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On Controllability of Ergodic System Lyapunov Exponents

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We consider a topological dynamical system (Ω, f^t). Ω is a metric complete separable space, f^t is a flow on Ω that is one-parameter group of transformation which continuous with respect to $(t,\omega) \in \mathbb{R} \times \Omega$. We will denote by $\gamma(\omega)$ the trajectory of motion $t \to f^t\omega$, $\overline{\gamma}(\omega)$ is the completion of $\gamma(\omega)$ by metric of the space Ω. The set $E \subset \Omega$ is said to be invariant if $f^t E \subset E$ for all $t \in \mathbb{R}$ and E is said to be minimal if it is invariant, compact and $\overline{\gamma}(\omega) = E$ for any $\omega \in E$.

To each pair $(\mathbb{A}, \omega)$, where $\mathbb{A} \doteq (A_0, A_1, \ldots, A_r) : \Omega \to M_{n,m}$, $m = n(r+1)$ and vector $u = (u_1, \ldots, u_r)$ we associate the equation

$$\dot{x} = A_0(f^t\omega)x + u_1 A_1(f^t\omega)x + \cdots + u_r A_r(f^t\omega)x, \quad x \in \mathbb{R}^n. \tag{1}$$

We suppose that $\mathbb{A}$ satisfies the following condition: for each fixed $\omega_0 \in \Omega$, the function $t \to |\mathbb{A}(f^t\omega_0)|$ is Lebesgue measurable, bounded on $\mathbb{R}$ and for any $\varepsilon > 0$ and $N > 0$ there exists

$\delta > 0$ such that $\rho(\omega, \omega_0) < \delta$ imply $\max_{|t| \leqslant N} \int_t^{t+1} |\mathbb{A}(f^s\omega) - \mathbb{A}(f^s\omega_0)|\, ds < \varepsilon$, where ρ is the metric in Ω.

We will identify (1) and $(\mathbb{A}, \omega)$.

Let $U \subset \mathbb{R}^r$ be convex, compact and $0 \in U$. We will call the set $\mathcal{U} \doteq \{u : \mathbb{R} \to U\}$, where u is Lebesgue measurable, the set of permissible controls. Let the equation $(\mathbb{A}, \omega)$ be fixed. To each permissible control $u = u(t)$ we associate Cauchy function $X_u(t, s, \omega)$ of the equation (1).

Let us consider the matrix equation

$$\begin{aligned} \dot{X} &= A_0(f^t\omega)X + u_1 A_1(f^t\omega)X + \cdots + u_r A_r(f^t\omega)X, \quad (2) \\ & \quad X \in M_n, \end{aligned}$$

which is corresponding to the equation (1). For any $\vartheta > 0$ we will construct the attainable set $\mathfrak{D}_\vartheta(\omega) = \{X \in M_n : X = X_u(\vartheta, 0, \omega),\ u(\cdot) \in \mathcal{U}\}$ of the equation (2).

Definition 1. The equation (1) is called

a) *local attainable* if there exists $\vartheta > 0$ and $\varepsilon > 0$ such that

$$\mathcal{B}_\varepsilon(I) X_0(\vartheta, 0, \omega) \subset \mathfrak{D}_\vartheta(\omega), \quad (3)$$

where $\mathcal{B}_\varepsilon(I) = \{H \in M_n : |H - I| \leqslant \varepsilon\}$, I is identity matrix.

b) *uniformly local attainable* if there exists $\vartheta > 0$ and $\varepsilon > 0$ such that for all $\omega_0 \in \overline{\gamma}(\omega)$ the inclusion (3) holds;

Theorem 1. *Let $\overline{\gamma}(\omega_0)$ be minimal. Then $(\mathbb{A}, \omega_0)$ is local attainable if and only if it is uniformly local attainable.*

Now, let the phase space Ω of the dynamical system (Ω, f^t) be compact on Borel σ-algebra $\mathfrak{B}$ of Ω, a probabilistic Borel measure μ is given, let it be invariant over flow f^t: $\mu(\Omega_0) = \mu(f^{-t}\Omega_0)$, $(t, \Omega_0) \in \mathbb{R} \times \mathfrak{B}$. Recall that the dynamical system is said to be ergodic if measure of any invariant set is null ore one.

We will denote by $\widehat{\sigma}$ the set of uniformly local attainable equations, by σ_0 the set of local attainable equations.

Theorem 2. *Let the dynamical system be ergodic and $\sigma_0 \neq \varnothing$. Then $\widehat{\sigma}$ is compact if and only if $\mu(\widehat{\sigma}) = \mu(\sigma_0)$ (and hence $\widehat{\sigma} = \sigma_0 = \Omega$).*

We consider whole spectrum $\lambda_1(\omega, u)$, ..., $\lambda_n(\omega, u)$ of equation (1) Lyapunov exponents.

Definition 2. The equation $(\mathbb{A}, \omega_0)$ possesses *uniformly local controllability of Lyapunov exponents* if there is $\delta > 0$ such that for any $\beta = (\beta_1, \ldots, \beta_n)$, $|\beta| \leqslant \delta$ and $\omega \in \overline{\gamma}(\omega_0)$ there exist a permissible control $u : \mathbb{R} \times \overline{\gamma}(\omega_0) \to U$ that imply $\lambda_j(\omega, u) = \lambda_j(\omega, 0) + \beta_j, \quad j = 1, \ldots, n$.

Definition 3. The matrix $A_0(f^t\omega_0)$ is said to be *diagonalizable* if we can reduce the equation

$$\dot{x} = A_0(f^t\omega_0)x$$

by Lyapunov transformation to an equation with the diagonal matrix.

Theorem 3. *Let $A_0(f^t\omega_0)$ be diagonalizable and $(\mathbb{A}, \omega_0)$ be uniformly local attainable. Then $(\mathbb{A}, \omega_0)$ possesses uniformly local controllability of Lyapunov exponents.*

Theorem 4. *Let the dynamical system be ergodic, σ_0 be not empty, $\widehat{\sigma}$ be compact and $A_0(f^t\omega)$ be diagonalizable for all $\omega \in \Omega$. Then $(\mathbb{A}, \omega)$ possesses uniformly local controllability of Lyapunov exponents for all $\omega \in \Omega$.*

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