(0) - \psi^{(-1)}_{\mathrm{h}\downarrow}(0)\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\
\begin{pmatrix} \psi^{(0)}_{\mathrm{e}\downarrow}(0) \\ \psi^{(0)}_{\mathrm{h}\uparrow}(0) \end{pmatrix} &= \frac{m(M-\Delta)}{4ia_0}\left(\psi^{(1)}_{\mathrm{e}\downarrow}(0) + \psi^{(-1)}_{\mathrm{e}\downarrow}(0) - \psi^{(1)}_{\mathrm{h}\uparrow}(0) - \psi^{(-1)}_{\mathrm{h}\uparrow}(0)\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
\begin{pmatrix} \psi^{(\pm1)}_{\mathrm{e}\uparrow}(0) \\ \psi^{(\pm1)}_{\mathrm{e}\downarrow}(0) \\ \psi^{(\pm1)}_{\mathrm{h}\uparrow}(0) \\ \psi^{(\pm1)}_{\mathrm{h}\downarrow}(0) \end{pmatrix} &= -\frac{(M-\Delta)m^2}{16a_0a_1}\begin{pmatrix} M-\Delta & \pm 2\pi i/W & \mp 2\pi i/W & -(M-\Delta) \\ \mp 2\pi i/W & -(M-\Delta) & M-\Delta & \pm 2\pi i/W \\ \pm 2\pi i/W & M-\Delta & -(M-\Delta) & \mp 2\pi i/W \\ -(M-\Delta) & \mp 2\pi i/W & \pm 2\pi i/W & M-\Delta \end{pmatrix} \times \\
&\quad \times \begin{pmatrix} \psi^{(1)}_{\mathrm{e}\uparrow}(0) + \psi^{(-1)}_{\mathrm{e}\uparrow}(0) - \psi^{(1)}_{\mathrm{h}\downarrow}(0) - \psi^{(-1)}_{\mathrm{h}\downarrow}(0) \\ -(\psi^{(1)}_{\mathrm{e}\downarrow}(0) + \psi^{(-1)}_{\mathrm{e}\downarrow}(0) - \psi^{(1)}_{\mathrm{h}\uparrow}(0) - \psi^{(-1)}_{\mathrm{h}\uparrow}(0)) \\ \psi^{(1)}_{\mathrm{e}\uparrow}(0) + \psi^{(-1)}_{\mathrm{e}\uparrow}(0) - \psi^{(1)}_{\mathrm{h}\downarrow}(0) - \psi^{(-1)}_{\mathrm{h}\downarrow}(0) \\ -(\psi^{(1)}_{\mathrm{e}\uparrow}(0) + \psi^{(-1)}_{\mathrm{e}\uparrow}(0) - \psi^{(1)}_{\mathrm{h}\downarrow}(0) - \psi^{(-1)}_{\mathrm{h}\downarrow}(0)) \end{pmatrix}.
\end{aligned} \tag{36}$$

From (36), we have

$$\psi^{(\pm1)}_{\mathrm{e}\uparrow}(0) = -\psi^{(\pm1)}_{\mathrm{h}\downarrow}(0), \qquad \psi^{(\pm1)}_{\mathrm{e}\downarrow}(0) = -\psi^{(\pm1)}_{\mathrm{h}\uparrow}(0). \tag{37}$$

Using (20), we obtain the system

$$\begin{aligned}
\psi^{(\pm1)}_{\mathrm{e}\uparrow}(0) &= \frac{\alpha}{2}\left((M-\Delta)(\psi^{(1)}_{\mathrm{e}\uparrow}(0) + \psi^{(-1)}_{\mathrm{e}\uparrow}(0)) \mp \frac{2\pi i}{W}(\psi^{(1)}_{\mathrm{e}\downarrow}(0) + \psi^{(-1)}_{\mathrm{e}\downarrow}(0))\right), \\
\psi^{(\pm1)}_{\mathrm{e}\downarrow}(0) &= \frac{\alpha}{2}\left(\mp\frac{2\pi i}{W}(\psi^{(1)}_{\mathrm{e}\uparrow}(0) + \psi^{(-1)}_{\mathrm{e}\uparrow}(0)) + (M-\Delta)(\psi^{(1)}_{\mathrm{e}\downarrow}(0) + \psi^{(-1)}_{\mathrm{e}\downarrow}(0))\right).
\end{aligned} \tag{38}$$

The matrix determinant of system (38) is $(1 - \alpha(M-\Delta))^2$; therefore, the existence condition for a bound state has the form

$$\frac{m^2|M-\Delta|}{2\sqrt{(2\pi/W)^2 + (M-\Delta)^2}} = 1 \tag{39}$$

and is independent of the phase. If (39) is satisfied, then it follows from (38) that

$$\psi^{(1)}_{\mathrm{e}\uparrow}(0) - \psi^{(-1)}_{\mathrm{e}\uparrow}(0) = -\frac{2\pi i}{W(M-\Delta)}(\psi^{(1)}_{\mathrm{e}\downarrow}(0) + \psi^{(-1)}_{\mathrm{e}\downarrow}(0)),$$
$$\psi^{(1)}_{\mathrm{e}\uparrow}(0) + \psi^{(-1)}_{\mathrm{e}\uparrow}(0) = -\frac{W(M-\Delta)}{2\pi i}(\psi^{(1)}_{\mathrm{e}\downarrow}(0) - \psi^{(-1)}_{\mathrm{e}\downarrow}(0)),$$

where $\psi^{(1)}_{\mathrm{e}\downarrow}(0)+\psi^{(-1)}_{\mathrm{e}\downarrow}(0) = C_1$ and $\psi^{(1)}_{\mathrm{e}\downarrow}(0)-\psi^{(-1)}_{\mathrm{e}\downarrow}(0) = C_2$ are arbitrary constants. Let $C_1 = 1$ and $C_2 = 0$, then

$$\psi^{(1)}_{\mathrm{e}\downarrow}(0) = \psi^{(-1)}_{\mathrm{e}\downarrow}(0) = \frac{1}{2}, \qquad \psi^{(1)}_{\mathrm{e}\uparrow}(0) = -\psi^{(-1)}_{\mathrm{e}\uparrow}(0) = -\frac{\pi i}{W(M-\Delta)}. \tag{40}$$

If $C_1 = 0$ and $C_2 = 1$, then

$$\psi^{(1)}_{\mathrm{e}\downarrow}(0) = -\psi^{(-1)}_{\mathrm{e}\downarrow}(0) = \frac{1}{2}, \qquad \psi^{(1)}_{\mathrm{e}\uparrow}(0) = \psi^{(-1)}_{\mathrm{e}\uparrow}(0) = -\frac{W(M-\Delta)}{4\pi i}. \tag{41}$$

Using two analogous wave functions, we write the function corresponding to (40). From (35), we have

$$\psi^{(0)}_{\mathrm{e}\uparrow}(0) = \psi^{(0)}_{\mathrm{h}\downarrow}(0) = 0, \qquad \psi^{(0)}_{\mathrm{e}\downarrow}(0) = -\psi^{(0)}_{\mathrm{h}\uparrow}(0) = -\frac{m}{2}\,\mathrm{sgn}(M-\Delta). \tag{42}$$

Using (10), (13), and (A.7), we write the wave function as

$$\begin{aligned}\Psi(x,y) = \frac{1}{\sqrt{W}}\Bigg(&\frac{m}{2}e^{-|M-\Delta||x|}(i\,\mathrm{sgn}(x), -\,\mathrm{sgn}(M-\Delta), \mathrm{sgn}(M-\Delta), -i\,\mathrm{sgn}(x)) - \\ &-\frac{m^2\,\mathrm{sgn}(M-\Delta)}{4\sqrt{(2\pi/W)^2+(M-\Delta)^2}}e^{-\sqrt{(2\pi/W)^2+(M-\Delta)^2}|x|}\times \\ &\times\sum_{j=1,2}(c_1, -(M-\Delta), M-\Delta, -c_1)e^{(-1)^{j+1}2\pi i y/W}\Bigg)^{\mathrm{T}},\end{aligned} \tag{43}$$

where $c_1 = i\sqrt{(2\pi/W)^2+(M-\Delta)^2}\,\mathrm{sgn}(x) + (-1)^j 2\pi i/W$ (see Fig. 2). Obviously, it does not satisfy conditions (29). This function is strongly localized with respect to x, unlike the function in Fig. 1. The second term in the right-hand side of (43) smoothes oscillations of $|\Psi|^2$ along the y axis.

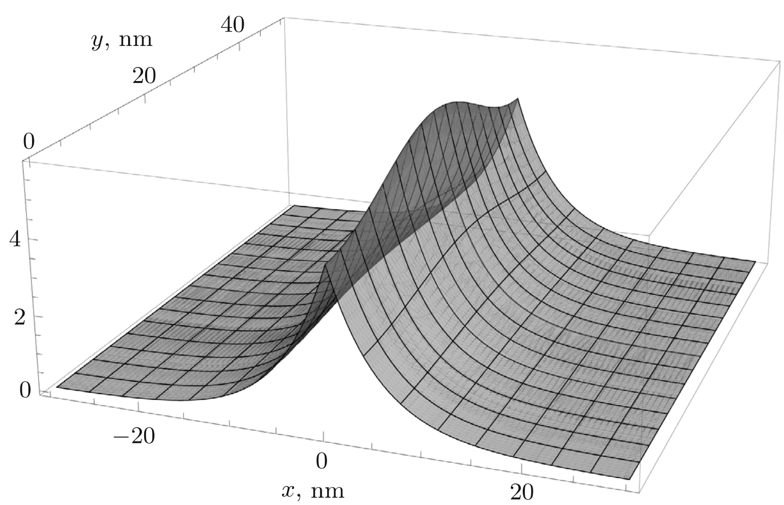

Fig. 2. Amplitude $|\Psi|^2$ of the ABS wave function with an energy that is close to zero. Here, $M-\Delta = 0,1$ meV and $m = 0,1$ meV.

Thus, the perturbation (local with respect to x) of the constant Zeeman field can generate ABSs with multiplicity 2 and near-zero energy; in contrast to the one-dimensional case [9], these are not Majorana-like states (cf. [10]).

5. Nonmagnetic impurity potential

We now consider the impurity potential $\mathcal{W}(x, y)$ of the form

$$\mathcal{W}(x,y) = w \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} (e^{-2\pi iy/W} + e^{2\pi iy/W})\delta(x). \tag{44}$$

First, let $E = M - \Delta + \varepsilon$, $M - \Delta > 0$, where $0 < \varepsilon \ll M - \Delta$. Using the line of reasoning as before, we obtain Eq. (13) with the spinor in the right-hand side replaced with

$$\begin{pmatrix} \psi_{e\uparrow}^{(1)}(0) + \psi_{e\uparrow}^{(-1)}(0) \\ \psi_{e\downarrow}^{(1)}(0) + \psi_{e\downarrow}^{(-1)}(0) \\ -(\psi_{h\uparrow}^{(1)}(0) + \psi_{h\uparrow}^{(-1)}(0)) \\ -(\psi_{h\downarrow}^{(1)}(0) + \psi_{h\downarrow}^{(-1)}(0)) \end{pmatrix}.$$

Instead of (16), (17), we have

$$\begin{gathered} \begin{pmatrix} \psi_{e\uparrow}^{(0)}(0) \\ \psi_{h\downarrow}^{(0)}(0) \end{pmatrix} = \frac{(M-\Delta)w}{2ia_0}(\psi_{e\uparrow}^{(1)}(0) + \psi_{e\uparrow}^{(-1)}(0) + \psi_{h\downarrow}^{(1)}(0) + \psi_{h\downarrow}^{(-1)}(0)) \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\ \psi_{e\downarrow}^{(0)}(0) = \psi_{h\uparrow}^{(0)}(0) = 0, \\ (\psi_{e\uparrow}^{(\pm 1)}(0), \psi_{e\downarrow}^{(\pm 1)}(0), \psi_{h\uparrow}^{(\pm 1)}(0), \psi_{h\downarrow}^{(\pm 1)}(0))^{\mathrm{T}} = \\ = -\frac{(M-\Delta)w^2}{2a_0a_1}\left(M-\Delta, \mp\frac{\pi i}{W}, \pm\frac{\pi i}{W}, -(M-\Delta)\right)^{\mathrm{T}} (\psi_{e\uparrow}^{(1)}(0) + \psi_{e\uparrow}^{(-1)}(0) + \psi_{h\downarrow}^{(1)}(0) + \psi_{h\downarrow}^{(-1)}(0)). \end{gathered} \tag{45}$$

Consequently, $\psi_{e\uparrow}^{(\pm 1)}(0) = -\psi_{h\downarrow}^{(\pm 1)}(0)$ and $\psi_{e\downarrow}^{(\pm 1)}(0) = -\psi_{h\uparrow}^{(\pm 1)}(0)$, and system (45) has no nonzero solutions for energies near the gap edge, and hence there are no states with such energies. It can be proved similarly that the nonmagnetic impurity generate no bound states with the energy that is close to zero. (In the one-dimensional case [9], the existence of the ABSs is phase-dependent.)

6. Conclusions

In this paper, we have considered the quasi-one-dimensional superconducting structure on the surface of a topological insulator in the presence of a perpendicular constant Zeeman field. We proved analytically that the local perturbation of the Zeeman field generates stable bound states with energies near the energy gap edges and also ABSs with near-zero energy. In contrast to the one-dimensional case, these states have no "particle–hole" symmetry. The nonmagnetic impurity generates no ABSs.

Appendix

To find the Green's function of the Hamiltonian $H^{(n)}$, we find the function Ψ from the equation $(H^{(n)}-E)\Psi=\Phi$. We rewrite this equation in the form

$$\begin{pmatrix} M-E & -i\partial_x+2\pi in/W & 0 & \Delta \\ -i\partial_x-2\pi in/W & -M-E & -\Delta & 0 \\ 0 & -\Delta & -M-E & -i\partial_x-2\pi in/W \\ \Delta & 0 & -i\partial_x+2\pi in/W & M-E \end{pmatrix}\times$$

$$\times\begin{pmatrix} \psi^{(n)}_{\mathrm{e}\uparrow}(x) \\ \psi^{(n)}_{\mathrm{e}\downarrow}(x) \\ \psi^{(n)}_{\mathrm{h}\uparrow}(x) \\ \psi^{(n)}_{\mathrm{h}\downarrow}(x) \end{pmatrix}=\begin{pmatrix} \varphi^{(n)}_{\mathrm{e}\uparrow}(x) \\ \varphi^{(n)}_{\mathrm{e}\downarrow}(x) \\ \varphi^{(n)}_{\mathrm{h}\uparrow}(x) \\ \varphi^{(n)}_{\mathrm{h}\downarrow}(x) \end{pmatrix}$$

or, after the Fourier transformation,

$$\widetilde{\varphi}(p)=F\varphi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ipx}\varphi(x)\,dx,$$

$$\begin{pmatrix} M-E & p+2\pi in/W & 0 & \Delta \\ p-2\pi in/W & -M-E & -\Delta & 0 \\ 0 & -\Delta & -M-E & p-2\pi in/W \\ \Delta & 0 & p+2\pi in/W & M-E \end{pmatrix}\begin{pmatrix} \widetilde{\psi}^{(n)}_{\mathrm{e}\uparrow}(p) \\ \widetilde{\psi}^{(n)}_{\mathrm{e}\downarrow}(p) \\ \widetilde{\psi}^{(n)}_{\mathrm{h}\uparrow}(p) \\ \widetilde{\psi}^{(n)}_{\mathrm{h}\downarrow}(p) \end{pmatrix}=$$

$$=(\widetilde{\varphi}^{(n)}_{\mathrm{e}\uparrow}(p),\widetilde{\varphi}^{(n)}_{\mathrm{e}\downarrow}(p),\widetilde{\varphi}^{(n)}_{\mathrm{h}\uparrow}(p),\widetilde{\varphi}^{(n)}_{\mathrm{h}\downarrow}(p))^{\mathrm{T}}. \tag{A.1}$$

The determinant of the matrix in (A.1) is

$$d=\left(p^2+\left(\frac{2\pi n}{W}\right)^2\right)^2+2\left(p^2+\left(\frac{2\pi n}{W}\right)^2\right)(M^2+\Delta^2-E^2)+$$
$$+(M^2-E^2)^2-2\Delta^2(M^2+E^2)+\Delta^4. \tag{A.2}$$

From (A.2), we find the dispersion law for the nth subband

$$E^2=(M\pm\Delta)^2+p^2+\left(\frac{2\pi n}{W}\right)^2 \tag{A.3}$$

and the equality

$$\frac{1}{d}=-\frac{1}{4M\Delta}\left(\frac{1}{p^2+(2\pi n/W)^2+(M+\Delta)^2-E^2}-\right.$$
$$\left.-\frac{1}{p^2+(2\pi n/W)^2+(M-\Delta)^2-E^2}\right). \tag{A.4}$$

Using Cramer's rule and neglecting quantities of the order of smallness of $(M-\Delta)^2$, we obtain the Green's function of the Hamiltonian $H^{(n)}$ in momentum representation,

$$\widetilde{G}^{(n)}(p,p',E)=\frac{\delta(p-p')}{2}\left(\frac{1}{p^2-a_n^2}-\frac{1}{p^2-b_n^2}\right)\times$$
$$\times\begin{pmatrix} M-\Delta+E & p+2\pi in/W & -(p+2\pi in/W) & -(M-\Delta+E) \\ p-2\pi in/W & -(M-\Delta-E) & M-\Delta-E & -(p-2\pi in/W) \\ -(p-2\pi in/W) & M-\Delta-E & -(M-\Delta-E) & p-2\pi in/W \\ -(M-\Delta+E) & -(p+2\pi in/W) & p+2\pi in/W & M-\Delta+E \end{pmatrix}, \tag{A.5}$$

where $a_n^2 = E^2 - (2\pi n/W)^2 - (M-\Delta)^2$ and $b_n^2 = E^2 - (2\pi n/W)^2 - (M+\Delta)^2$. To pass to the coordinate representation, we use the known formulas

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ipx}\widetilde{\varphi}(p)\,dp}{p^2-a^2} &= -\frac{1}{2ia}\int_{-\infty}^{\infty} e^{ia|x-x'|}\varphi(x')\,dx',\\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{pe^{ipx}\widetilde{\varphi}(p)\,dp}{p^2-a^2} &= -\frac{1}{2i}\int_{-\infty}^{\infty} e^{ia|x-x'|}\,\mathrm{sgn}(x-x')\varphi(x')\,dx',\end{aligned} \tag{A.6}$$

assuming that $a = a_n$ or b_n.

To find the Green's function in the coordinate representation using formulas (A.6), we integrate equalities (A.5) over p. Because of the conditions for the quantities $1/W$ and $|M-\Delta|$ (see Sec. 2), when integrating over the region where the p are small, the denominator of the second term in the factor before the matrix in (A.5) is much larger than that of the first term, and the integrals over the region where p are rather large are small. Thus, the second term in (A.5) can be neglected. Although the remaining integrals are taken over the entire number line, the smallness of their denominators for small p under the adopted conditions for the parameters makes these integrals mainly saturated by small p, and the condition for the smallness of the momenta p (see the assumptions after (11)) is satisfied. This reasoning is easily confirmed by numerical estimations.

Based on the foregoing, we use (A.5) and (A.6) to obtain the Green's function of the Hamiltonian $H^{(n)}$ in the coordinate representation:

$$G^{(n)}(x-x',E) = -\frac{e^{ia_n|x-x'|}}{4ia_n}\begin{pmatrix} M-\Delta+E & b_n^+ & -b_n^+ & -(M-\Delta+E)\\ b_n^- & -(M-\Delta-E) & M-\Delta-E & -b_n^-\\ -b_n^- & M-\Delta-E & -(M-\Delta-E) & b_n^-\\ -(M-\Delta+E) & -b_n^+ & b_n^+ & M-\Delta+E \end{pmatrix}, \tag{A.7}$$

where $b_n^{\pm} = a_n\,\mathrm{sgn}(x-x') \pm 2\pi i n/W$.

Conflicts of interest. The authors declare no conflicts of interest.

REFERENCES

1. J. Alicea, "New directions in the pursuit of Majorana fermions in solid state systems," *Rep. Prog. Phys.*, **75**, 076501, 36 pp. (2012).
2. F. von Oppen, Y. Peng, and F. Pientka, "Topological superconducting phases in one dimension," in: *Topological Aspects of Condensed Matter Physics: Lecture Notes of the Les Houches Summer School* (École de Physique des Houches, Session CIII, 4–29 August, 2014) (C. Chamon, M. O. Goerbig, R. Moessner, L. F. Cugliandolo, eds.), Oxford Univ. Press, Oxford (2017), pp. 389–449.
3. M. Sato and S. Fujimoto, "Majorana fermions and topology in superconductors," *J. Phys. Soc. Japan*, **85**, 072001, 32 p. (2016).
4. C.-X. Liu, J. D. Sau, T. D. Stanescu, and S. Das Sarma, "Andreev bound states versus Majorana bound states in quantum dot-nanowire-superconductor hybrid structures: Trivial versus topological zero-bias conductance peaks," *Phys. Rev. B*, **96**, 075161, 29 pp. (2017).
5. T. D. Stanescu and S. Tewari, "Robust low-energy Andreev bound states in semiconductor-superconductor structures: Importance of partial separation of component Majorana bound states," *Phys. Rev. B*, **100**, 155429, 21 pp. (2019), arXiv: 1811.02557.
6. C. Moore, C. Zeng, T. D. Stanescu, and S. Tewari, "Quantized zero bias conductance plateau in semiconductor-superconductor heterostructures without non-Abelian Majorana zero modes," *Phys. Rev. B*, **98**, 155314, 6 pp. (2018), arXiv: 1804.03164.

7. J. Cayao and A. M. Black-Schaffer, “Distinguishing trivial and topological zero-energy states in long nanowire junctions,” *Phys. Rev. B*, **104**, L020501, 6 pp. (2021).
8. B. D. Woods, S. Das Sarma, and T. D. Stanescu, “Subband occupation in semiconductor-superconductor nanowires,” *Phys. Rev. B*, **101**, 045405, 13 pp. (2020), arXiv: 1910.04362.
9. Yu. P. Chuburin and T. S. Tinyukova, “The emergence of bound states in a superconducting gap at the topological insulator edge,” *Phys. Lett. A*, **384**, 126694, 7 pp. (2020).
10. B. D. Woods, J. Chen, S. M. Frolov, and T. D. Stanescu, “Zero-energy pinning of topologically trivial bound states in multiband semiconductor-superconductor nanowires,” *Phys. Rev. B*, **100**, 125407, 17 pp. (2019), arXiv: 1902.02772.
11. Z. Hou and J. Klinovaja, “Zero-energy Andreev bound states in iron-based superconductor Fe(Te,Se),” arXiv: 2109.08200.
12. Yu. P. Chuburin and T. S. Tinyukova, “Interaction between subbands in a quasi-one-dimensional superconductor,” *Theoret. and Math. Phys.*, **210**, 398–410 (2022).
13. T. D. Stanescu, J. D. Sau, R. M. Lutchyn, and S. Das Sarma, “Proximity effect at the superconductor-topological insulator interface,” *Phys. Rev. B*, **81**, 241310, 4 pp. (2010), arXiv: 1002.0842.
14. J. Linder, Yu. Tanaka, T. Yokoyama, A. Sudbo, and N. Nagaosa, “Interplay between superconductivity and ferromagnetism on a topological insulator,” *Phys. Rev. B*, **81**, 184525, 11 pp. (2010).
15. C. T. Olund and E. Zhao, “Current-phase relation for Josephson effect through helical metal,” *Phys. Rev. B*, **86**, 214515, 7 pp. (2012).
16. F. Crepin, B. Trauzettel, and F. Dolcini, “Signatures of Majorana bound states in transport properties of hybrid structures based on helical liquids,” *Phys. Rev. B*, **89**, 205115, 12 pp. (2014).
17. A. C. Potter and P. A. Lee, “Multichannel generalization of Kitaev’s Majorana end states and a practical route to realize them in thin films,” *Phys. Rev. Lett.*, **105**, 227003, 4 pp. (2010), arXiv: 1007.4569.
18. J. R. Taylor, *Scattering Theory: The Quantum Theory of Nonrelativistic Collisions*, Wiley, New York (1972).
19. P. Szumniak, D. Chevallier, D. Loss, and J. Klinovaja, “Spin and charge signatures of topological superconductivity in Rashba nanowires,” *Phys. Rev. B*, **96**, 041401, 5 pp. (2017).
20. M. Serina, D. Loss, and J. Klinovaja, “Boundary spin polarization as a robust signature of a topological phase transition in Majorana nanowires,” *Phys. Rev. B*, **98**, 035419, 10 pp. (2018).
21. Yu. P. Chuburin and T. S. Tinyukova, “Behaviour of Andreev states for topological phase transition,” *Theoret. and Math. Phys.*, **208**, 977–992 (2021).