# FUNCTIONAL DIFFERENTIAL EQUATIONS

Dedicated to the memory of
N.V. AZBELEV

Guest Editors
V.P. MAKSIMOV and L.F. RAKHMATULLINA

VOLUME 15, 2008
No. 1-2

ARIEL UNIVERSITY CENTER
OF SAMARIA, ARIEL, ISRAEL

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VOLUME 15
2008, NO 1–2
PP. 159–177

# ONE EXTENSION OF THE SPACE OF DISTRIBUTIONS

V. DERR *

**Abstract.** In what follows, we introduce an extension of the classical space of distributions which allows us to define the correct operations of multiplication of distributions by discontinuous functions and differentiation of distributions, as well as to pose correctly the Cauchy problem for the ordinary linear differential equation with distributions in coefficients, which was the subject of research of many authors. We define this extension using a modification of the Perron-Stieltjes integral, whose properties are also studied in the present paper.

**Key Words.** Regulated functions, distributions, multiplication of distributions, alpha-integral, principal solution.

**AMS(MOS) subject classification.** 46F10

## 1. Introduction.

The subject of the ordinary differential equations with distributions continues to attract the interest of many authors, e.g., see [1-5]. Many authors (see [6-11]) consider the Cauchy problem for the linear systems with distributions of the form

$$(1) \qquad x' = M'(t)x + f'(t) \quad (t \in I), \quad x(t_0) = x_0 \quad (t_0 \in I, x_0 \in \mathbb{R}^n),$$

where $I$ is an open interval in $\mathbb{R}$ (in particular, $I = \mathbb{R}$), $M(\cdot)$ $\big(x(\cdot), f(\cdot)\big)$ are matrix-valued and vector-valued functions on $I$, respectively, where $'$ denotes the differentiation in the sense of distribution theory.

If $M(\cdot)$ is locally absolutely-continuous on $I$ (that is, $M'(\cdot)$ is locally summable on $I$), then the right-hand side of system in (1) is an ordinary function which satisfies Caratheodory conditions. In other cases $M'$ is a distribution. Then it is natural to assume that the solution of the problem

* Faculty of Mathematics, Udmurtia State University, Universiteskaya Str., 1 (Bld. 4), Izhevsk, 426034, Russia.

(1) is a discontinuous function, so the product $M'x$ is undefined, since it is impossible to define correctly the operation of multiplication by discontinuous functions in the classical theory of distributions. For this reason we need not only to define what is called the solution of the Cauchy problem (1), but also to specify in what sense understand the notation (1). Clearly, every definition of solution produces implicitly the definition of the product of a distribution $M'$ and a discontinuous function (in the general case this definition is specific for every $M'$). The ambiguity in definition of the product of distribution by discontinuous function leads to various definitions of solution of the problem (1), so the same Cauchy problem may have different solutions.

In [12] we have introduced the space of distributions with discontinuous test functions, whose elements admit continuous multiplication by regulated functions (that is, the functions which may have discontinuities only of the first kind). In [12] this space was used to provide existence of the Nash equilibrium in some zero-sum games with discontinuous payoff functions, which do not have equilibrium in any classical sense.

The extension of the classical space of distributions, which we introduce in the present paper, is analogous to the one defined in[12] but it is oriented for applications to study of the ordinary linear differential equations of the form (1) such that the elements of $M$ and $f$ are the regulated functions. The proposed extension is based on one modification of Perron-Stieltjes integral introduced in [12] (we call it alpha-integral), so certain part of the present paper is devoted to the study of this construction.

**2. Functions Spaces.** Let $I = (a, b)$ be a fixed open interval (in particular, $a = -\infty$ and (or) $b = \infty$). In what follows, a bounded function $f : I \mapsto \mathbf{R}$ possessing one-sided limits $f(t-)$, $f(t+)$ $(t \in I)$ is called *regulated* [13]; the set of regulated functions is denoted by $\tilde{\mathbf{R}}$. As is well-known (e.g., see [14, p.17], [15]), any regulated function may possess at most countable set of points of discontinuity. We denote the set of points of discontinuity of $f \in \tilde{\mathbf{R}}$ by $T(f)$.

We call two functions in $\tilde{\mathbf{R}}$ equivalent if they differ only by their values at the points of discontinuity; further, if $f(t+) = f(t-)$, then we put $f$ to be continuous at $t$. We denote the set of equivalence classes in $\tilde{\mathbf{R}}$ by $\mathbf{R}$. Clearly, $\mathbf{R}$ is an algebra with respect to the ordinary pointwise operations of addition, multiplication on the elements of $\mathbb{R}$ and multiplication. We endow algebra $\mathbf{R}$ with the norm

$$(2) \qquad \|f\| = \sup_{t \in I} \max\{|f(t-)|, |f(t+)|\},$$

so that $\mathbf{R}$ becomes a Banach algebra.

So, we *ignore the value of* $f(\cdot) \in \mathbf{R}$ *at a given point;* $f$ *may possess a value at the point if and only if* $f$ *is continuous at this point.* We use notation $f(t)$ in the following two cases: a) to underline the independent variable (e.g., under the sign of integral) b) $t$ is a point of continuity of $f$. If necessary, we use the operator

$$(3) \qquad (^\alpha) : \mathbf{R} \mapsto \tilde{\mathbf{R}}, \quad (^\alpha f)(t) = \alpha(f)f(t+) + (1 - \alpha(t))f(t-),$$

where $\alpha \in \tilde{\mathbf{R}}$ is a continuous function. In what follows, we call the elements of $\mathbf{R}$ (equivalence classes!) analogously as functions (let us note that in [15] algebra $\mathbf{R}$ is denoted by $DC$, while in [12] and some other papers it is denoted by $\mathbb{G}$).

Given $f \in \mathbf{R}$ and a partition of $I$ generated by $\tau = \{t_k\}_{k=1}^n$, $a < t_0 < t_1 < ... < t_n < b$, we define

$$v_\tau(f) \doteq \sum_{k=1}^{n} |f(t_k-) - f(t_{k-1}+)|.$$

We call the *total variation of* $f$ the value

$$(4) \qquad \bigvee_a^b(f) \doteq \sup_\tau v_\tau(f).$$

The set of functions $f$ of finite total variation is denoted by $\mathbf{BV}$. By definition, $\mathbf{BV} \subset \mathbf{R}$, however it is more convenient to use in $\mathbf{BV}$ the classical norm: if $f \in \mathbf{BV}$, then we define

$$(5) \qquad \|f\| = |f(a+)| + \bigvee_a^b(f).$$

Then $\mathbf{BV}$ is a Banach algebra with respect to this norm. The set of functions having finite total variation on each closed subinterval $[a', b'] \subset I$ is denoted by $\mathbf{BV}\ell$. We say that $x_n \to x$ in $\mathbf{BV}\ell$ ($x_n$, $x \in \mathbf{BV}\ell$, $n \in \mathbb{N}$) if for every closed subinterval $[a', b'] \subset I$ we have $\|x_n - x\|_{\mathbf{BV}(a',b')} \to 0$ as $n \to \infty$.

Let us denote by $\mathbf{C} \subset \mathbf{R}$ the algebra of continuous functions with the ordinary sup-norm, which in this case coincides with (2). Also, we put $\mathbf{CBV} := \mathbf{C} \cap \mathbf{BV}$. If $f \in \mathbf{CBV}$, then we define

$$(6) \qquad \|f\| = \sup_{t \in I} |f(t)| + \bigvee_a^b(f).$$

The restriction of the norm (5) to **CBV** is equivalent to the norm (6). Further, we denote by **H** the set of elements $f \in \mathbf{BV}$ taking at most countable set of values and such that $f(a+) = 0$, with the norm

$$\|f\| = \sum_{t \in T(f)} \sigma_t(f) \quad \Big(= \bigvee_a^b (f)\Big), \tag{7}$$

where $\sigma_t(f) \doteq f(t+) - f(t-)$.

The algebras **CBV** and **H** are Banach with respect to their norms.

As is well-known (e.g., see [17]), every function $f \in \mathbf{BV}$ admits representation

$$f = f_c + f_d, \tag{8}$$

where $f_c \in \mathbf{CBV}$ is the continuous part of $f$, and $f_d \in \mathbf{H}$ is the discrete part of $f$ (due to condition $f(a+) = 0$ this representation is unique). Thus, **BV** can be represented as the direct sum: $\mathbf{BV} = \mathbf{CBV} \dotplus \mathbf{H}$, so that (see [15])

$$\bigvee_a^b (f) = \bigvee_a^b (f_c) + \bigvee_a^b (f_d).$$

Also, let us define $\mathbf{HC} := \mathbf{H} \dotplus \mathbf{C}$.

## 3. Riemann-Stieltjes integral for the elements of the algebra R.

*1.* In this section we introduce a modification of the definition of the Riemann-Stiltjes intergral (RS-integral) which allows us to integrate the elements of the algebra **R** over an open subinterval of $I$.

Let $-\infty < a < b < +\infty$, $g, f : I \to \mathbb{R}$ and $\tau = \{t_k\}_{k=1}^n$ be such that

$$a < t_0 < t_1 < \cdots < t_n < b, \quad \tau \subset I \setminus \big(T(f) \cup T(g)\big). \tag{9}$$

Denote by $d(\tau)$ the so-called diameter of $\tau$,

$$d(\tau) = \max_{0 \leqslant k \leqslant n+1} (t_k - t_{k-1}) \quad (t_{-1} = a, \quad t_{n+1} = b).$$

We define the Stieltjes sums by the formula

$$\mathfrak{S}_\tau(g, f) = \sum_{k=1}^n g(\xi_k)\big(f(t_k) - f(t_{k-1})\big),$$

where $\xi_k \in [t_{k-1}, t_k] \setminus T(g)$ $(k = 1, 2, \dots, n)$. By definition, $J$ is the value of RS-integral of $g$ by $f$ over the open interval $(a, b)$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $\tau$ of the form (9) with $d(\tau) < \delta$, the inequality $|\mathfrak{S}_\tau(g, f) - J| < \varepsilon$ holds. We denote

$$J = \int\limits_{(a,b)} g(t)\, df(t) = \int\limits_{(a,b)} g\, df$$

and use the notation

$$J = \lim_{d(\tau)\to 0} \mathfrak{S}_\tau(g, f).$$

*2.* Below we list some properties of this integral:

1) The integrals $\int_{(a,b)} g(t)\, df(t)$ and $\int_{(a,b)} f(t)\, dg(t)$ either exist both, or do not exist together; the proof of this statement is analogous to the proof in [17, p.250]; furthermore (see [18, p.172]);

$$\int_{(a,b)} g(t)\, df(t) + \int_{(a,b)} f(t)\, dg(t) = f(t)g(t)|_{a+}^{b-}. \tag{10}$$

2) Let $f$ be an increasing function. Then the integral $\int_{(a,b)} g\, df$ exists if and only if $\mu_f(T(g)) = 0$, where $\mu_f(B)$ is the Lebesgue-Stieltjes measure of the set $B$ generated by function $g$ (see [20]).

3) If $g \in \mathbf{R}$, $f \in \mathbf{CBV}$ (or $g \in \mathbf{CBV}$, $f \in \mathbf{R}$), then the integral $\int_{(a,b)} g\, df$ exists; the proof of this result is based on statement 2), see [15].

4) If $g \in \mathbf{R}$, $f \in \mathbf{CBV}$, then

$$\Big|\int_{(a,b)} g(t)\, df(t)\Big| \leqslant \int_{(a,b)} |g(t)|\, d\bigvee_a^t(f) \leqslant \|g\|_{\mathbf{R}} \bigvee_a^b(f); \tag{11}$$

5) Let $g \in \mathbf{R}$, $f \in \mathbf{CBV}$; let $F(t) = \int_{(a,t)} g(s)\, df(s)$; then $F \in \mathbf{CBV}$;

6) The integral is additive in the following sense. Suppose that $T(g) \cap T(f) = \emptyset$. Then the existence of the integral $\int_{(a,b)} g\, df$ implies that the integrals $\int_{(a,c)} g\, df$ and $\int_{(c,b)} g\, df$ exist, where $(c \in (a, b))$, and

$$\int_{(a,b)} g\, df = \int_{(a,c)} g\, df + \int_{(c,b)} g\, df + g(c)\sigma_c(f).$$

If $a = -\infty$ and (or) $b = +\infty$, then we put

$$\int_a^b g(t)\, df(t) \doteq \lim_{\substack{\alpha\to-\infty \\ \beta\to+\infty}} \int_{(\alpha,\beta)} g(t)\, df(t).$$

In this case the properties 1)–6) remain valid.

THEOREM 1. *Suppose that* $g, g_n \in \mathbf{R}$ $(n \in \mathbb{N})$, $f \in \mathbf{CBV}$. *Then:*

*1) If* $g_n \to g$ *in* $\mathrm{R}$, *that is,* $\|g_n - g\|_{\mathbf{R}} \to 0$, *then*

$$\lim_{n\to\infty} \int_{(a,b)} g_n\, df = \int_{(a,b)} g\, df. \tag{12}$$

*2) If* $g_n(t+) \to g(t+)$ $(n \to \infty, t \in [a,b))$, $g_n(t-) \to g(t-)$ $(n \to \infty, t \in (a,b])$ *and* $\|g_n\|_{\mathbf{R}} \leqslant K$ *for certain* $K$, *then we also have* (*12*).

*Proof.* We may use the estimation (11) in order to get statement 1). The proof of statement 2) is similar to the proof of analogous statement for the ordinary Riemann-Stieltjes integral (e.g., see [19, c.119]). □

THEOREM 2. *If one of* $f$, $g$, $h$ *is in* $\mathbf{R}$, *while two others are in* $\mathbf{CBV}$, *then*

$$\int_I h(t)\, dg(t) f(t) = \int_I h(t) g(t)\, df(t) + \int_I h(t) f(t)\, dg(t). \tag{13}$$

The proofs of Theorems 2–6 are provided in Section 6.

**4. Alpha-integral of Stieltjes type.**

*1.* Let $g \in \mathbf{R}$, $f \in \mathbf{BV}$. We define

$$(\alpha) \int g(t)\, df(t) \doteq \int_I g(t)\, df_c(t) + \sum_{t\in T(f)} (^{\alpha}g)(t)\sigma_t(f), \tag{14}$$

where $\alpha \in \mathbf{C}$ is a fixed function (see (3)). The first summand in (14) is the aforementioned modification of the Riemann-Stieltjes integral. The convergence of the series in the second summand follows from boundedness of $g$ and the finiteness of the total variation of $f$.

It follows immediately from the definition that the integral (14) exists even in the case when $g \in \mathbf{BV}$, $f \in \mathbf{HC}$.

For an open interval $(a', b') \subset I$ the alpha-integral is defined as follows:

$$(\alpha) \int_{(a',b')} g(t)\, df(t) \doteq \int_{(a',b')} g(t)\, df_c(t) + \sum_{t\in T_{(a',b')}(f)} (^{\alpha}g)(t)\sigma_t(f).$$

Clearly, the continuous part of $f$ on $(a', b')$ differs from the continuous part of $f$ on $I$ by a constant. Since this does not affect the value of the integral, we still use the notation $df_c(t)$ under the sign of the integral for any open subinterval of $I$. Let us note also that for $f \in \mathbf{BV}\ell$, $g(t) = \chi_{(a',b')}(t)$, where $\chi_A(\cdot)$ is the characteristic function of $A \subset I$, we have

$$(\alpha) \int_{(a',b')} g(t)\, df(t) = f(b'-) - f(a'+). \tag{15}$$

*2.* As it follows immediately from the definition (14), alpha-integral possesses the following properties.

a) If $g \in \mathbf{R},\ f \in \mathbf{BV}$, then we have

$$\tag{16} \left|(\alpha)\int g(t)\,df(t)\right| \leqslant A\|g\|_{\mathbf{R}} \bigvee_a^b (f),$$

where $A$ depends only on $\alpha$ (see [16]).

b) Alpha-integral is additive in the following sense: let $c' \in I$; then

$$(\alpha)\int g\,df = (\alpha)\int_{(a,c')} g\,df + (\alpha)\int_{(c',b)} g\,df + ({}^{\alpha}g)(c')\sigma_{c'}(f);$$

Indeed, due to property 6) in Section 2 and definition (14) we have

$$(\alpha)\int g(t)\,df(t) = \int_{(a,c')} g(t)\,df_c(t) + \int_{(c',b)} g(t)\,df_c(t) + g(c')\sigma_{c'}(f_c) +$$

$$\sum_{t\in T_{(a,c')}(f)} ({}^{\alpha}g)(t)\sigma_t(f) + \sum_{t\in T_{(c',b)}(f)} ({}^{\alpha}g)(t)\sigma_t(f) + ({}^{\alpha}g)(c')\sigma_{c'}(f);$$

since $\sigma_{c'}(f_c) = 0$; this gives us the required result.

c) Suppose that $a' \geq a$; let us denote

$$\Phi_\alpha(a',t) \doteq (\alpha)\int_{(a',t)} g(s)\,df(s), \quad \Phi_\alpha(t) \doteq \Phi_\alpha(a,t).$$

Then for every $t_0 > a', t_0 \in I$ we have the following equalities

$$\lim_{t\to t_0+} \Phi_\alpha(t_0,t) = 0, \ \lim_{t\to t_0-} \Phi_\alpha(a',t) = \Phi_\alpha(a',t_0),$$

$$\lim_{t\to t_0+} \Phi_\alpha(a',t) = \Phi_\alpha(a',t_0) + ({}^{\alpha}g)(t_0)\sigma_{t_0}(f),$$

that is,

$$\tag{17} \sigma_t(\Phi_\alpha) = ({}^{\alpha}g)(t)\sigma_t(f).$$

d) If $g \in \mathbf{R},\ f \in \mathbf{BV}$, then it is easy to see that $\Phi_\alpha \in \mathbf{BV}$,

$$\bigvee_a^b (\Phi_\alpha) \leqslant (1+A)\|g\|_{\mathbf{R}} \bigvee_a^b (f)$$

and (see the representation (8))

$$\Phi_{\alpha c}(t) = f(a+) + \int_{(a,t)} g(s)\,df(s), \quad \Phi_{\alpha d}(t) = \sum_{s\in T_{(a,t)}(f)} ({}^{\alpha}g)(s)\sigma_t(f),$$

while $\Phi_\alpha$ is continuous at the points of continuity of $f$.

*3.* Below we formulate the main results on the properties of alpha-integral.

THEOREM 3. *(see [21, c.48]). Suppose that either $g$ or $f$ is in* **BV**, *while the other function is in* **HC**. *Then*

$$(\alpha)\int g(t)\,df(t) + (1-\alpha)\int f(t)\,dg(t) = f(t)g(t)|_{a+}^{b-}. \tag{18}$$

Let us note that the formula (18) has certain advantages in comparison with the analogous formula for Perron-Stieltjes integral given in [21, p.48]. Namely, in contrast to [21, p.48] in (18) there are no the additional summands, the sums of products of jumps of the functions at their points of discontinuity. Also, we do not require that both functions have finite total variation.

We use formula (18) to define the value of the alpha-integral in the case when $g \in \mathbf{BV}$, $f \in \mathbf{R}$. Namely, let $g \in \mathbf{BV}$, $f \in \mathbf{R}$. We define

$$(\alpha)\int g(t)\,df \doteq g(t)f(t)|_{a+}^{b-} - (1-\alpha)\int f(t)\,dg(t). \tag{19}$$

Thus, alpha-integral is defined and exists when one of integrated functions is in **R** and the other is in **BV**. In both cases we have formula (18).

THEOREM 4. *Suppose that $\alpha, \beta \in \mathbf{C}$, $f, g \in \mathbf{BV}$, $h(\cdot, s)$, $h(t, \cdot) \in \mathbf{R}$ $\big(f, g \in \mathbf{R}$, $h(\cdot, s)$, $h(t, \cdot) \in \mathbf{BV}\big)(t \in I, s \in (c,d))$, and there exists $M > 0$ such that $\|h(\cdot, s)\|_{\mathbf{R}} \leqslant M \quad (s \in (c,d))$. Then*

$$(\alpha)\int_{(a,b)} \left( (\beta)\int_{(c,d)} h(t,s)\,dg(s) \right) df(t) = (\beta)\int_{(c,d)} \left( (\alpha)\int_{(a,b)} h(t,s)\,df(t) \right) dg(s)$$

THEOREM 5. *(see[21, c.40]). Let $\alpha \in \mathbf{C}$, $h, g \in \mathbf{R}$, $f \in \mathbf{BV}$ $\big(h, g \in \mathbf{BV}$, $f \in \mathbf{R}\big)$. Then*

$$(\alpha)\int h(t)\,d\Big((\alpha)\int_{(a,t)} g(s)\,df(s)\Big) = \tag{20}$$

$$= (\alpha)\int h(t)g(t)\,df(t) - \alpha(1-\alpha)\sum_{t \in T(f)} \sigma_t(h)\sigma_t(g)\sigma_t(f).$$

THEOREM 6. *If $g_n \to g$ in* **R**, *$f_n \to f$ in* **BV** $\big($*or $g_n \to g$ in* **BV**, *$f_n \to f$ in* **R**$\big)$, *$\alpha_n \to \alpha$ in* **C**, *then*

$$\lim_{n \to \infty} (\alpha_n)\int g_n\,df_n = (\alpha)\int g\,df.$$

## 5. Distributions.

*1.* Let us denote by $\mathcal{D}$ the topological vector space of finite continuous functions endowed with the standard topology (e.g., see [22]). By definition, the elements of the "classical" space of distributions are the continuous linear functionals defined on $\mathcal{D}$. We denote this space by $\mathcal{D}'$.

Let us remark that we do not require the differentiability of our test functions (i.e., the elements of $\mathcal{D}$) since in what follows we consider the systems of differential equations of the first order only. The operation of differentiation in $\mathcal{D}'$ is defined only on the regular distributions in $\mathcal{D}'$ which are generated by the elements of $\mathbf{BV}\ell$.

Let $\mathbf{X}$ denote either $\mathbf{R}$ or $\mathbf{BV}\ell$, while the notation $\mathbf{Y}$ stands for the other space. Let $\mathcal{X}$ be the topological vector space of finite elements in $\mathbf{X}$ (i.e., the test functions) endowed with the following definition of convergence: we say that $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{X}$ converges to $\varphi \in \mathcal{X}$ if there exists a closed interval $[\alpha, \beta] \subset I$ such that $supp\, \varphi_n \subset [\alpha, \beta]$ $(n = 1, 2, \dots)$ and $\varphi_n \to \varphi$ in $\mathbf{X}$.

Further, let us consider the linear continuous functionals on $\mathcal{X}$ (distributions); we denote the set of such functionals by $\mathcal{X}'$, the value of $f \in \mathcal{X}'$ on the test function $\varphi \in \mathcal{X}$ is denoted by $(f, \varphi)$.

EXAMPLE 1 (REGULAR DISTRIBUTIONS). *Suppose that $f$ is a locally summable on $I$ function, $\varphi \in \mathcal{X}$; then $f \cdot \varphi$ is summable on $I$, so we may define*

$$(f, \varphi) \doteq \int_a^b f(t)\varphi(t)\, dt. \tag{21}$$

*The functional (21) is an extension of the regular functional from $\mathcal{D}$ to $\mathcal{X}$.*

Since the elements of the space $\mathbf{X}$ are locally summable, each one of them determines according to (21) a regular linear continuous functional on $\mathcal{X}$, so $\mathbf{X} \subset \mathcal{X}'$.

EXAMPLE 2. *Suppose that $f \in \mathbf{Y}$; given $\varphi \in \mathcal{X}$, we define*

$$(g, \varphi) = (\alpha) \int \varphi(t)\, df(t), \tag{22}$$

*where $\alpha \in \mathbf{C}$ is fixed. The linearity of the functional $g$ is obvious; if $\varphi_n \to \varphi$ in $\mathcal{X}$, then according to Theorem 6 we have the convergence $(g, \varphi_n) \to (g, \varphi)$, so $g$ is a continuous functional, $g \in \mathcal{X}'$.*

The functional (22) is also an extension of certain linear functional from $\mathcal{D}$ to $\mathcal{X}$: if $\varphi \in \mathcal{D}$, then as it follows from the definition of alpha-integral we have for any $\alpha \in \mathbf{C}$

$$(\alpha) \int \varphi(t)\, df(t) = \int_I \varphi(t)\, df(t).$$

As follows from (22), the extension $g \in \mathcal{X}'$ depends on $\alpha \in \mathbf{C}$.

EXAMPLE 3. *Given* $\varphi \in \mathcal{X}$, $s \in I$, *we put* $(f, \varphi) \doteq \varphi(s+)$; *clearly,* $f$ *is a linear continuous* $\mathcal{X}$, $f \in \mathcal{X}'$; *let us call this distribution the* right delta-function *concentrated at* $s \in I$; *thus,*

$$(23) \qquad (\delta_s^+, \varphi) = \varphi(s+) \quad (\delta_0^+ = \delta^+);$$

*the left delta-function is defined analogously*

$$(24) \qquad (\delta_s^-, \varphi) = \varphi(s-) \quad (\delta_0^- = \delta^-).$$

Generally, given $\alpha \in \mathbb{R}$ and $\varphi \in \mathcal{X}$, we define

$$(25) \qquad (\delta_s^\alpha, \varphi) = ({}^\alpha\varphi)(s) \quad \big(= \alpha\varphi(s+) + \big(1-\alpha\big)\varphi(s-)\big) \quad (\delta_0^\alpha = \delta^\alpha).$$

Then $\delta_s^+ = \delta_s^1$, $\delta_s^- = \delta_s^0$. If we take $\varphi \in \mathcal{D}$ in (23)–(25) then we obtain the equality $(\delta_s, \varphi) = \varphi(s)$, that is, the definition of the classical delta-function. Thus, $\delta_s^+, \delta_s^-, \delta_s^\alpha$ are the extensions of the delta-function $\delta_s$ from $\mathcal{D}$ to $\mathcal{X}$.

Let us show that the example 3 is a particular case of 2. Let $\theta_s(t) = 0$ if $t < s$, $\theta_s(t) = 1$ if $t > s$ $(s \in I)$. Then $(\alpha(t) = \alpha)$

$$(26) \qquad (\alpha)\int \varphi(t)\, d\theta_s(t) = \alpha\varphi(s+) + \big(1-\alpha\big)\varphi(s-) = (\delta_s^\alpha, \varphi).$$

In what follows, we sometimes consider $\alpha$ in (25) as the value of certain function $\alpha \in \mathbf{C}$:

$$(\delta_s^\alpha, \varphi) = \alpha(s)\varphi(s+) + \big(1 - \alpha(s)\big)\varphi(s-).$$

***2.*** The space $\mathcal{X}'$ is endowed with the ordinary operations of summation and multiplication by elements of $\mathbb{R}$, so that $\mathcal{X}'$ becomes the vector space. Further, we define the convergence in $\mathcal{X}'$ in the standard way: we say that $f_n \to f$ $n \to \infty$ $(f, f_n \in \mathcal{X}', n \in \mathbb{N})$, if $(f_n, \varphi) \to (f, \varphi)$ $(n \to \infty)$. The convergence determines a topology in $\mathcal{X}'$, so that $\mathcal{X}'$ is the topological vector space.

The proof of the following statement for $\mathcal{X}$ and $\mathcal{X}'$ is similar to the proof of analogous statement in [22].

LEMMA 1. If $\{f_n\}_{n=1}^\infty$ converges in $\mathcal{X}'$, and $\varphi_n \to 0$ in $\mathcal{X}$, then $(f_n, \varphi_n) \to 0$ $(n \to \infty)$.

This lemma allows us to prove the completeness of $\mathcal{X}'$ in the following sense.

THEOREM 7. *Suppose that the sequence* $\{f_n\}_{n=1}^{\infty}$ *is such that, given any* $\varphi \in \mathcal{X}$, *the sequence* $\{(f_n, \varphi)\}_{n=1}^{\infty}$ *converges when* $n \to \infty$. *Then the functional* $f$ *defined on* $\mathcal{X}$ *by the equality*

$$(f, \varphi) = \lim_{n\to\infty} (f_n, \varphi) \tag{27}$$

*is linear and continuous, i.e.,* $f \in \mathcal{X}'$.

*Proof.* The linearity of the limit functional (27) is obvious. It suffices to prove its continuity at $\varphi = 0$. Let $\varphi_n \to 0$ in $\mathcal{X}'$ $(n \to \infty)$. Suppose the contrary: $(f, \varphi_n)$ does not converge to 0 $(n \to \infty)$. Then, taking a subsequence if necessary, we may assume that for any $n \in \mathbb{N}$ we have the inequality $|(f, \varphi_n)| > \varepsilon_0$ for certain $\varepsilon_0 > 0$. According to (27) we have that for each $k \in \mathbb{N}$ there exists $n_k$ such that $|(f_{n_k}, \varphi_k)| > \frac{\varepsilon_0}{2}$. Without loss of generality we may put $n_k = k$, i.e., $|(f_k, \varphi_k)| > \frac{\varepsilon_0}{2}$ for all $k \in \mathbb{N}$. But the latter contradicts Lemma 1 which states that $|(f_k, \varphi_k)| \to 0 \quad (k \to \infty)$. Consequently, $(f, \varphi_n) \to 0$ $(n \to \infty)$, so the limit functional is continuous. □

*3.* Now we define in $\mathcal{X}'$ the operation of multiplication by the elements of **X** (let us note that, generally speaking, these are the discontinuous functions). Since, given any $g \in \mathbf{X}$ and $\varphi \in \mathcal{X}$, we have $g\varphi \in \mathcal{X}$, we put $(f \in \mathcal{X}')$

$$(gf, \varphi) = (fg, \varphi) \doteq (f, g\varphi). \tag{28}$$

THEOREM 8. Let $f_n \to f$ in $\mathcal{X}'$, $g_n \to g$ in $\mathbf{X}$ $(n \to \infty)$. *Then* $g_n f_n \to gf$ *in* $\mathcal{X}'$ $(n \to \infty)$.

*Proof.* Let us note first that $g_n\varphi \to g\varphi$ in $\mathcal{X}$ $(n \to \infty)$ for every $\varphi \in \mathcal{X}$. So,

$$\begin{aligned} |(g_n f_n, \varphi) - (gf, \varphi)| &= |(f_n, g_n\varphi) - (f, g\varphi)| \leqslant \\ &\leqslant |(f_n, g_n\varphi) - (f_n, g\varphi)| + |(f_n, g\varphi) - (f, g\varphi)| \leqslant \\ &\leqslant |(f_n, g_n\varphi - g\varphi)| + |(f_n, g\varphi) - (f, g\varphi)| \to 0 \quad (n \to \infty) \end{aligned}$$

according to Lemma 1 and due to convergence $f_n \to f$ in $\mathcal{X}'$. □

Thus, the operation of multiplication defined above is continuous.

EXAMPLE 4. *Let us find the product* $\theta_s\delta_s^\alpha$. *We have for the test function* $\varphi \in \mathcal{X}$

$$\begin{aligned} (\theta_s\delta_s^\alpha, \varphi) = (\delta_s^\alpha, \theta_s\varphi) &= \alpha(s)\theta_s(s+)\varphi(s+) + \big(1 - \alpha(s)\big)\theta_s(s-)\varphi(s-) = \\ &= \alpha(s)\varphi(s+) = (\alpha\delta_s^+, \varphi); \end{aligned}$$

*in particular* $(\theta_s\delta_s^+, \varphi) = \varphi(s+)$, $(\theta_s\delta_s^-, \varphi) = 0$. *Thus,*

$$\theta_s\delta_s^\alpha = \alpha(s)\delta_s^+, \quad \theta_s\delta_s^+ = \delta_s^+, \quad \theta_s\delta_s^- = 0. \tag{29}$$

Let us note that on the test functions in $\mathcal{D}$ the equality (29) gives us the "usual" equality (e.g., see [6]), $\theta_s\delta_s = \alpha\delta_s$.

*4.* The restriction of the functional $f \in \mathcal{X}'$ from $\mathcal{X}$ to $\mathcal{D}$ is a ("classical") distribution. As a result, $\mathcal{X}'$ can be viewed as a set of extensions of the distributions from $\mathcal{D}$ to $\mathcal{X}$. The extension of any distribution in $\mathcal{D}'$ is non-unique. For example, the family of extensions of the functional $\delta_s$ from $\mathcal{D}$ to $\mathcal{X}$ contains the family of delta-functions $\{\delta_s^\alpha\}_{\alpha\in\mathbb{R}}$. Furthermore, given any $\beta \in \mathbb{R}$, $\xi \in I$, the functional $\delta_s^\alpha + \beta(\delta_\xi^+ - \delta_\xi^-)$ is also an extension of $\delta_s$ from $\mathcal{D}$ to $\mathcal{X}$.

Let $\Gamma : \mathcal{X}' \to \mathcal{D}'$ be the operator of restriction which associates to each distribution $f \in \mathcal{X}'$ its restriction $f|_\mathcal{D}$ to $\mathcal{D}$. Clearly, $\Gamma$ is a linear continuous operator; according to the Hahn-Banach Theorem [23, p.49] every distribution $g \in \mathcal{D}'$ possesses its extension $f$ to $\mathcal{X}$, i.e., $\Gamma f = g$, so $\Gamma$ is surjective; its kernel $\ker \Gamma \neq \{0\}$ is a linear (closed) subspace; $\mathcal{X}'$; $f = \beta(\delta_\xi^+ - \delta_\xi^-) \in \ker \Gamma$; moreover, if $\beta(\cdot, s) \in \mathbf{X}$ $(s \in I)$, $\beta(t, \cdot) \in \mathbf{Y}$ $(t \in I)$, $f(t) = \int_a^b \beta(t,s)\big(\delta_s^+(t) - \delta_s^-(t)\big)\,ds$, where the "integral" is understood in the following sense: for every $\varphi \in \mathcal{X}$
$\left(\int_a^b \beta(t,s)\big(\delta_s^+(t) - \delta_s^-(t)\big)\,ds, \varphi\right) = \sum_{s\in T(\varphi)} \beta(t,s)\sigma_s(\varphi)$, then $f \in \ker \Gamma$.

Let us denote by $P : \mathcal{D}' \to \mathcal{X}'$ the (set-valued) operator of extension of linear continuous functionals from $\mathcal{D}$ to $\mathcal{X}$: for $f \in \mathcal{D}'$ $P(f)$ is the set of all extensions of $f$ from $\mathcal{D}$ to $\mathcal{X}$.
THEOREM 9. *For any* $f \in \mathcal{D}'$ $P(f) \in \mathcal{X}'/\ker\Gamma$.

*Proof.* We have to demonstrate that the family of extensions of $f \in \mathcal{D}'$ from $\mathcal{D}$ to $\mathcal{X}$ forms a class of equivalence. Let us denote $A \doteq P(f)$; let $g_1, g_2 \in A$. Then by the definition $\Gamma$ $\Gamma(g_i) = f$ $(i = 1, 2)$; consequently, $\Gamma(g_1 - g_2) = 0$, i.e., $g_1 - g_2 \in \ker\Gamma$. This implies that $A$ is the equivalence class in $\mathcal{X}'/\ker\Gamma$. □

We also use notation $P$ to denote the single-valued operator of extension of distributions in $\mathcal{D}'$ in $\mathcal{X}'/\ker\Gamma$ which associated to each functional $f \in \mathcal{D}'$ whole class $P(f)$. Thus, for every $f \in \mathcal{X}'$ $P(\Gamma(f)) = \widehat{f}$, where $\widehat{f}$ is the equivalence class containing $f$ ($P\Gamma$ is the natural homomorphism from $\mathcal{X}'$ to $\mathcal{X}'/\ker\Gamma$), for any $f \in \mathcal{D}'$ $\Gamma(P(f)) = f$, so $\Gamma P$ can be viewed as an identity operator on $\mathcal{D}'$.

***5.*** Let us define in $\mathcal{X}'$ the operation of differentiation. We identify $f \in \mathbf{Y}$ with the corresponding element $f \in \mathcal{X}'$.

Suppose that $f \in \mathbf{Y}$; we put $(\varphi \in \mathcal{X},\ \alpha \in \mathbf{C})$

$$(f'_\alpha, \varphi) \doteq (\alpha)\int \varphi(t)\,df(t). \tag{30}$$

As it follows from Example 2 and Theorem 6, $f'_\alpha$ is a linear and continuous functional on $\mathcal{X}$, $f'_\alpha \in \mathcal{X}'$. Further, it follows straightforwardly from the definition (30) that the operation of differentiation is not single-valued: it depends on certain $\alpha \in \mathbf{C}$. Also, $\Gamma f'_\alpha \in \mathcal{D}'$, so all extensions of the derivative in $\mathcal{D}'$, defined by

$$(f', \varphi) \doteq \int_I \varphi(t)\, df(t) \quad (\varphi \in \mathcal{D}),$$

from $\mathcal{D}$ to $\mathcal{X}$ have form $\quad Pf' = f'_\alpha + ker\Gamma$.

Note that according to (26) $(\theta_s)'_\alpha(t) = \delta^\alpha_s(t)$.

If $f \in \mathbf{CBV}$, then $f'$ does not depend on the "extending" function $\alpha \in \mathbf{C}$, since in this case for every $\varphi \in \mathcal{X}$ $(\alpha)\int \varphi\, df = \int_I \varphi(t)\, df(t)$; if $f$ is locally absolutely continuous on $I$, then $(\alpha)\int \varphi\, df = \int_{(a,b)} \varphi f'\, dt$, so $f'$ is identified with the regular distribution generated by the locally-summable function $f'$, i.e., $f'$ coincides with the ordinary derivative.

**6. Systems of differential equations.** Let $Z$ denote one of spaces considered above. In what follows, we denote by $Z^n$ $(Z^{n\times n})$ the spaces of $n$-vectors ($n \times n$-matrices) whose components are in $Z$ (in the case $Z = \mathcal{X}'$ we write $\mathcal{X}^{n'}$ $(\mathcal{X}^{n\times n'})$); the linear operations, the relation of identity, the multiplication by a scalar function, the differentiation and integration are defined component-wisely.

Let us consider the Cauchy problem

$$x'_\alpha = M'_\alpha x + f'_\alpha \quad (t \in I,\ t > t_0), \tag{31}$$

$$x(t_0+) = x_0 \quad (t_0 \in I,\ x_0 \in \mathbb{R}^n), \tag{32}$$

where $\alpha \in \mathbf{C}$ is a fixed function, $M \in \mathbf{Y}^{n\times n}$, $f \in \mathbf{Y^n}$. The identities which arise in (31) were correctly defined above.

By the solution of the problem (31), (32) we understand an $n$-vector function $x \in \mathbf{BV}\ell^n$ that satisfies the equality (31) in the sense of operation in $\mathcal{X}^{n'}$ and such that the equality (32) is satisfied. In other words, given any $\varphi \in \mathcal{X}$, we have the equality $(x'_\alpha,\ \varphi) = (M'_\alpha x,\ \varphi) + (f'_\alpha,\ \varphi)$.

Now, taking the definitions of the operation of multiplication and differentiation in $\mathcal{X}^{n'}$ defined above (see (28) and (30)), as well as the equality (15), we obtain that with precision up to the elements of $(\ker \Gamma)^n$ the Cauchy problem (31), (32) is equivalent to the alpha-integral equation of the Volterra-Stieltjes type

$$x(t-) = (\alpha) \int_{(t_0,t)} dM(s) \cdot x(s) + F(t-) \qquad \big(F(t) = x_0 + f(t) + f(t_0+)\big) \tag{33}$$

in the space $\mathbf{BV}\ell^n$.

Let us note that the "parameter of multiplicity" $\alpha$ is a priori introduced in the statement of the Cauchy problem, so there is not any multiplicity in Cauchy problem itself. Furthermore, instead of system (31) we could consider the system $x'_\beta = M'_\alpha x + f'_\gamma \quad (t \in I,\ t > t_0)$, where $\alpha, \beta, \gamma \in \mathbf{C}$, which also leads to the integral equation (33).

If $\hat{x}$ is the solution of the equation (33), then all solutions of the problem (31), (32) have form $x = \hat{x} + \tilde{x}$, where $\tilde{x} \in (\ker \Gamma)^n$. We call $\hat{x}$ the *principal solution* of the problem (31), (32).

If $\mathbf{X} = \mathbf{R}$, $\mathbf{Y} = \mathbf{BV}\ell$, then the question of existence and uniqueness of solution of the equation (33) (i.e., the principal solution of the problem (31), (32)) can be solved similarly to how it was done in [12]. Conversely, if $\mathbf{X} = \mathbf{BV}\ell$, $\mathbf{Y} = \mathbf{R}$, then this question can be reduced to the investigation of the properties of the alpha-integral equation of Volterra-Stieltjes type (33), which is the subject of our next paper.

**7. Proofs of Theorems 2–6.**

*Proof of 2.* First, let us prove statement (13) under the following conditions:

$$h \in \mathbf{R},\ g, f \in \mathbf{CBV}. \tag{34}$$

It suffices to prove (13) for the case when $g$ and $f$ are increasing. According to property 3) all integrals in (13) exist. Let $\varepsilon > 0$ be arbitrarily fixed. According to the necessary and sufficient condition of integrability (see [16]) there exists $\delta > 0$ such that the inequalities

$$\sum_{k=1}^{n} \omega_k(f)(g(t_k)-g(t_{k-1})) < \frac{\varepsilon}{2\|h\|}, \quad \sum_{k=1}^{n} \omega_k(g)(f(t_k)-f(t_{k-1})) < \frac{\varepsilon}{2\|h\|} \tag{35}$$

hold for any (9) $\tau = \{t_k\}_{k=0}^n$ with $d(\tau) < \delta$ ($\omega_k(f) = \sup_{t,s\in[t_{k-1},t_k]} |f(t) - f(s)|$). Let us consider the Stieltjes sum

$$\mathfrak{S}_\tau(h, gf) = \sum_{k=1}^{n} h(\xi_k)\big(g(t_k)f(t_k) - g(t_{k-1})f(t_{k-1})\big) =$$

$$\sum_{k=1}^{n} h(\xi_k)f(t_k)\big(g(t_k) - g(t_{k-1})\big) + \sum_{k=1}^{n} h(\xi_k)g(t_{k-1})\big(f(t_k) - f(t_{k-1})\big).$$

Let us denote the first summand by $S_1$, and the second summand by $S_2$. For the divisions given above, (35) implies that

$$\big|\mathfrak{S}_\tau(hf, g) - S_1\big| \leqslant \|h\|_{\mathbf{R}} \sum_{k=1}^{n} \omega_k(f)(g(t_k) - g(t_{k-1})) < \frac{\varepsilon}{2}$$

and

$$\left|\mathfrak{S}_\tau(hg,f)-S_2\right|\leqslant\|h\|_{\mathbf{R}}\sum_{k=1}^{n}\omega_k(g)(f(t_k)-f(t_{k-1}))<\frac{\varepsilon}{2}.$$

Thus, if $d(\tau)<\delta$, then

$$\left|\mathfrak{S}_\tau(h,gf)-\big(\mathfrak{S}_\tau(hf,g)+\mathfrak{S}_\tau(hg,f)\big)\right|<\varepsilon;$$

so since $\varepsilon$ is arbitrary, we have that (13) is true under conditions (34).

Suppose that $h,g\in\mathbf{CBV}$, $f\in\mathbf{R}$. Let us note that all integrals in (13) also exist. According to (10) we have the equalities

$$\int\limits_{(a,b)} h\,dgf=h(t)g(t)f(t)|_{a+}^{b-}-\int\limits_{(a,b)} gf\,dh,\ \int\limits_{(a,b)} hg\,df=h(t)g(t)f(t)|_{a+}^{b-}-\int\limits_{(a,b)} f\,dhg.$$

Let us deduce the second equality from the first equality. Further, let us extract from the both sides of the equality the value of integral $\int_{(a,b)} hf\,dg$; as a result, we will obtain the identity

$$\int\limits_{(a,b)} h\,dgf-\int\limits_{(a,b)} hg\,df-\int\limits_{(a,b)} hf\,dg=-\int\limits_{(a,b)} gf\,dh+\int\limits_{(a,b)} f\,dhg-\int\limits_{(a,b)} hf\,dg,$$

whose the right-hand side is zero as it was shown above. This implies that (13) is valid in this case also.

The case $h,f\in\mathbf{CBV}$, $g\in\mathbf{R}$ can be considered analogously. □

*Proof of Theorem 3.* Denote $\beta=1-\alpha$, $T=T(f)\cup T(g)$,

$$K=\sum_{t\in T}\big(({}^{\alpha}g)(t)\sigma_t(f)+({}^{\beta}f)(t)\sigma_t(g)\big).$$

Then, using the definition (14), the representation (8), the formula of integration by parts (18), the formula for evaluation of $RS$-integral of a continuous function by a step function, and the formula for representation of a step function (see [16]), we have

$$S\doteq(\alpha)\int g\,df+(\beta)\int f\,dg=\int_{(a,b)} g\,df_c+\int_{(a,b)} f\,dg_c+K=$$

$$\int_{(a,b)} g_c\,df_c+\int_{(a,b)} f_c\,dg_c+\int_{(a,b)} g_d\,df_c+\int_{(a,b)} f_d\,dg_c+K=$$

$$(g_cf_c)|_{a+}^{b-}+(g_df_c)|_{a+}^{b-}+(f_dg_c)|_{a+}^{b-}-\int_{(a,b)} f_c\,dg_d-\int_{(a,b)} g_c\,df_d+K=$$

$$g_c(b-)f_c(b-) - g(a+)f(b-) + g_d(b-)f_c(b-) + f_d(b-)g_c(b-) + L,$$

where $L = K - \sum\limits_{t\in T} f_c(t)\sigma_t(g) - \sum\limits_{t\in T} g_c(t)\sigma_t(f)$. Thus,

$$S = (fg)|_{a+}^{b-} - f_d(b-)g(b-) + L = (fg)|_{a+}^{b-} - \sum_{t\in T}\sigma_t(f)\sum_{s\in T}\sigma_t(g) + L.$$

Consequently,

$$L = \sum_{t\in T}\sigma_t(g)\sigma_t(f) + \sum_{t\in T}\sigma_t(g)\sum_{s\in T, s<t}\sigma_s(f) + \sum_{t\in T}\sigma_t(f)\sum_{s\in T, s<t}\sigma_s(g).$$

Now we may change the order of summation in the last summand (which is possible since this series converges absolutely) to get the equality

$$L = \sum_{t\in T}\sigma_t(f)\sum_{s\in T}\sigma_s(g);$$

this implies (18). □

*Proof of Theorem 4.* Let $f, g \in \mathbf{BV}$, $h(\cdot, s)$, $h(t, \cdot) \in \mathbf{R}$. We denote

$$H(t) \doteq \int_{(c,d)} h(t,s)\,dg_c(s),\ \mathcal{H}(s) \doteq \int_{(a,b)} h(t,s)\,df_c(t),$$

$$H_\beta(t) \doteq \int_{(c,d)} h(t,s)\,dg(s) = H(t) + \sum_{s\in T}({}^\beta(h(t,\cdot))(s)\sigma_s(g),$$

$$\mathcal{H}_\alpha(s) \doteq (\alpha)\int_{(a,b)} h(t,s)\,df(t) = \mathcal{H}(s) + \sum_{t\in T}({}^\alpha(h(\cdot,s))(t)\sigma_t(f).$$

Then we may reformulate the statement of the theorem using these notations:

$$(36)\qquad (\alpha)\int_{(a,b)} H_\beta(t)\,df(t) = (\beta)\int_{(c,d)} \mathcal{H}_\alpha(s)\,dg(s)$$

According to the theorem on change of integration order for $RS$-integral (see [16]) we have that

$$(37)\qquad \int_{(a,b)} H(t)\,df_c(t) = \int_{(c,d)} \mathcal{H}(s)\,dg_c(s).$$

Let us prove (36):

$$S \doteq (\alpha)\int_{(a,b)} H_\beta(t)\,df(t) = \int_{(a,b)} H_\beta(t)\,df_c(t) + \sum_{t\in T(f)}({}^\alpha H_\beta)(t)\sigma_t(f) =$$

$$\int_{(a,b)} \left( H(t) + \sum_{s \in T(g)} ({}^{\beta}h(t,\cdot))(s)\sigma_s(g) \right) df_c(t) + \sum_{t \in T(f)} ({}^{\alpha}H_{\beta})(t)\sigma_t(f) =$$

$$\int_{(a,b)} \left( H(t)\, df_c(t) + \sum_{s \in T(g)} ({}^{\beta}\mathcal{H})(s)\sigma_s(g) \right) + \sum_{t \in T(f)} ({}^{\alpha}H)(t)\sigma_t(f) + Q,$$

where

$$Q = \sum_{t \in T(f)} \sum_{s \in T(g)} \left( {}^{\alpha} \left( {}^{\beta}h(\cdot, s) \right) (s) \right) (t) \sigma_s(g) \sigma_t(f).$$

The above integration of elements of the series by $t$ is possible due to its absolute and uniform convergence with respect to this variable. Further, the operators $({}^{\alpha})$ (acting on the first argument) and $({}^{\beta})$ (acting on the second argument) commute; the summation by $t$ and by $s$ can be reversed due to absolute convergence of the corresponding series, so

$$Q = \sum_{s \in T(g)} \sum_{t \in T(f)} \left( {}^{\beta} \left( {}^{\alpha}h(\cdot, s) \right) (t) \right) (s) \sigma_t(f) \sigma_s(g).$$

According to (37) we now have

$$S = \int_{(c,d)} \mathcal{H}(s)\, dg_c(s) + \sum_{t \in T(f)} ({}^{\alpha}H)(t)\sigma_t(f) +$$

$$\sum_{s \in T(g)} ({}^{\beta}\mathcal{H})(s)\sigma_s(g) + Q = (\beta) \int_{(c,d)} \mathcal{H}_{\alpha}(s)\, dg(s),$$

so (36) is proved.

The case $f, g \in \mathbf{R}$, $h(\cdot, s)$, $h(t, \cdot) \in \mathbf{BV}$ can be reduced to the one considered above using the formula (18). □

*Proof of Theorem 5.* Suppose that $h, g \in \mathbf{R}$, $f \in \mathbf{BV}$. Then according to properties d) and f) the integrals in both sides exist, and the convergence of the series in the second line of this equality is provided by conditions of the present statement. Then according to (17) and be the theorem on substitution for $RS$-integral (see [16]), we have

$$(\alpha) \int h(t)\, d\Big((\alpha) \int_{(a,t)} g(s)\, df(s)\Big) = (\alpha) \int h(t)\, d\Phi_{\alpha}(t) = \int_{(a,b)} h(t)\, d\Phi_{\alpha c}(t)$$

$$- \sum_{t \in T(\Phi_{\alpha})} ({}^{\alpha}h)(t)\sigma_t(\Phi_{\alpha}) = \int_{(a,b)} h(t)\, d \int_{(a,t)} g(s)\, df_c(s) +$$

$$\sum_{t\in T(f)} ({}^{\alpha}h)(t)({}^{\alpha}g)(t)\sigma_t(f) = \int_{(a,b)} h(t)g(t)\,df_c(t)+$$

$$\sum_{t\in T(f)} \big(({}^{\alpha}(hg)\big)(t)\sigma_t(f) + \mathcal{R}_\alpha = (\alpha)\int h(t)g(t)\,df(t) + \mathcal{R}_\alpha,$$

where $\mathcal{R}_\alpha = \sum\limits_{t\in T(f)} \big(({}^{\alpha}h)(t)({}^{\alpha}g)(t) - ({}^{\alpha}(hg))(t)\big)\sigma_t(f)$. Finally, we obtain $\mathcal{R}_\alpha = -\alpha(1-\alpha)\sum\limits_{t\in T(f)} \sigma_t(h)\sigma_t(g)\sigma_t(f)$.

The case $h, g \in \mathbf{BV}$, $f \in \mathbf{R}$ can be reduced to the previous one using the result (18). □

*Proof of Theorem 6.* Let $g_n \to g$ $\mathbf{R}$, $f_n \to f$ $\mathbf{BV}$. Let us show first that

$$\lim_{n\to\infty} (\alpha_n)\int g_n\,df = (\alpha)\int g\,df. \tag{38}$$

There exists $K \geqslant A$ such that $\|g_n\| \leqslant K$ $(n \in \mathbb{N})$. Then due to (16)

$$\Big|(\alpha_n)\int g_n\,df - (\alpha)\int g\,df\Big| \leqslant \Big|(\alpha_n)\int g_n\,df - (\alpha_n)\int g\,df\Big|+$$

$$\Big|(\alpha_n)\int gdf - (\alpha)\int gdf\Big| \leqslant K\|g_n - g\|_{\mathbf{R}}\bigvee_a^b(f) + \|\alpha_n - \alpha\|_{\mathbf{C}} \sum_{t\in T(f)} |\sigma_t(g)\|\sigma_t(f)|.$$

Now since the sequence $\{|\sigma_t(g)|\}_{t\in T(f)}$ is bounded, the series in the second summand converges. The estimation obtained gives us the required statement (38).

Further, according to (16)

$$\Big|(\alpha_n)\int g_n\,df_n - (\alpha)\int g\,df\Big| \leqslant \Big|(\alpha_n)\int g_n\,df_n - (\alpha_n)\int g_n\,df\Big|+$$

$$+\Big|(\alpha_n)\int g_n\,df - (\alpha)\int g\,df\Big| \leqslant K^2\bigvee_a^b(f_n - f) + \Big|(\alpha_n)\int g_n df - (\alpha)\int g\,df\Big|.$$

The convergence of the first summand to zero follows from the convergence $\{f_n\}_{n=1}^{\infty}$ in $\mathbf{BV}$, the convergence of the second summand follows from (38).

If $g_n \to g$ in $\mathbf{BV}$, $f_n \to f$ in $\mathbf{R}$, then we may use Theorem 3 and (19) to reduce our consideration to the case considered above. □