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ON THE EXTENSION OF SCHWARTZ DISTRIBUTIONS TO THE SPACE OF DISCONTINUOUS TEST FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT. The present paper is devoted to the investigation of the properties of the space of distributions with discontinuous test functions of several variables. The consideration of *discontinuous test functions allows us to define the* operations of integration of distributions and multiplication of distributions by discontinuous functions which are continuous, defined everywhere and coinciding with the ordinary ones for regular distributions. These operations are undefined in the classical space $\mathcal{D}'$ of distributions with continuous test functions, yet necessary in many applications of distribution theory: in what follows, we consider a class of zero-sum games with discontinuous payoff functions; these games may have no solution in the set of pure strategies or in the set of classical mixed strategies, but may possess the solution in the set of $\mathcal{R}'$-mixed strategies which are the elements of the new space of distributions.

1. Introduction. For the past decades the progress of distribution theory was highly motivated by efforts to overcome well-known insufficiencies of the classical space $\mathcal{D}'$ of distributions with continuous test functions [**16**]: the impossibility to define in the space $\mathcal{D}'$ the correct operation of integration of distributions, as well as the correct operation of multiplication of distributions by discontinuous functions [**2, 12, 13, 14, 15**] (the operation is said to be *correct* if it is defined everywhere, continuous and coincides with the ordinary one for regular distributions [**16**]). Numerous applications of distribution theory to ordinary and partial differential equations [**2, 5, 14, 15**], where the necessity to integrate distributions and to multiply distributions by discontinuous functions arise, demonstrate the importance of the definition of these operations.

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In the present paper we study the space $\mathcal{R}'$ of distributions with *discontinuous test functions* of several variables, where the correct operation of integration of distributions, as well as the correct, commutative and associative operation of multiplication of distributions by discontinuous functions, is defined. In the space $\mathcal{R}'$, the family of delta-functions $\delta_p^\alpha \in \mathcal{R}'$ is defined. The parameter α is called the *characteristics of the shape*: if $n = 1$ (the number of variables), then $\alpha : \{-1, 1\} \to \mathbf{R}$, $\alpha(1) + \alpha(-1) = 1$, and

$$\theta\delta^\alpha = \alpha(1)\delta^+, \tag{1}$$

where θ is the Heaviside function and $\delta^+ \in \mathcal{R}'$ is the *right delta-function* (the notation δ^+ stands for δ^α with $\alpha(1) = 1$, $\alpha(-1) = 0$). Due to continuity of the operation of multiplication in $\mathcal{R}'$, the equality (1) can also be obtained if the delta-functions are replaced by terms of the corresponding delta-sequences in $\mathcal{R}'$. The equality (1) resembles the empiric definitions of the product in the space $\mathcal{D}'$ of the form

$$\theta\delta = \beta\delta, \quad \beta \in \mathbf{R}, \tag{2}$$

where $\delta \in \mathcal{D}'$ ($\beta = 1/2$ in [**5**]; other values of $\beta \in \mathbf{R}$ are considered in [**5, 12, 13, 14, 15**], see further references therein). Despite resemblance with (1), the operation of multiplication (2) is neither continuous nor associative (if $\beta \neq 0$, $\beta \neq 1$). Consideration of the discontinuous test functions allows us to overcome these and other insufficiencies of the classical space of distributions $\mathcal{D}'$.

The space of distributions with discontinuous test functions of one variable was constructed in [**8, 9**], where the test functions are infinitely differentiable on $\mathbf{R} \setminus \{0\}$ and possess a discontinuity of the first kind (together with all their derivatives) at the point $x = 0$. In the present paper we consider the case of several variables and, in contrast to [**8, 9**], do not pose any restrictions on the set of points of discontinuity of the test functions (equivalently, on the set of points of discontinuity of an ordinary multiple). We show that every distribution in $\mathcal{D}'$ admits a linear continuous extension from the classical space of continuous test functions $\mathcal{D}$ to the space of discontinuous test functions $\mathcal{R}$. Let us note that the definition of the derivative of a locally-summable function $f \in \mathbf{L}^1_{\mathrm{loc}}(\mathbf{R})$ given in [**8, 9**] by the formula

$$(f', \varphi) := -\int_{\mathbf{R}} f(x)\varphi'(x)\,dx,$$

where φ' is defined on $\mathbf{R}\backslash\{0\}$, gives rise to the operator of differentiation which is neither continuous nor linear, e.g., $\theta' \neq -(1-\theta)'$.

In what follows, we propose another definition of the derivative, which agrees with the topology in $\mathcal{R}'$. Since the distributions in $\mathcal{R}'$ arise as the continuations of the distributions in $\mathcal{D}'$, the elements of the space $\mathcal{R}'$ do not allow unlimited differentiation and, similarly to the elements of the space $\mathcal{D}'$, may be viewed as measures.

Let us mention that every Schwartz distribution can also be integrated, multiplied by a discontinuous function and, furthermore, multiplied by another distribution, in the algebra of *Colombeau generalized functions* [**1**, **2**]. However, in contrast to our approach, in the general case the value of the integral of a distribution is not an ordinary real or complex number, the value of the product is not a distribution, but *the Colombeau generalized function* [**1**].

As one of the applications of the results obtained, in the last section we consider the family of zero-sum games with discontinuous payoff functions (for more details on noncooperative games with discontinuous payoff functions, see [**4**, **11**] and further references therein), which in the general case do not possess the solutions in the set of the pure strategies or in the set of the classical mixed strategies [**7**], yet possess solutions in the set of the so-called $\mathcal{R}'$*-mixed strategies*, which have an obvious probabilistic interpretation.

2. Regulated functions. In what follows, the algebra of *regulated functions* is used to define the space of *discontinuous test functions*.

Let $\Omega \subset \mathbf{R}^n$ be an open set. Let $\mathcal{F}$ be the family of finite unions and differences of convex subsets of Ω. We call $\mathcal{F}$ the *appropriate family*. Following [**3**], we call a bounded function $g : \Omega \mapsto \mathbf{R}$ the *regulated function* if, for every $x \in \overline{\Omega}$ (where $\overline{\Omega}$ stands for the closure of Ω) and any $\varepsilon > 0$ there exist a neighborhood $U_x = U_x(\varepsilon) \in \mathcal{F}$ and $\{S_i\}_{i=1}^m \subset \mathcal{F}$ such that $U_x = \cup_{i=1}^m S_i$, and $|g(y_1) - g(y_2)| < \varepsilon$ for every $y_1, y_2 \in S_i$, $1 \leqslant i \leqslant m$. The algebra of regulated functions is denoted by $\widehat{\mathbf{G}}(\Omega)$ and endowed with the supremum-norm [**3**]. A regulated function $g \in \widehat{\mathbf{G}}(\Omega)$ is called *piecewise-constant* if, for every bounded open subset $\Gamma \subset \Omega$, the restriction $g|_\Gamma$ is a linear combination of the characteristic functions χ_S, where $S \in \{A \cap \Gamma : A \in \mathcal{F}\}$ [**3**]. We denote the algebra of piecewise-constant functions by $\widehat{\mathbf{PC}}(\Omega)$.

Lemma 1 [**3**]. *A function* $g : \Omega \to \mathbf{R}$ *is in* $\widehat{\mathrm{G}}(\Omega)$ *if and only if there exists a sequence* $\{g_k\}_{k=1}^{\infty} \subset \widehat{\mathbf{PC}}(\Omega)$ *such that*

$$\|g - g_k\|_{\widehat{\mathrm{G}}(\Omega)} := \sup_{x\in\Omega}(g(x) - g_k(x)) \to 0. \tag{3}$$

If $n = 1$, then $\widehat{\mathrm{G}}(\Omega)$ is the algebra of bounded functions possessing the one-sided limits $g(x+)$, $g(x-)$ for each $x \in \overline{\Omega}$ (at the boundary points of Ω the existence of one of the one-sided limits is assumed) [**3**].

Let us define the notion of the *surrounding value* of a regulated function $g \in \widehat{\mathrm{G}}(\Omega)$ at the point $x_0 \in \Omega$. Let S^{n-1} be the unit sphere in $\mathbf{R}^n$ centered at 0. For each $s \in S^{n-1}$ we define

$$\gamma_g(x_0)(s) := \lim_{t\to 0+} g(x_0 + ts). \tag{4}$$

Let us show that the function $\gamma_g(x_0)(\cdot)$ is defined everywhere on S^{n-1}. Let $s \in S^{n-1}$ be given. We denote $L = \{x_0 + ts : 0 < t < 1\}$. Let $\mathcal{F}_L := \{A \cap L : A \in \mathcal{F}\}$. Then, as follows from the definition of $\mathcal{F}$, $\mathcal{F}_L$ is an appropriate family for the interval L. By virtue of the remark above on the algebra $\mathbf{G}(\Omega)$ for $n = 1$, the limit (4) exists.

We denote $g(x)(\cdot) := \gamma_g(x)(\cdot)$. We call $g(x)(\cdot) : S^{n-1} \mapsto \mathbf{R}$ the *surrounding value* of g at x. If g is continuous at x, then $g(x)(\cdot) \equiv g(x)$, i.e., the surrounding value is identically equal to the ordinary value.

Example 1. If $n = 1$, then $S^0 = \{-1, 1\}$, and the surrounding value can be identified with the ordered pair of one-sided limits. If $n = 2$, then we identify S^1 and the interval $[0, 2\pi)$, so the surrounding value is the mapping $[0, 2\pi) \mapsto \mathbf{R}$. For $n = 3$, the surrounding value is the mapping of the unit sphere $S^2 \mapsto \mathbf{R}$.

Remark 1. Consideration of the surrounding value of a regulated function at a point of discontinuity allows us (roughly speaking) to estimate the values of the function in a sufficiently small neighborhood, which is important for further construction of the space of distributions with discontinuous test functions.

Lemma 2. *For every* $x \in \Omega$, *the surrounding value* $g(x)(\cdot) \in \mathbf{L}^{\infty}(S^{n-1})$.

Proof. Suppose that $x \in \Omega$ is given. Clearly, $g(x)(\cdot)$ is bounded. Let $\{g_k\}_{k=1}^{\infty} \subset \mathbf{PC}(\Omega)$ be the sequence in the formulation of Lemma 1. Then $g_k(x)(\cdot)$ is Lebesgue measurable on S^{n-1}. Due to the uniform convergence (3) we may change the order of limits in (3) and (4), so

$$\sup_{s \in S^{n-1}} \big(g_k(x)(s) - g(x)(s)\big) \longrightarrow 0. \tag{5}$$

Consequently, $g(x)(\cdot)$ is Lebesgue measurable on S^{n-1} as the limit of a uniformly convergent sequence of the Lebesgue measurable functions. As a result, $g(x)(\cdot) \in \mathbf{L}^{\infty}(S^{n-1})$. □

Let $J = \{g \in \widehat{\mathbf{G}}(\Omega) : g(x)(\cdot) = 0 \ (x \in \Omega)\}$. Then J is a closed ideal in $\widehat{\mathbf{G}}(\Omega)$. We define the factor-algebra $\mathbf{G}(\Omega) = \widehat{\mathbf{G}}(\Omega)/J$, so, every element of the algebra $\mathbf{G}(\Omega)$ is uniquely determined by its surrounding values on Ω. The algebra $\mathbf{G}(\Omega)$ is endowed with the norm

$$\|g\|_{\mathbf{G}(\Omega)} = \sup_{x \in \Omega}\{\|g(x)(\cdot)\|_{\mathbf{L}^{\infty}(S^{n-1})}\}.$$

Lemma 3. *The mapping* $\mathbf{G}(\Omega) \to \mathbf{L}^{\infty}(S^{n-1}) : g \to g(x)(\cdot)$ *is the continuous algebra homomorphism.*

Proof. As follows from the arithmetic properties of the limit, this mapping is a homomorphism. The continuity follows from (5), see the proof of Lemma 2. □

Let us define the set of points of discontinuity of $g \in \mathbf{G}(\Omega)$ by $T(g) = \{x \in \Omega : g(x)(\cdot) \not\equiv \text{const in } \mathbf{L}^{\infty}(S^{n-1})\}$. In what follows, by the use of notation $g(x)$ we assume that $x \in \Omega \setminus T(g)$.

Let us define the support $\operatorname{supp}(g) = \operatorname{cl}\{x \in \Omega : g(x)(\cdot) \neq 0 \text{ in } \mathbf{L}^{\infty}(S^{n-1})\}$.

Lemma 4. $\mathbf{G}(\Omega)$ *is the Banach algebra.*

Proof. Since $\mathbf{G}(\Omega) = \widehat{\mathbf{G}}(\Omega)/J$, where J is a closed ideal, and $\widehat{\mathbf{G}}(\Omega)$ is Banach [**3**], according to a known statement we have that the factor-algebra $\mathbf{G}(\Omega)$ is also Banach. □

Let us denote by $\mathbf{PC}(\Omega) \hookrightarrow \mathbf{G}(\Omega)$ the image of the algebra of piecewise-constant functions under the canonical mapping $\widehat{\mathbf{G}}(\Omega) \to \mathbf{G}(\Omega)$, where the notation $\hookrightarrow$ stands for the embedding, i.e., the injective map preserving the linear and the topological structure.

Let $\mathbf{C}(\Omega) \hookrightarrow \mathbf{G}(\Omega)$ be the algebra of continuous elements of $\mathbf{G}(\Omega)$.

Lemma 5. *The closure of* $\mathbf{PC}(\Omega)$ *coincides with* $\mathbf{G}(\Omega)$.

Proof. Let $\hat{g} \in \widehat{\mathbf{G}}(\Omega)$ be given. Let $g \in \mathbf{G}(\Omega)$ be the corresponding equivalence class of $\hat{g}$ in $\mathbf{G}(\Omega)$. Then $\|g\|_{\mathbf{G}(\Omega)} \leqslant \|\hat{g}\|_{\widehat{\mathbf{G}}(\Omega)}$ by the definition of the norm in the factor-algebra $\mathbf{G}(\Omega)$. It suffices to apply Lemma 1 to complete the proof. □

Lemma 6. *Suppose that* $g \in \mathbf{G}(\Omega)$. *Then* $T(g) \subset \cup_{k=1}^{\infty} \partial S_k$ *for certain* $S_k \in \mathcal{F}$.

Proof. Let us denote by $\widehat{T}(\hat{g})$ the set of points of discontinuity of a representative $\hat{g} \in \widehat{\mathbf{G}}(\Omega)$. Clearly, $T(g) \subset \widehat{T}(\hat{g})$. According to [**3**] there exist $S_k \in \mathcal{F}$, $k \in \mathbf{N}$, such that $\widehat{T}(\hat{g}) = \cup_{k=1}^{\infty} \partial S_k$, so $T(g) \subset \cup_{k=1}^{\infty} \partial S_k$. □

Lemma 6 implies that, if $n = 1$, then the set $T(g)$ of points of discontinuity of $g \in \mathbf{G}(\Omega)$ is at most countable.

3. Distributions. Let $\mathcal{D}(\Omega)$ be the space of functions $\varphi \in \mathbf{C}(\Omega)$ having compact support $\operatorname{supp}(\varphi) \subset \Omega$ and endowed with the standard locally-convex topology [**16**] (the space of *continuous test functions*).

Let $\mathcal{R}(\Omega)$ be the space of functions $\varphi \in \mathbf{G}(\Omega)$ having compact support $\operatorname{supp}(\varphi) \subset \Omega$ (the space of *discontinuous test functions*). The basis of neighborhoods of zero in $\mathcal{R}(\Omega)$ consists of the neighborhoods $U_\gamma = \{\varphi \in \mathcal{R}(\Omega) : |\varphi(x)(\cdot)| < \gamma(x)\ (x \in \Omega)\}$, where $\gamma \in \mathbf{C}(\Omega)$, $\gamma(x) > 0$ for all $x \in \Omega$. Here $\varphi(x)(\cdot)$ is the surrounding value of $\varphi \in \mathcal{R}(\Omega)$ at the point $x \in \Omega$; we put $|\varphi(x)(\cdot)| < \gamma(x)$ if $|\varphi(x)(s)| < \gamma(x)$ for every $s \in S^{n-1}$. As follows from the definition of the topology in $\mathcal{D}(\Omega)$ [**16**], we have $\mathcal{D}(\Omega) \hookrightarrow \mathcal{R}(\Omega)$.

Theorem 1. *The space $\mathcal{R}(\Omega)$ is a locally-convex topological vector space.*

Proof. Let us show that, for any two neighborhoods U_{γ_1}, U_{γ_2}, there exists a neighborhood U_{γ_3} such that $U_{\gamma_3} \subset U_{\gamma_1} \cap U_{\gamma_2}$. Clearly, $U_{\gamma_1} \cap U_{\gamma_2}$ consists of the test functions $\varphi \in \mathcal{R}(\Omega)$ such that $|\varphi(x)(\cdot)| < \gamma_1(x)$, $|\varphi(x)(\cdot)| < \gamma_2(x)$ for every $x \in \Omega$. So, it suffices to put $\gamma_3(x) = \min\{\gamma_1(x), \gamma_2(x)\}$, $x \in \Omega$, where $\gamma_3 \in \mathbf{C}(\Omega)$, $\gamma_3(x) > 0$ for every $x \in \Omega$ and $U_{\gamma_3} = U_{\gamma_1} \cap U_{\gamma_2}$.

Further, given $\lambda \in \mathbf{R}$, $|\lambda| \leqslant 1$, and a neighborhood U_γ, we have $\lambda U_\gamma \subset U_\gamma$ since, for each $\varphi \in U_\gamma$, $|\lambda\varphi(x)(\cdot)| = |\lambda||\varphi(x)(\cdot)| \leqslant |\varphi(x)(\cdot)| < \gamma(x)$ for every $x \in \Omega$.

Suppose that $C \subset \Omega$ is compact. Let $\varphi \in \mathcal{R}(\Omega)$, $\operatorname{supp}(\varphi) \subset C$, and $\gamma \in \mathbf{C}(\Omega)$, $\gamma > 0$, be given. Since $\min_{x\in C}\{|\gamma(x)|\} > 0$, we may define $\lambda = \max_{x\in C}\{\|\varphi(x)(\cdot)\|_{\mathbf{L}^\infty(S^{n-1})}\}/\min_{x\in C}\{|\gamma(x)|\} \geqslant 0$. Then, clearly, $\varphi \in \mu U_\gamma$ for every $\mu \in \mathbf{R}$, $|\mu| \geqslant \lambda$.

Also, for every neighborhood U_γ there exists a neighborhood $U_{\gamma'}$ such that $U_{\gamma'} + U_{\gamma'} \subset U_\gamma$. Indeed, we may put $\gamma' = \gamma/2$. According to [7] the space $\mathcal{R}(\Omega)$ is a topological vector space.

Further, consider a sequence of neighborhoods $\{U_{\gamma_k}\}_{k=1}^\infty$, where $\gamma_k \equiv 1/k$ on Ω. Then $\cap_{k=1}^\infty U_{\gamma_k} = \{0\}$, so according to [7] the space $\mathcal{R}(\Omega)$ is Hausdorff.

Given a neighborhood U_γ and the test functions $\varphi, \psi \in U_\gamma$, we have $|\lambda\varphi(x)(\cdot) + (1-\lambda)\psi(x)(\cdot)| \leqslant \lambda|\varphi(x)(\cdot)| + (1-\lambda)|\psi(x)(\cdot)| \leqslant \lambda\gamma(x) + (1-\lambda)\gamma(x) = \gamma(x)$, $x \in \Omega$, for every $0 \leqslant \lambda \leqslant 1$, so U_γ is convex. Since $\mathcal{R}(\Omega)$ is Hausdorff, $\mathcal{R}(\Omega)$ is locally-convex. □

The same argument as for the space $\mathcal{D}'$ [**16**] allows us to show that $\varphi_k \to \varphi$ in $\mathcal{R}(\Omega)$ if and only if there exists a compact subset $C \subset \Omega$ such that $\operatorname{supp}(\varphi_k) \subset C$, $k \in \mathbf{N}$, and $\varphi_k \to \varphi$ in $\mathbf{G}(\Omega)$.

Let $\mathcal{D}'(\Omega)$ and $\mathcal{R}'(\Omega)$ be the spaces topologically adjoint to $\mathcal{D}(\Omega)$ and $\mathcal{R}(\Omega)$ (over $\mathbf{R}$), respectively. By definition, the elements of $\mathcal{D}'(\Omega)$ and $\mathcal{R}'(\Omega)$ are the linear continuous functionals defined on $\mathcal{D}(\Omega)$ and $\mathcal{R}(\Omega)$, respectively. The elements of $\mathcal{D}'(\Omega)$ and $\mathcal{R}'(\Omega)$ are called the *distributions*.

Theorem 2. *Every distribution in $\mathcal{D}'(\Omega)$ admits a linear continuous extension from $\mathcal{D}(\Omega)$ to $\mathcal{R}(\Omega)$.*

Proof. We have the embedding $\mathcal{D}(\Omega) \hookrightarrow \mathcal{R}(\Omega)$. According to Theorem 1 the space $\mathcal{R}(\Omega)$ is locally-convex, so the linear and continuous extension exists by the Hanh-Banach theorem [**7**]. □

The space $\mathcal{R}'(\Omega)$ is endowed with linear operations and weak* topology, so $f_k \to f$ in $\mathcal{R}'(\Omega)$ if and only if $(f_k, \varphi) \to (f, \varphi)$ for every $\varphi \in \mathcal{R}(\Omega)$.

Example 2. Let $f \in \mathbf{L}^1_{\text{loc}}(\Omega)$. Let us define the *regular distribution* $f \in \mathcal{R}'(\Omega)$ by the formula

$$(f, \varphi) = \int_\Omega f(x)\varphi(x)\,dx, \quad \varphi \in \mathcal{R}(\Omega). \tag{6}$$

Since $\mathcal{D}(\Omega) \hookrightarrow \mathcal{R}(\Omega)$ and the canonical mapping $\mathbf{L}^1_{\text{loc}}(\Omega) \to \mathcal{D}'(\Omega)$ is injective [**16**], we may identify the elements of $\mathbf{L}^1_{\text{loc}}(\Omega)$ and the regular distributions in $\mathcal{R}'(\Omega)$.

Example 3. Suppose that $p \in \Omega$. Let $\alpha \in \mathbf{L}^1(S^{n-1})$, $\int_{S^{n-1}} \alpha(s)\,ds = 1$ be given. We define

$$(\delta_p^\alpha, \varphi) = \int_{S^{n-1}} \alpha(s)\varphi(p)(s)\,ds, \tag{7}$$

where $\varphi(p)(\cdot)$ is the surrounding value of the test function $\varphi \in \mathcal{R}(\Omega)$ at $p \in \Omega$. The linearity and continuity of the functional δ_τ^α follows from Lemma 3, so $\delta_p^\alpha \in \mathcal{R}'(\Omega)$. We call $\delta_p^\alpha \in \mathcal{R}'(\Omega)$ the *delta-function possessing the characteristics of the shape* α. For any $\varphi \in \mathcal{D}(\Omega)$ we have that the surrounding value $\varphi(p)(\cdot) \equiv \varphi(p)$, so

$$(\delta_p^\alpha, \varphi) = \int_{S^{n-1}} \alpha(s)\varphi(p)\,ds = \varphi(p).$$

Thus, $\delta_p^\alpha \in \mathcal{R}'(\Omega)$ is an extension of $\delta_p \in \mathcal{D}'(\Omega)$ from $\mathcal{D}(\Omega)$ to $\mathcal{R}(\Omega)$.

Example 4. Let $n = 1$, $\beta = \alpha(1)$ (see Example 1). Then $(\delta_p^\alpha, \varphi) = \beta\varphi(p+) + (1-\beta)\varphi(p-)$. We define the *right delta-function* by $(\delta_p^+, \varphi) = \varphi(p+)$, and we define the *left delta-function* by $(\delta_p^-, \varphi) = \varphi(p-)$, so

$$\delta_p^\alpha = \beta\delta_p^+ + (1-\beta)\delta_p^-.$$

Example 5. Let $n = 1$, $\beta \in \mathbf{R}$. As follows from the properties of the integral, the sequence $\{\omega_k\}_{k=1}^\infty$,

$$\omega_k := k\Big(\beta\chi_{(p,p+(1/k))} + (1-\beta)\chi_{(p-(1/k),p)}\Big),$$

converges to a delta-function, i.e., $\omega_k \to \delta_p^\alpha$, where $\beta = \alpha(1)$ (see Example 4); (this explains the use of the term "characteristics of the shape of the delta-function").

The derivative of $f \in \mathcal{R}'(\Omega)$ is a distribution $f' \in \mathcal{R}'(\Omega)$ such that $f'|_{\mathcal{D}(\Omega)} = \big(f|_{\mathcal{D}(\Omega)}\big)' \in \mathcal{D}'(\Omega)$.

Example 6. Let $n = 1$. Since $\delta_p^\alpha|_{\mathcal{D}(\Omega)} = \delta_p \in \mathcal{D}'(\Omega)$, and $\theta_p' = \delta_p$ in $\mathcal{D}'(\Omega)$, we have that $\delta_p^\alpha = \theta_p'$ for any characteristics of the shape α (so, the operation of differentiation in $\mathcal{R}'(\Omega)$ is multi-valued). Analogously,

$$\delta_p^\alpha + c(\delta_q^+ - \delta_q^-) = \theta_p'$$

for every $q \in \Omega$, $c \in \mathbf{R}$. Note that for any $q \in \Omega$, $c \in \mathbf{R}$, there exists a sequence $\{f_k\}_{k=1}^\infty$ of continuously differentiable functions $f_k : \Omega \to \mathbf{R}$ such that $f_k \to \theta_p$ and $f_k' \to \delta_p^\beta + c(\delta_q^+ - \delta_q^-) = \theta_p'$.

We call the support $\operatorname{supp}(f) \subset \Omega$ of $f \in \mathcal{R}'(\Omega)$ the minimal closed set such that for every test function $\varphi \in \mathcal{R}(\Omega)$ possessing the property $\operatorname{supp}(f) \cap \operatorname{supp}(\varphi) = \varnothing$ we have the equality $(f, \varphi) = 0$.

The distribution $f \in \mathcal{R}'(\Omega)$ is called nonnegative ($f \geqslant 0$), if $(f, \varphi) \geqslant 0$ for any $\varphi \geqslant 0$, $\varphi \in \mathcal{R}(\Omega)$. The regular distribution $f \in \mathcal{R}'(\Omega)$ is nonnegative if and only if $f(\cdot) \geqslant 0$ in $\mathbf{L}^1_{\mathrm{loc}}(\Omega)$. The latter follows from the embedding $\mathcal{D}(\Omega) \hookrightarrow \mathcal{R}(\Omega)$ and the fact that the analogous statement is true for $\mathcal{D}'(\Omega)$ [**16**].

The proof of the next lemma is similar to the proof of the analogous statement for the space $\mathcal{D}'(\Omega)$ [**16**].

Lemma 7. *If* $\{f_k\}_{k=1}^{\infty}$ *converges in* $\mathcal{R}'(\Omega)$, $\varphi_k \to 0$ *in* $\mathcal{R}(\Omega)$, *then* $(f_k, \varphi_k) \to 0$.

Proof. Suppose the contrary. Then we may assume that (let us consider a subsequence, if necessary) there exists a $c > 0$ such that $|(f_k, \varphi_k)| \geqslant c > 0$, $k \in \mathbf{N}$. Since $\varphi_k \to 0$ in $\mathcal{R}(\Omega)$, then we may suppose (also, if necessary, let us consider a subsequence) that $\|\varphi_k\|_{\mathbf{G}(\Omega)} \leqslant 1/(4^k)$. We define $\zeta_k = 2^k \varphi_k \in \mathcal{R}(\Omega)$. Then we have

$$\|\zeta_k\|_{\mathbf{G}(\Omega)} \leqslant \frac{1}{2^k}, \tag{8}$$

so $\zeta_k \to 0$ in $\mathcal{R}(\Omega)$, though

$$|(f_k, \zeta_k)| = 2^k |(f_k, \varphi_k)| \geqslant 2^k c \to \infty.$$

Let us choose f_{k_1}, ζ_{k_1} so that $|(f_{k_1}, \zeta_{k_1})| > 1$. Suppose that f_{k_j}, ζ_{k_j} are chosen, $1 \leqslant j \leqslant l-1$. Then, for $k \geqslant k'$, $|(f_{k_j}, \zeta_k)| < 1/(2^{l-j})$, $1 \leqslant j \leqslant l-1$. There exists $k_l \geqslant k'$ such that

$$|(f_{k_l}, \zeta_{k_l})| > \sum_{j=1}^{l-1} |(f_{k_l}, \zeta_{k_j})| + l \tag{9}$$

since $|(f_k, \zeta_k)| \to \infty$, $(f_k, \zeta_{k_j}) \to 0$, $k \to \infty$. Suppose that the sequence $\{\zeta_{k_l}\}_{l=1}^{\infty}$ is constructed. Let us define $\zeta = \sum_{j=1}^{\infty} \zeta_{k_j}$, where the series converges due to (8), so $\zeta \in \mathcal{R}(\Omega)$. Consequently,

$$(f_{k_l}, \zeta) = \sum_{j=1}^{l-1} (f_{k_l}, \zeta_{k_j}) + (f_{k_l}, \zeta_{k_l}) + \sum_{l+1}^{\infty} (f_{k_l}, \zeta_{k_j}).$$

Since (9) holds and

$$\sum_{j=l+1}^{\infty} (f_{k_l}, \zeta_{n_j}) < \sum_{j=l+1}^{\infty} \frac{1}{2^{j-l}} = 1,$$

we obtain that $|(f_{k_l}, \zeta)| > l-1$. This contradicts $\lim_{k\to\infty} (f_k, \zeta) = (f, \zeta)$, where $f = \lim_{k\to\infty} f_k$. □

Let us define the product of $f \in \mathcal{R}'(\Omega)$ and $g \in \mathbf{G}(\Omega)$ by the equality

$$(gf, \varphi) := (f, g\varphi), \tag{10}$$

where $\varphi \in \mathcal{R}(\Omega)$ and, clearly, $g\varphi \in \mathcal{R}(\Omega)$. The operation of multiplication defined by (10) is commutative and associative in the sense that $(gh)f = g(hf)$ for any g, $h \in \mathbf{G}(\Omega)$, $f \in \mathcal{R}'(\Omega)$.

Example 7. Let $f \in \mathcal{R}'(\Omega)$ be a regular distribution and let $g \in \mathbf{G}(\Omega)$. Then

$$(gf, \varphi) := (f, g\varphi) = \int_\Omega f(x)g(x)\varphi(x)\,dx,$$

i.e., for the regular distributions the operation of multiplication in $\mathcal{R}'(\Omega)$ coincides with the ordinary operation of multiplication.

Theorem 3. *Suppose that $g_k \to g$ in $\mathbf{G}(\Omega)$, $f_k \to f$ in $\mathcal{R}'(\Omega)$. Then $g_k f_k \to gf$ in $\mathcal{R}'(\Omega)$.*

Proof. Let us note that $g_k\varphi \to g\varphi$ in $\mathcal{R}(\Omega)$ for every $\varphi \in \mathcal{R}(\Omega)$. Consequently,

$$\begin{aligned} |(g_k f_k, \varphi) - (gf, \varphi)| &= |(f_k, g_k\varphi) - (f, g\varphi)| \\ &\leqslant |(f_k, g_k\varphi) - (f_k, g\varphi)| + |(f_k, g\varphi) - (f, g\varphi)| \\ &\leqslant |(f_k, g_k\varphi - g\varphi)| + |(f_k, g\varphi) - (f, g\varphi)| \to 0, \end{aligned}$$

by virtue of Lemma 7 and convergence $f_k \to f$ in $\mathcal{R}'(\Omega)$. □

Example 8. Let $n = 1$, $p \in \Omega$. Let us define the *Heaviside function* $\theta_p \in \mathbf{G}(\Omega)$ by the equalities $\theta_p(x) = 1$, $x > p$, $\theta_p(x) = 0$, $x < p$. Then

$$(\theta_p\delta_p^\alpha, \varphi) := (\delta_p^\alpha, \theta_p\varphi) = \beta\theta_p(p+)\varphi(p+) + (1-\beta)\theta_p(p-)\varphi(p-) = \beta\varphi(p+),$$

i.e., $\theta_p\delta_p^\alpha = \beta\delta_p^+$, where $\beta = \alpha(1)$, see Example 1, Example 4 and the Introduction.

Example 9. Let $n = 2$, $p \in \Omega$. Let us find the product of the function $g \in \mathbf{G}(\Omega)$,

$$g(x) = \begin{cases} \mu & x^1 > p^1,\, x^2 > p^2, \\ \nu & x^1 < p^1,\, x^2 < p^2, \\ 0 & \text{otherwise}, \end{cases}$$

where $x = (x^1, x^2)$, $p = (p^1, p^2) \in \Omega$, and the delta-function $\delta_p^\alpha \in \mathcal{R}'(\Omega)$ ($\alpha : [0, 2\pi) \mapsto \mathbf{R}$, $\int_0^{2\pi} \alpha(s)\, ds = 1$). We have

(11)
$$(g\delta_p^\alpha, \varphi) := (\delta_p^\alpha, g\varphi) = \int_0^{\pi/2} \alpha(s)\varphi(p)(s)\, ds + \int_\pi^{3\pi/2} \alpha(s)\varphi(p)(s)\, ds.$$

Let us denote

$$\rho = \int_0^{\pi/2} \alpha(s)\, ds + \int_\pi^{3\pi/2} \alpha(s)\, ds \in \mathbf{R}.$$

If $\rho \neq 0$, then the equality (11) can be rewritten as

$$g\delta_p^\alpha = \rho\delta_p^\gamma,$$

where $\gamma : [0, 2\pi) \mapsto \mathbf{R}$, $\int_0^{2\pi} \gamma(s)\, ds = 1$ are the characteristics of the shape of the delta-function δ_p^γ given by

$$\gamma(s) = \begin{cases} (\sigma\alpha(s))/\rho & 0 < s < \pi/2, \\ (\mu\alpha(s))/\rho & \pi < s < (3\pi)/2, \\ 0 & \text{otherwise}. \end{cases}$$

Let us define the integral of $f \in \mathcal{R}'(\Omega)$ over $S \in \mathcal{F}_c := \{S \in \mathcal{F} : \overline{S} \subset \Omega\}$ by the formula

$$\int_S f\, dx := (f, \chi_S), \tag{12}$$

where, clearly, $\chi_S \in \mathcal{R}(\Omega)$. The integral (12) exists for every distribution, is linear with respect to the distribution and coincides with the Lebesgue integral for regular distributions in $\mathcal{R}'(\Omega)$.

Theorem 4. *Suppose that $S \in \mathcal{F}_c$, $f_k \to f$ in $\mathcal{R}'(\Omega)$. Then $\int_S f_k\,dx \to \int_S f\,dx$.*

Proof. We have $\int_S f_k\,dx = (f_k, \chi_S) \to (f, \chi_S) = \int_S f\,dx$. □

Example 10. Let $n = 1$, p, $x_0 \in \Omega$, $x_0 < p$. Then

$$\int_{x_0}^{x} \delta_p^\alpha\,ds = \begin{cases} \theta(x) & x \neq p, \\ 1 - \alpha(1) & x = p. \end{cases}$$

Example 11. Let $n = 2$, $p \in \Omega$, $B_p^2 \subset \Omega$ is a disk centered at p, and $S_p(r) \subset B_p^2$ is a sector possessing the central angle $r \in [0, 2\pi)$. Then

$$\int_{S_p(r)} \delta_p^\alpha\,dx := (\delta_p^\alpha, \chi_{S_p(r)}) = \int_0^{2\pi} \alpha(s)\chi_{S_p(r)}(p)(s)\,ds = \int_0^r \alpha(s)\,ds,$$

where $r \in [0, 2\pi)$, and $\chi_{S_p(r)}(p)(\cdot)$ is the surrounding value of the characteristic functions $\chi_{S_p(r)}$ at $p \in \Omega$.

Remark 2. As is well known, the definition of the correct operation of multiplication of distributions by the elements of the algebra $\mathbf{G}(\Omega)$, as well as the correct operation of integration of distributions, is impossible in the space $\mathcal{D}'(\Omega)$ [**16**], see the Introduction. Along with that, as follows from Example 7, Theorem 3 and Theorem 4, the operations of multiplication and integration in the space $\mathcal{R}'(\Omega)$ are correct (let us note that the definition of the incorrect operation of integration in the space $\mathcal{D}'(\Omega)$ can be found in [**6**]).

4. The family of zero-sum games with discontinuous payoff functions. Let $\Omega = \Omega_1 \times \Omega_2 \subset \mathbf{R}^2$, where Ω_1, Ω_2 are the open interval. Let us consider the following zero-sum game

$$G = (X_1, X_2, \rho), \tag{13}$$

where X_1, X_2 are the open intervals, $\overline{X}_1 \subset \Omega_1$, $\overline{X}_2 \subset \Omega_2$, $\rho \in \mathbf{G}(\Omega)$.

4.1. Pure and mixed strategies. The elements $x_1 \in X_1$ and $x_2 \in X_2$ such that $(x_1, x_2) \notin T(\rho)$ are called the *pure strategies* of the first and the second player, respectively, the function ρ is called the payoff function of the first player.

Also, we consider game G in the set of the *mixed strategies*, i.e., we consider the game $G^L = (X_1^L, X_2^L, \rho^L)$, where

$$X_1^L := \left\{ u_1 \in \mathbf{L}(X_1) : u_1 \geqslant 0, \int_{X_1} u_1(x_1)\, dx_1 = 1 \right\},$$
$$X_2^L := \left\{ u_2 \in \mathbf{L}(X_2) : u_2 \geqslant 0, \int_{X_2} u_2(x_2)\, dx_2 = 1 \right\}$$

are the sets of the mixed strategies of the first and the second player, respectively, the mapping

$$\rho^L(u_1, u_2) := \int_{X_1 \times X_2} \rho(x_1, x_2) u_1(x_1) u_2(x_2)\, dx_1\, dx_2.$$

is the payoff function of the first player.

As the following example shows, G may have no solution in the set of the pure strategies or in the set of the mixed strategies.

Example 12. Let $X_1 = X_2 = (-1, 1)$,

$$\rho(x_1, x_2) = \begin{cases} 1 & x_1, x_2 > 0,\ x_1 + x_2 < 1 \text{ or} \\ & x_1, x_2 < 0,\ x_1 + x_2 > -1, \\ 0 & \text{otherwise.} \end{cases}$$

The game $G = (X_1, X_2, \rho)$ does not have a solution in the set of pure strategies. Indeed, for any $x_2 \in X_2 \setminus \{0\}$, we have $\sup_{x_1} \rho(x_1, x_2) = 1$, i.e., $\inf_{x_2} \sup_{x_1} \rho(x_1, x_2) = 1$. Similarly, for any $x_1 \in X_1 \setminus \{0\}$ we have $\inf_{x_2} \rho(x_1, x_2) = 0$, so $\sup_{x_1} \inf_{x_2} \rho(x_1, x_2) = 0$. According to [7] the solution of G does not exist.

Let us show that the game G does not have a solution in the set of mixed strategies. Let us define

$$\sigma_{u_1}(x_2) = \int_{X_1} \rho(x_1, x_2) u_1(x_1)\, dx_1,$$

where $u_1 \in X_1^L$. Clearly, the function σ_{u_1} is nonnegative, monotonically increasing on the interval $(-1, 0) \subset X_2$, monotonically decreasing on the interval $(0,1) \subset X_2$ and is such that $\sigma_{u_1}(1-) = \sigma_{u_1}(-1) = 0$. So, for any $\varepsilon > 0$ there exists the strategy $u_2^\varepsilon \in X_2^L$ possessing the support $\operatorname{supp}(u_2^\varepsilon) \subset (1-\varepsilon, 1)$ such that

$$\rho^L(u_1, u_2^\varepsilon) = \int_{X_2} \sigma_{u_1}(x_2) u_2^\varepsilon(x_2)\, dx_2 < \varepsilon.$$

Consequently, $\inf_{u_2} \rho^L(u_1, u_2) = 0$ for any $u_1 \in X_1^L$. Then

$$\sup_{u_1} \inf_{u_2} \rho^L(u_1, u_2) = 0. \tag{14}$$

Analogously, given $u_2 \in X_2^L$, we define

$$\tau_{u_2}(x_1) = \int_{X_2} \rho(x_1, x_2) u_2(x_2)\, dx_2.$$

Clearly, the function τ_{u_2} is nonnegative, monotonically increasing on the interval $(-1,0) \subset X_1$, monotonically decreasing on the interval $(0,1) \subset X_1$, and is such that the equality $\tau_{u_2}(0+) + \tau_{u_2}(0-) = 1$ holds. Consequently, for any $\varepsilon > 0$ the strategy $u_1^\varepsilon \in X_1^L$ exists possessing the support $\operatorname{supp}(u_1^\varepsilon) \subset (-\varepsilon, \varepsilon)$ such that

$$\rho^L(u_1^\varepsilon, u_2) = \int_{X_1} \tau_{u_2}(x_1) u_1^\varepsilon(x_1)\, dx_1 > 1 - \varepsilon.$$

Consequently, $\sup_{u_1} \rho^L(u_1, u_2) = 1$ for any $u_2 \in X_2^L$. Then

$$\inf_{u_2} \sup_{u_1} \rho^L(u_1, u_2) = 1. \tag{15}$$

Comparison of (14) and (15) shows that the game G^L does not a have solution [**7**].

4.2. $\mathcal{R}'$-mixed strategies. In order to provide the existence of a solution, let us consider the game (13) in the set of *$\mathcal{R}'$-mixed strategies*, i.e., let us consider the game $G^R = (X_1^R, X_2^R, \rho^R)$, where

$$X_1^R := \left\{ v_1 \in \mathcal{R}'(\Omega_1) : v_1 \geqslant 0, \int_{X_1} v_1\, dx_1 = 1 \right\},$$

$$X_2^R := \left\{ v_2 \in \mathcal{R}'(\Omega_2) : v_2 \geqslant 0, \int_{X_2} v_2\, dx_2 = 1 \right\}$$

are sets of the $\mathcal{R}'$-mixed strategies of the first and the second player, respectively, the mapping

$$\rho^R(v_1, v_2) := \int_{X_1} \left(\int_{X_2} \rho(x_1, x_2) v_2 \, dx_2 \right) v_1 \, dx_1 = \int_{X_2} \left(\int_{X_1} \rho(x_1, x_2) v_1 \, dx_1 \right) v_2 \, dx_2 \tag{16}$$

is the payoff function of the first player (the definitions of nonnegative distribution and the integral of a distribution in the distribution space $\mathcal{R}'$ were given above).

Definition 1. The elements $v_1 \in X_1^R$, $v_2 \in X_2^R$ such that

$$x_1 \longrightarrow \int_{X_2} \rho(x_1, x_2) v_2 \, dx_2 \in \mathbf{G}(\Omega_1),$$
$$x_2 \longrightarrow \int_{X_1} \rho(x_1, x_2) v_1 \, dx_1 \in \mathbf{G}(\Omega_2)$$

and the equality in (16) hold are called *$\mathcal{R}'$-mixed strategies.*

We call the mapping ρ^R the payoff function of the first player. Let us note that $X_1^L \subset X_1^R$, $X_2^L \subset X_2^R$, see Example 2, and $\rho^R|_{X_1^L \times X_2^L} = \rho^L$. We denote

$$\rho(x_1^*\pm, x_2) := \lim_{x_1 \to x_1^*\pm} \rho(x_1, x_2), \qquad \rho(x_1, x_2^*\pm) := \lim_{x_2 \to x_2^*\pm} \rho(x_1, x_2).$$

Theorem 5. *Suppose that $\rho \geqslant 0$. If there exists $(x_1^*, x_2^*) \in X_1 \times X_2$ such that*

$$a_\pm^r := \lim_{x_1 \to x_1^*+} \rho(x_1, x_2^*\pm) \geqslant \rho(x_1, x_2^*\pm), \quad x_1 > x_1^*, \tag{17}$$

$$a_\pm^l := \lim_{x_1 \to x_1^*-} \rho(x_1, x_2^*\pm) \geqslant \rho(x_1, x_2^*\pm), \quad x_1 < x_1^*, \tag{18}$$

$$b_\pm^r := \lim_{x_2 \to x_2^*+} \rho(x_1^*\pm, x_2) \leqslant \rho(x_1^*\pm, x_2), \quad x_2 > x_2^*, \tag{19}$$

$$b_\pm^l := \lim_{x_2 \to x_2^*-} \rho(x_1^*\pm, x_2) \leqslant \rho(x_1^*\pm, x_2), \quad x_2 < x_2^*, \tag{20}$$

$$(21) \qquad b^r_+ = a^r_+, \quad b^l_- = a^l_-, \quad b^r_- = a^l_+, \quad b^l_+ = a^r_-,$$

$$(22) \qquad a^r_+ \geqslant a^r_-, \quad a^l_- \geqslant a^l_+, \quad a^l_- \geqslant a^r_-, \quad a^r_+ \geqslant a^l_+,$$

and $a^r_+ - a^l_+ \neq a^r_- - a^l_-$, *then the pair of delta-functions* $\delta^{\alpha_1^*}_{x_1^*} \in X_1^R$, $\delta^{\alpha_2^*}_{x_2^*} \in X_2^R$, *where*

$$\alpha_1^*(1) = \frac{a^l_- - a^l_+}{a^r_+ - a^l_+ - a^r_- + a^l_-}, \qquad \alpha_2^*(1) = \frac{a^l_- - a^r_-}{a^r_+ - a^l_+ - a^r_- + a^l_-},$$

is the solution of the game G in the set of the $\mathcal{R}'$*-mixed strategies.*

Since the function ρ is bounded on Ω, the case $\rho \not\geqslant 0$ can be reduced to the case above by consideration of the payoff function $\rho + C$ for certain $C > 0$.

Proof. 1) Let us show that $\delta^{\alpha_1^*}_{x_1^*}$, $\delta^{\alpha_2^*}_{x_2^*}$ are the $\mathcal{R}'$-mixed strategies, i.e., the value $\rho(\delta^{\alpha_1^*}_{x_1^*}, \delta^{\alpha_2^*}_{x_2^*})$ is correctly defined. Let us denote $A = a^r_+ - a^l_+ - a^r_- + a^l_- \neq 0$. Observe that

$$\int_{X_2} \rho(x_1, x_2)\delta^{\alpha_2^*}_{x_2^*}\, dx_2 = \frac{a^l_- - a^r_-}{A}\rho(x_1, x_2^*+) + \left(1 - \frac{a^l_- - a^r_-}{A}\right)\rho(x_1, x_2^*-) \in \mathbf{G}(\Omega_1).$$

Consequently, we have the following equality:

$$\begin{aligned}\int_{X_1} &\left(\int_{X_2} \rho(x_1, x_2)\delta^{\alpha_2^*}_{x_2^*}\, dx_2\right)\delta^{\alpha_1^*}_{x_1^*}\, dx_1 \\ &= \frac{a^l_- - a^l_+}{A}\left(\frac{a^l_- - a^r_-}{A}a^r_+ + \left(1 - \frac{a^l_- - a^r_-}{A}\right)a^r_-\right) \\ &\quad + \left(1 - \frac{a^l_- - a^l_+}{A}\right)\left(\frac{a^l_- - a^r_-}{A}a^l_+ + \left(1 - \frac{a^l_- - a^r_-}{A}\right)a^l_-\right) \\ &= \frac{a^r_+ a^l_- - a^r_- a^l_+}{A}.\end{aligned}$$

Analogously,

$$\int_{X_1} \rho(x_1,x_2)\delta_{x_1^*}^{\alpha_1^*}\,dx_1 = \frac{a_-^l - a_+^l}{A}\rho(x_1^*+,x_2) + \left(1-\frac{a_-^l - a_+^l}{A}\right)\rho(x_1^*-,x_2) \in \mathbf{G}(\Omega_2).$$

According to (21), we have

$$\begin{aligned}\int_{X_2}\left(\int_{X_1} \rho(x_1,x_2)\delta_{x_1^*}^{\alpha_1^*}\,dx_1\right)\delta_{x_2^*}^{\alpha_2^*}\,dx_2 &= \frac{a_-^l - a_-^r}{A}\left(\frac{a_-^l - a_+^l}{A}b_+^r + \left(1-\frac{a_-^l - a_+^l}{A}\right)b_-^r\right)\\ &\quad + \left(1-\frac{a_-^l - a_-^r}{A}\right)\left(\frac{a_-^l - a_+^l}{A}b_+^l + \left(1-\frac{a_-^l - a_+^l}{A}\right)b_-^l\right)\\ &= \frac{a_+^r a_-^l - a_-^r a_+^l}{A}.\end{aligned}$$

Then by Definition 1, $\delta_{x_1^*}^{\alpha_1^*}$, $\delta_{x_2^*}^{\alpha_2^*}$ are the $\mathcal{R}'$-mixed strategies for G.

2) Observe that, see [7],

$$\inf_{v_2}\sup_{v_1}\rho^R(v_1,v_2) \geqslant \sup_{v_1}\inf_{v_2}\rho^R(v_1,v_2). \tag{23}$$

Let $X_2 = (p_1,p_2) \subset \mathbf{R}$. Then we have the equality

$$\begin{aligned}\rho(\delta_{x_1^*}^{\alpha_1^*},v_2) &= \int_{X_2}\left(\int_{X_1}\rho(x_1,x_2)\delta_{x_1^*}^{\alpha_1^*}\,dx_1\right)v_2\,dx_2\\ &= \int_{X_2}\left(\frac{a_-^l - a_+^l}{A}\rho(x_1^*+,x_2) + \left(1-\frac{a_-^l - a_+^l}{A}\right)\rho(x_1^*-,x_2)\right)v_2\,dx_2\end{aligned} \tag{24}$$

Further, according to the definitions of the product and the integral in the space $\mathcal{R}'$,

$$\begin{aligned}\int_{p_1}^{x_2^*}\rho(x_1^*+,x_2)v_2\,dx_2 &= \int_{p_1}^{x_2^*} b_+^l v_2\,dx_2 + \int_{p_1}^{x_2^*}(\rho(x_1^*+,x_2) - b_+^l)v_2\,dx_2\\ &= \int_{p_1}^{x_2^*} b_+^l v_2\,dx_2 + (v_2, (\rho(x_1^*+,x_2) - b_+^l)\chi_{(p_1,x_2^*)})\\ &\geqslant \int_{p_1}^{x_2^*} b_+^l v_2\,dx_2\end{aligned} \tag{25}$$

since $v_2 \geqslant 0$ in $\mathcal{R}'(\Omega_2)$, $\rho(x_1^*, x_2) \geqslant b_+^l$, $x < x_2^*$, according to (20). Since $\rho \geqslant 0$, $A \neq 0$ and (22) holds, we obtain that $A > 0$. Then (19)–(21), the equality (24) and the argument similar to (25) gives us the inequality

$$\begin{aligned}
\rho(\delta_{x_1^*}^{\alpha_1^*}, v_2) \geqslant & \int_{p_1}^{x_2^*} \left(\frac{a_-^l - a_+^l}{A} b_+^l + \left(1 - \frac{a_-^l - a_+^l}{A}\right) b_-^l \right) v_2 \, dx_2 \\
& + \int_{x_1^*}^{p_2} \left(\frac{a_-^l - a_+^l}{A} b_+^r + \left(1 - \frac{a_-^l - a_+^l}{A}\right) b_-^r \right) v_2 \, dx_2 \\
= & \int_{p_1}^{x_2^*} \left(\frac{a_-^l - a_+^l}{A} a_-^r + \left(1 - \frac{a_-^l - a_+^l}{A}\right) a_-^l \right) v_2 \, dx_2 \\
& + \int_{x_2^*}^{p_2} \left(\frac{a_-^l - a_+^l}{A} a_+^r + \left(1 - \frac{a_-^l - a_+^l}{A}\right) a_+^l \right) v_2 \, dx_2 \\
= & \frac{a_+^r a_-^l - a_+^l a_-^r}{A}.
\end{aligned}$$

for any $\mathcal{R}'$-mixed strategy v_2. Consequently,

$$\inf_{v_2} \rho^R(\delta_{x_1^*}^{\alpha_1^*}, v_2) \geqslant \frac{a_+^r a_-^l - a_+^l a_-^r}{A}, \text{ i.e., } \sup_{v_1} \inf_{v_2} \rho^R(v_1, v_2) \geqslant \frac{a_+^r a_-^l - a_+^l a_-^r}{A}.$$

Analogously, due to inequalities (17), (18) and the equalities (21), we have

$$\inf_{v_2} \sup_{v_1} \rho^R(v_1, v_2) \leqslant \frac{a_+^r a_-^l - a_+^l a_-^r}{A}.$$

Then (23) implies that

$$\max_{v_1} \inf_{v_2} \rho^R(v_1, v_2) = \min_{v_2} \sup_{v_1} \rho^R(v_1, v_2), \tag{26}$$

i.e., the solution of G^R exists, the maximum and the minimum in (26) are attained at

$$v_1^* = \delta_{x_1^*}^{\alpha_1^*} \text{ and } v_2^* = \delta_{x_2^*}^{\alpha_2^*}, \tag{27}$$

respectively. According to [7] the pair (27) is the solution of the game G^R. □

Example 13. Let us consider the game $G = (X_1, X_2, \rho)$ of Example 12. Let us put $(x_1^*, x_2^*) = (0, 0)$. Then the conditions of Theorem 5 are satisfied, where

$$a_+^r = 1, \quad a_+^l = 0, \quad a_-^r = 0, \quad a_-^l = 1,$$
$$b_+^r = 1, \quad b_+^l = 0, \quad b_-^r = 0, \quad b_-^l = 1,$$

so the pair

$$\delta_0^{\alpha_1^*} \in X_1^R, \quad \delta_0^{\alpha_2^*} \in X_2^R, \tag{28}$$

where the characteristics of the shapes α_1^*, α_2^* of the delta-functions are given by

$$\alpha_1^*(1) = \alpha_2^*(1) = \frac{1}{2},$$

is the solution of game G in the set of the $\mathcal{R}'$-mixed strategies.

Let us note that the solution (28) admits approximation by the mixed strategies, see Example 5, and thus, possesses an obvious probabilistic interpretation.

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