Consistency and Pole Assignment in Linear Systems With Incomplete Feedback ⋆

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Abstract: New necessary and sufficient conditions in the pole assignment problem are obtained for linear systems with incomplete feedback. The concept of consistency is brought in, which is generalization the concept of complete controllability to systems with incomplete feedback. The conditions are obtained under which the property of consistency is equivalent to global controllability of spectrum for systems with incomplete feedback.

Keywords: Feedback control, pole assignment, stabilization.

1. INTRODUCTION

Consider a linear control system

$$\dot{x} = A(t)x + B(t)u, \quad (t, x, u) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m, \qquad (1)$$

$$y = C^*(t)x, \quad y \in \mathbf{R}^k \qquad (2)$$

with bounded continuous matrix functions. Let the control in system (1), (2) be constructed as linear incomplete feedback $u = U(t)y$, where U is a bounded piecewise continuous function. Then the closed-loop system is

$$\dot{x} = (A(t) + B(t)U(t)C^*(t))x, \quad x \in \mathbf{R}^n. \qquad (3)$$

If feedback is complete, i.e. $C(t) \equiv I$, then closed-loop system is

$$\dot{x} = (A(t) + B(t)U(t))x, \quad x \in \mathbf{R}^n. \qquad (4)$$

Let system (4) be stationary. By $\lambda_1(A+BU), \ldots, \lambda_n(A+BU)$ denote the eigenvalue spectrum of the matrix $A+BU$. The problem of stabilization for system (4) consists in constructing a stationary control U such that $\operatorname{Re}\lambda_i(A+BU) \leq -\alpha < 0$ for all $i = \overline{1,n}$. If for any given μ_i there exists an U such that $\lambda_i(A+BU) = \mu_i$ for all $i = \overline{1,n}$ then system (4) is stabilizable. This more general problem is called *pole assignment problem*. In Russian-speaking literature it is said to be *problem of global control over spectrum*.

It was shown (see Popov (1964)) that the spectrum for system (4) is globally controllable iff system (1) is completely controllable (i.e. $\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n$). It was shown later (Wonham (1967)) that if the set $\{\mu_1, \ldots, \mu_n\}$ is the spectrum of the real type (i.e. there exists a real matrix having this spectrum) then one can choose a feedback matrix U which is *real*.

A sufficient condition for nonstationary system (4) to be uniform stabilizable is uniform complete controllability of system (4) (Tonkov (1979)).

For systems with incomplete feedback, the above problems is more complicated. In this work, new results about global control over spectrum for system (3) is obtained. The main results are contained in the theorem 8.

2. CONSISTENT SYSTEMS

The system (1), (2) is said to be *consistent on* $[t_0, t_0+\vartheta]$ if there exists $l > 0$ such that for any $(n \times n)$-matrix G we can find a piesewise continuous control $U_G(t)$, $t \in [t_0, t_0+\vartheta]$ such that the solution of the matrix initial value problem

$$\dot{Z} = A(t)Z + B(t)U_G(t)C^*(t)X(t, t_0), \quad Z(t_0) = 0$$

satisfies condition $Z(t_0) = G$, and the inequality $|U_G(t)| \leq l|G|$, $t_0 \leq t \leq t_0+\vartheta$ is satisfied; $X(t, s)$ denotes the Cauchy matrix of the system of equations $\dot{x} = A(t)x$. The system (1), (2) is said to be *uniformly consistent* if there exists $\vartheta > 0$ such that it is consistent on every segment $[t_0, t_0+\vartheta]$, $t_0 \in \mathbf{R}$, and the last estimate is uniform, i.e. l does not depend on t_0.

The concept of consistency is generalization of the concept of complete controllability to systems with incomplete feedback. If $C(t) \equiv I$ then these properties are equivalent. The definition of consistency has been given in Popova and Tonkov (1994a). On the basis of this property results about local control over Lyapunov exponents of system (3) have been obtained in Popova and Tonkov (1994b), Popova and Tonkov (1995).

Consistent systems have following properties.

Proposition 1. If system (1), (2) is consistent on $[t_0, t_0+\vartheta]$, then system (1) is completely controllable on $[t_0, t_0+\vartheta]$, and the system

$$\dot{x} = A(t)x, \quad y = C^*(t)x$$

is completely observable on $[t_0, t_0+\vartheta]$.

In general, the converse statement is not true.

Let us construct so-called "big system"

$$\dot{q} = F(t)q + G(t)v \qquad (5)$$

from system (1), (2), where the $(n^2 \times n^2)$-matrix F and the $(n^2 \times mr)$-matrix G are defined as

$$F(t) = A(t) \otimes I - I \otimes A^*(t), \quad G(t) = B(t) \otimes C(t), \qquad (6)$$

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and the symbol $\otimes$ denotes the direct (Kronecker) product of matrices.

Proposition 2. System (1), (2) is consistent on $[t_0, t_0 + \vartheta]$ if and only if system (5) is completely controllable on $[t_0, t_0 + \vartheta]$.

Now, let us assume that system (1), (2) is stationary. Consider the system

$$\dot{x} = Ax + Bu, \quad (x, u) \in \mathbf{R}^n \times \mathbf{R}^m, \tag{7}$$

$$y = C^* x, \quad y \in \mathbf{R}^k. \tag{8}$$

One can formulate following criteria of consistency for stationary systems.

Proposition 3. Suppose $\operatorname{rank} C = n$. Then system (7), (8) is consistent if and only if system (7) is completely controllable.

Proposition 4. Suppose $\operatorname{rank} B = n$. Then system (7), (8) is consistent if and only if the system

$$\dot{x} = Ax, \quad y = C^* x \tag{9}$$

is completely observable.

Proposition 5. System (7), (8) is consistent if and only if

$$\operatorname{rank}[G, FG, \ldots, F^{n^2-1}G] = n^2,$$

where F and G defined by (6).

The propositions 3, 4 are clear. However, if $\operatorname{rank} B = m < n$ and $\operatorname{rank} C = k < n$ then these statements are inapplicable. The proposition 5 follows from the proposition 2. But this criterion is ineffective owing to the big dimensions. The following necessary condition is more convenient for application.

Theorem 6. If the minimal polynomial of matrix A has degree n and the system (7), (8) is consistent then matrices

$$C^*B, C^*AB, \ldots, C^*A^{n-1}B \tag{10}$$

are linearly independent. In general, the converse statement is not true.

3. GLOBAL CONTROL OVER SPECTRUM

Consider system (7), (8), and stationary closed-loop system

$$\dot{x} = (A + BUC^*)x \tag{11}$$

We say *the spectrum of system (11)* instead of *the eigenvalue spectrum of the matrix of system (11)*. We shall say that *the spectrum of system (11) is globally controllable* if for any polynomial $p(\lambda) = \lambda^n + \gamma_1 \lambda^{n-1} + \ldots + \gamma_n$ with $\gamma_i \in \mathbf{R}$ there exists a real constant control U_γ such that the characteristic polynomial $\chi(A + BU_\gamma C^*; \lambda)$ is equal to $p(\lambda)$. The following statements are clear.

Proposition 7. 1. Suppose $\operatorname{rank} C = n$. Then the spectrum of system (11) is globally controllable iff system (7) is completely controllable.
2. Suppose $\operatorname{rank} B = n$. Then the spectrum of system (11) is globally controllable iff system (9) is completely observable.
3. If the spectrum of system (11) is globally controllable then system (7) is completely controllable and system (9) is completely observable.

E.L. Tonkov has asked a following question. Is consistency of system (7), (8) equivalent to global controllability of the spectrum of system (11)? On this question the following answer has been received (Zaitsev (1999)). If $n = 2$ then these properties are equivalent and reduced to one of two above trivial cases, $\operatorname{rank} B = n$ or $\operatorname{rank} C = n$. If $n > 2$ then neither implication is true.

Suppose $m < n$, $k < n$. One has a question: what necessary and (or) sufficient conditions for global controllability of the spectrum of system (11) exist there? Various conditions which are either necessary or sufficient are known. For example, the sufficient condition in typical case is the condition $m + k > n$ (Davison and Wang (1975)), and the necessary condition in typical case is the condition $mk \geq n$. In general, there are no conditions which would be simultaneously necessary and sufficient in contrast to the system with complete feedback. The following theorem shows that in a special case such conditions exist.

Theorem 8. Suppose the matrices of system (7), (8) are

$$A = \left\| \begin{array}{ccccc} a_{11} & a_{12} & 0 & \ldots & 0 \\ a_{21} & a_{22} & a_{23} & \ldots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ a_{n-1,1} & a_{n-1,2} & \ldots & \ldots & a_{n-1,n} \\ a_{n1} & a_{n2} & \ldots & \ldots & a_{nn} \end{array} \right\|,$$

$$B = \left\| \begin{array}{ccc} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ b_{p1} & \ldots & b_{pm} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nm} \end{array} \right\|, \qquad C = \left\| \begin{array}{ccc} c_{11} & \ldots & c_{1k} \\ \vdots & & \vdots \\ c_{p1} & \ldots & c_{pk} \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{array} \right\|,$$

where $a_{i,i+1} \neq 0$ for all $i = \overline{1, n-1}$; $p \in \{1, \ldots, n\}$. Then following statements are equivalent.
1. System (7), (8) is consistent.
2. The spectrum of system (11) is globally controllable.
3. Matrices (10) are linearly independent.

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