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The Adjoint Riemann–Stieltjes Integral

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1. THE ALGEBRA G$[a, b]$ OF DISCONTINUOUS FUNCTIONS

Fix a segment $K = [a, b]$ and denote by $\mathrm{G} = \mathrm{G}[a, b]$ the space of *discontinuous* ([1], p. 16) functions, i. e., the functions $x : K \to \mathbb{C}$ which have finite limits $x(t-0) = \lim\limits_{\tau\to t-0} x(\tau)$ for all $t \in (a, b]$ and $x(t+0) = \lim\limits_{\tau\to t+0} x(\tau)$ for all $t \in [a, b)$. Denote by G_0 the space of functions $x : K \to \mathbb{C}$ such that for each $\varepsilon > 0$ the set $\{t \in K : |x(t)| \geqslant \varepsilon\}$ is finite. The inclusion $\mathrm{G}_0 \subset \mathrm{G}$ is true. A function $x : K \to \mathbb{C}$ is said to be *staircase*, if a finite decomposition $a = \tau_0 < \tau_1 < \cdots < \tau_n = b$ exists such that on each interval (τ_{k-1}, τ_k), $k = 1, \dots, n$, the function x identically equals a constant value $c_k \in \mathbb{C}$. A staircase function is discontinuous.

Theorem 1.1 ([1], p. 16). *If $x \in \mathrm{G}[a, b]$, then x is bounded and measurable, and the space $\mathrm{G}[a, b]$ is a Banach one with respect to the norm $\|x\| = \sup\limits_{t\in[a,b]} |x(t)|$ (moreover, $\mathrm{G}[a, b]$ is a Banach algebra) and represents the closure of the space of staircase functions with respect to the* sup*-norm.*

Theorem 1.2 (ibid., p. 17). *The set $T(x)$ of all discontinuity points of a function $x \in \mathrm{G}[a, b]$, is at most countable.*

Theorem 1.3 ([2]). *For any function $x \in \mathrm{G}[a, b]$ and any continuous function of bounded variation $y \in \mathrm{CBV}[a, b]$ the Riemann–Stieltjes integrals $\int\limits_a^b x\,dy$ and $\int\limits_a^b y\,dx$ exist.*

2. ALGEBRAS $\mathrm{G}^T[a, b]$, $\Gamma[a, b]$, AND BV$[a, b]$

We call an arbitrary finite or countable set $T \doteq \{\tau_1, \tau_2, \dots\}$ of pairwise different points $\tau_k \in K$ a *decomposition* of the segment K; we denote the set of all decompositions of K by $\mathbb{T}(K)$. We also include the empty set into the family $\mathbb{T}(K)$.

Fix $T \in \mathbb{T}(K)$ and for any $x \in \mathrm{G}$ and $\tau_k \in T$ introduce the notation: $x_k^- \doteq x(\tau_k - 0) - x(\tau_k)$, $x_k^+ \doteq x(\tau_k + 0) - x(\tau_k)$. Assume that $x_k^- = 0$ with $\tau_k = a$ and $x_k^+ = 0$ with $\tau_k = b$. Denote by $\lceil x \rceil_T$ the series (and its sum, if the series converges) $\lceil x \rceil_T \doteq \sum\limits_{\tau_k\in T} (|x_k^-| + |x_k^+|)$; let $\mathrm{G}^T \doteq \mathrm{G}^T[a, b]$ stand for the family of all $x \in \mathrm{G}$ such that the series $\lceil x \rceil_T$ converges. With respect to the natural operations of addition and multiplication G^T is an algebra over $\mathbb{C}$.

If T is a finite set, then the equality $\mathrm{G}^T = \mathrm{G}$ is true, in particular, $\mathrm{G}^\emptyset = \mathrm{G}$. Any function of bounded variation belongs to G^T with any $T \in \mathbb{T}(K)$. Between the algebras BV and G^T an algebra $\Gamma \doteq \Gamma[a, b]$ exists, consisting of functions $x \in \mathrm{G}$ such that the series $\lceil x \rceil_{T(x)}$ converges. Therefore, $\mathrm{BV} \subset \Gamma \subset \mathrm{G}^T \subseteq \mathrm{G}$.

For any $\tau_k \in T$ two staircase functions are defined: $\xi_k(t) = -1$ for $t < \tau_k$ and $\xi_k(t) = 0$ for $t \geqslant \tau_k$; $\eta_k(t) = 0$ for $t \leqslant \tau_k$ and $\eta_k(t) = 1$ for $t > \tau_k$. For any $\alpha \in K$ and $x \in \mathrm{G}^T$ the functional series

$$x_T(t) \doteq -\sum_{\tau_k \in T} x_k^- \int_\alpha^t d\xi_k + \sum_{\tau_k \in T} x_k^+ \int_\alpha^t d\eta_k \tag{2.1}$$

converges on K absolutely and uniformly (because the series $\lceil x \rceil_T$ majorizes that composed of the modules). Denote the sum of a series, as the series itself, by $x_T(t)$. For $T = \emptyset$ we put $x_T(t) \equiv 0$. In accordance with [3] (p. 336) the functions of the form (1) are said to be *saltus functions*. Ibid one notes that $x_T \in \mathrm{BV}$ and $\operatorname{Var} x_T = \lceil x \rceil_T$ (we denote by $\operatorname{Var} y$ the overall variation of a function y on the segment K). Together with (1) the function $x^T(t) \doteq x(t) - x_T(t)$ is defined. It is continuous at any point $\tau_k \in T$.

For any $x \in \mathrm{G}^T$ the inclusions $x_T,\ x^T \in \mathrm{G}^T$ are true and each operator

$$P_T : x \to x_T \ \text{ and } \ P^T : x \to x^T$$

is a projector in G^T.

If $x \in \Gamma$ (or BV), then for all T such that $T \supseteq T(x)$ we have $x_T = x_{T(x)}$ and $x^T = x^{T(x)}$; in addition, $x^T \in \mathrm{C}$ (or $x^T \in \mathrm{CBV}$). Denoting $x_c \doteq x_{T(x)}$ and $x^c \doteq x^{T(x)}$, we establish that the representation $x = x^{T(x)} + x_{T(x)}$ with $x \in \mathrm{BV}$ coincides with the known decomposition of the function x into the sum of a continuous component and a saltus function $x = x^c + x_c$. Therefore, in Γ (or in BV) the projectors $P_c : x \to x_c$ and $P^c : x \to x^c$ are also defined.

3. TOPOLOGICAL PROPERTIES

With respect to the sup-norm the family of algebras $\{\mathrm{G}^T\}_{T \in \mathbb{T}(K)}$ contains both complete and incomplete algebras.

Theorem 3.1. *Algebras* $\{\mathrm{G}^T[a,b], \|\cdot\|_T\}$ *and* $\{\Gamma[a,b], \|\cdot\|_\Gamma\}$ *with the norms*

$$\|x\|_T \doteq \|x^T\| + \lceil x \rceil_T = \|x^T\| + \operatorname{Var} x_T \quad \textit{and} \quad \|x\|_\Gamma \doteq \sup_{T \in \mathbb{T}(K)} \|x\|_T = \|x^c\| + \operatorname{Var} x_c$$

are commutative Banach algebras with the unit element.

Remark. The algebra $\mathrm{BV}[a,b]$ with the norm $\|x\|_{\mathrm{BV}} \doteq |x(\alpha)| + \operatorname{Var} x$ is also a commutative Banach algebra with the unit element, and for all $x \in \mathrm{BV}$ and $T \in \mathbb{T}(K)$,

$$\|x\| \leqslant \|x\|_T \leqslant \|x\|_\Gamma \leqslant \|x\|_{\mathrm{BV}}.$$

4. THE ADJOINT MULTIPLICATION AND INTEGRAL IN $\mathrm{G}^T[a,b]$

The projectors $P_T : x \to x_T$ and $P^T : x \to x^T$ are endomorphisms of the space G^T, but not those of the algebra G^T. They become so, if we introduce a new multiplication operation in G^T.

Definition 4.1. If $x, y \in \mathrm{G}^T$, then the function $z \doteq x \cdot y \doteq x^T y^T - x_T y_T$ is said to be the *adjoint product* of functions x and y, and the binary operation "$\cdot$" is said to be the *adjoint multiplication* in G^T.

Theorem 4.1. *The space* $\mathrm{G}^T[a,b]$ *endowed with the operation of the adjoint multiplication is a commutative associative algebra* (*in general, without the unit element*). *It is a Banach one with respect to the norm* $\|\cdot\|_T$.

Theorem 4.2. *Each operator* $P_T : x \to x_T$ *and* $P^T : x \to x^T$ *represents an endomorphism of the algebra* G^T *with the adjoint multiplication. The image* $\operatorname{Im} P_T$ $(= \operatorname{Ker} P^T)$ *and the kernel* $\operatorname{Ker} P_T$ $(= \operatorname{Im} P^T)$ *are bilateral ideals of the algebra. The operators* P_T *and* P^T *are continuous orthogonal* (*with respect to the adjoint multiplication*) *projectors of the algebra.*

Definition 4.2. Let $T \in \mathbb{T}(K)$ and $x, y \in \mathrm{G}^T$. If the integral $\int\limits_K x\,dy$ exists, then the function $\int\limits_\alpha^t x \cdot dy \doteq \int\limits_\alpha^t x^T dy^T - \int\limits_\alpha^t x_T dy_T$ is said to be the *indefinite adjoint integral* of the function x with respect to that y.

Theorem 4.3. *Let $x, y, z \in \mathrm{G}^T$. The existence of one of the integrals $\int\limits_\alpha^t x \cdot dy$ or $\int\limits_\alpha^t y \cdot dx$ implies the existence of the other one and the equality $\int\limits_\alpha^t x \cdot dy + \int\limits_\alpha^t y \cdot dx = x \cdot y|_\alpha^t$. Let the integral $w(t) \doteq \int\limits_\alpha^t y \cdot dz$ exist. Both integrals $\int\limits_\alpha^t x \cdot dw$ and $\int\limits_\alpha^t (x \cdot y) \cdot dz$ either exist or not concurrently. If the integrals exist, then $\int\limits_\alpha^t x(s) \cdot d\Big(\int\limits_\alpha^s y \cdot dz\Big) = \int\limits_\alpha^t (x \cdot y) \cdot dz$.*

5. THE ADJOINT MULTIPLICATION AND INTEGRAL IN $\Gamma[a, b]$ AND IN $\mathrm{BV}[a, b]$

Definition 5.1. If $x, y \in \Gamma$ (or BV), then the function $z \doteq x \circ y \doteq x^c y^c - x_c y_c$ is said to be the *adjoint product* of the functions x and y, and the operation "$\circ$" is said to be the *adjoint multiplication* in Γ (or in BV).

Theorem 5.1. *The space Γ (or BV) endowed with the operation of adjoint multiplication is a commutative associative algebra with the unit element. It is a Banach one with respect to the norm $\|\cdot\|_\Gamma$ (or $\|\cdot\|_{\mathrm{BV}}$).*

Theorem 5.2. *Each operator $P_c : x \to x_c$ and $P^c : x \to x^c$ represents an endomorphism of the algebra Γ (or BV) with the adjoint multiplication. The image $\operatorname{Im} P_c$ $(= \operatorname{Ker} P^c)$ and the kernel $\operatorname{Ker} P_c$ $(= \operatorname{Im} P^c)$ are bilateral ideals of the algebra. The operators P_c and P^c are continuous orthogonal (with respect to the adjoint multiplication) projectors.*

Definition 5.2. Let $x, y \in \Gamma$ (or BV). If the integral $\int\limits_K x\,dy$ exists, then the function $\int\limits_\alpha^t x \circ dy \doteq \int\limits_\alpha^t x^c dy^c - \int\limits_\alpha^t x_c dy_c$ is said to be the *indefinite adjoint integral* of the function x with respect to that y.

Theorem 5.3. *Let $x, y, z \in \Gamma$ (or BV). The existence of one of the integrals $\int\limits_\alpha^t x \circ dy$ or $\int\limits_\alpha^t y \circ dx$ implies the existence of the other one and the equality $\int\limits_\alpha^t x \circ dy + \int\limits_\alpha^t y \circ dx = x \circ y\big|_\alpha^t$. Let $w(t) \doteq \int\limits_\alpha^t y \circ dz$ exist. Both integrals $\int\limits_\alpha^t x \circ dw$ and $\int\limits_\alpha^t (x \circ y) \circ dz$ either exist or not concurrently. If the integrals exist, then $\int\limits_\alpha^t x(s) \circ d\Big(\int\limits_\alpha^s y \circ dz\Big) = \int\limits_\alpha^t (x \circ y) \circ dz$.*

6. DISCONTINUOUS FUNCTIONS DEFINED ON AN INTERVAL

Fix an interval $K \doteq (a, b)$ (not necessarily bounded) and denote by $\mathrm{G} \doteq \mathrm{G}(a, b)$ the space (the algebra) of *discontinuous* functions, i. e., the functions $x : K \to \mathbb{C}$ which have finite limits $x(t - 0)$ and $x(t + 0)$ for all $t \in K$. Let $\mathrm{G}_0^{\mathrm{loc}} \doteq \mathrm{G}_0^{\mathrm{loc}}(a, b)$ stand for the algebra of functions $x : K \to \mathbb{C}$ such that for any segment $[\alpha, \beta] \subset K$ the contraction $x : [\alpha, \beta] \to \mathbb{C}$ belongs to $\mathrm{G}_0[\alpha, \beta]$. The inclusion $\mathrm{G}_0^{\mathrm{loc}} \subset \mathrm{G}$ is true. We call the functions $x, y \in \mathrm{G}$ *equivalent* ($x \sim y$), if $x - y \in \mathrm{G}_0^{\mathrm{loc}}$. Denote by $\mathrm{H}^{\mathrm{loc}}$ the algebra of functions $x : K \to \mathbb{C}$ such that for any $[\alpha,\beta] \subset K$ the contraction $x : [\alpha,\beta] \to \mathbb{C}$ is a saltus function. For any $M \subseteq K$ the algebra $\mathrm{H}^{\mathrm{loc}}[M]$ consists of $x \in \mathrm{H}^{\mathrm{loc}}$ such that $T(x) \subseteq M$.

We call a finite or countable set $T \doteq \{\tau_1, \tau_2, \dots\}$ of pairwise different points $\tau_k \in K$ a *decomposition* of the interval K and denote the totality of all decompositions of K by $\mathbb{T}(K)$. Let $\mathrm{G}^T_{\mathrm{loc}}$ (Γ^{loc}) stand for the algebra of functions $x : K \to \mathbb{C}$ such that for any segment $[\alpha,\beta] \subset K$ the contraction $x : [\alpha,\beta] \to \mathbb{C}$ belongs to the algebra $\mathrm{G}^S[\alpha,\beta]$ (respectively, $\Gamma[\alpha,\beta]$), where $S \doteq T \cap [\alpha,\beta]$.

7. GENERALIZED DISCONTINUOUS FUNCTIONS

We call the space $D \doteq D(a,b)$ of finite functions of the space $\mathrm{CBV}^{\mathrm{loc}}(a,b)$ the space of *principal* functions. We say that a sequence of principal functions $\{\varphi_n\}$, $\varphi_n \in D$, converges to a principal function $\varphi \in D$ (we write $\varphi_n \xrightarrow{D} \varphi$), if all functions φ_n and φ have a common support $[\alpha,\beta] \subset K$ and $\operatorname*{Var}_{[\alpha,\beta]}(\varphi_n - \varphi) \underset{n}{\to} 0$. Let D' stand for the space of linear continuous functionals $\ell : D(a,b) \to \mathbb{C}$, we call its elements *generalized functions*. For any $x \in \mathrm{G}(a,b)$ the generalized functions $\varphi \to (x,\varphi) \doteq \int_K \varphi(t)x(t)dt$ and $\varphi \to (x',\varphi) \doteq \int_K \varphi\, dx$ are defined.

Theorem 7.1. *Let $x \in \mathrm{G}(a,b)$. The equality $(x',\varphi) = 0$ is true for all $\varphi \in D(a,b)$ if and only if $x \sim \mathrm{const}$.*

8. ADJOINT GENERALIZED FUNCTIONS

Fix a decomposition $T \in \mathbb{T}(K)$. For any $x \in \mathrm{G}^T_{\mathrm{loc}}$ the linear continuous functional $\varphi \to (\dot{x},\varphi) \doteq (\dot{x},\varphi)^T \doteq \int_K \varphi \cdot dx$ is defined. Since φ is continuous, we have $\varphi_T = 0$, therefore $\int_K \varphi \cdot dx = \int_K \varphi\, dx^T$, and the identity $(\dot{x},\varphi) \equiv 0$ is true if and only if $x^T \sim \mathrm{const}$. For $T = \emptyset$ we have $\dot{x} = x'$.

For functions x from Γ^{loc} (or $\mathrm{BV}^{\mathrm{loc}}$) and $\varphi \in D$ the adjoint integral $(\overset{\circ}{x},\varphi) \doteq \int_K \varphi \circ dx$ exists which generates the functional $\varphi \to (\overset{\circ}{x},\varphi)$ in D'. Due to the continuity of φ the equality $\int_K \varphi \circ dx = \int_K \varphi\, dx^c$ is true and the identity $(\overset{\circ}{x},\varphi) \equiv 0$ takes place if and only if $x^c = \mathrm{const}$.

Theorem 8.1. *Let $\alpha \in K$, $Q \in \mathrm{BV}^{\mathrm{loc}}$; let A be a quadratic matrix of the order n with elements $A_{ij} \in \mathbb{C}$. For an operator $\mathrm{V} : X^n \to \Gamma^{\mathrm{loc}}_n$ such that $X = \{z \in \Gamma^{\mathrm{loc}} : T(z) \cap T(Q) = \emptyset\}$, $(\mathrm{V}x)(t) \doteq x(t) - \int_\alpha^t Ax\, dQ$, and for any $y \in \Gamma^{\mathrm{loc}}_n$ the family of solutions to the equation $\int_K \varphi \circ d\mathrm{V}x \equiv \int_K \varphi \circ dy$ has the form*

$$x(t) = e^{AQ^c(t)}\left[h(t) + \int_\alpha^t e^{-AQ^c(\cdot)} dy^c\right] \ \forall\, h \in \mathrm{H}^{\mathrm{loc}}_n[K \setminus T(Q)].$$

The totality of $x(t) = e^{AQ^c(t)}\left[c + \int_\alpha^t e^{-AQ^c(\cdot)} dy^c\right]$, $c \in \mathbb{C}^n$, represents the family of all continuous solutions to this equation.

AN EXAMPLE

The impulse equation $x' = \delta(t)x$ written in terms of the traditional generalized functions ($\varphi \in \mathrm{C}^\infty$) has the form $-\int_K x(t)\varphi'(t)dt \equiv \int_K \varphi x\, d\theta$ or $\int_K \varphi\, d\mathrm{V}x \equiv 0$, where $(\mathrm{V}x)(t) \doteq x(t) - \int_\alpha^t x\, d\theta$, $\theta(\cdot)$ is the Heaviside function, i. e., $\theta(t) = 0$ with $t \leqslant 0$, and $\theta(t) = 1$, otherwise. The saltus functions continuous at zero ($x \in \mathrm{H}^{\mathrm{loc}}$), and only they, are solutions to the equation $\int_K \varphi \circ d\mathrm{V}x \equiv 0$.

In this paper we do not consider the solvability of the equation $\int_K \varphi \cdot d\mathrm{V}x \equiv \int_K \varphi \cdot dy$.

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