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Quasi-levels of the two-particle discrete Schrödinger operator with a perturbed periodic potential

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Abstract
We consider the two-particle discrete Schrödinger operator H with a periodic potential perturbed by an exponentially decreasing interaction potential. This operator can be considered as the Hamiltonian of the two-magnon states of ferromagnets with periodically arranged impurities. The operator H can be naturally decomposed in the direct integral of spaces that is related to the analogous direct integral for the periodic operator. We show that the essential spectrum of H in the cell coincides with the band spectrum of the corresponding periodic operator. It is proved that for sufficiently small coupling constants there exists a unique quasi-level (an eigenvalue or a resonance) near the non-degenerate stationary points of eigenvalues of the periodic Schrödinger operator with respect to the chosen component of the quasimomentum. The asymptotic behavior of these quasi-levels for the coupling constant tending to zero is investigated. We obtain the simple sufficient condition when a quasi-level is an eigenvalue.

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1. Introduction

We consider the Hamiltonian on $l^2(\mathbb{Z}^2)$ given by

$$H = H_0 + V(n) + W(n_1 - n_2) \tag{1}$$

with $n = (n_1, n_2) \in \mathbb{Z}^2$, where

$$(H_0\psi)(n_1, n_2) = \psi(n_1 + 1, n_2) + \psi(n_1 - 1, n_2) + \psi(n_1, n_2 + 1) + \psi(n_1, n_2 - 1),$$

$V(n)$ is a real function periodic in n_1, n_2 with period $T > 0$ (if $V(n)$ is periodic in n_j with period $T_j > 0$, $j = 1, 2$, then we set $T = T_1T_2$), and $W(n_1)$ is a real function satisfying the estimate

$$|W(n_1)| \leqslant Ce^{-a|n_1|}, \qquad a > 0. \tag{2}$$

We now discuss the physical interpretation of H. A two-magnon state for ferromagnets (or antiferromagnets) may by written in the form

$$\Psi = \sum_{n_1>n_2,\,|n_1-n_2|=1} \psi(n_1, n_2) S^+_{n_1} S^+_{n_2}|0\rangle \tag{3}$$

(see [1]). Here $\psi(n_1, n_2)$ are amplitudes, $S^+_{n_j}$ is the atomic spin creation operator for the atom located at the lattice position n_j, and $|0\rangle$ is the ground state. The state Ψ is the eigenvector of the Heisenberg Hamiltonian [1]. By means of the Bethe ansatz, it can be proved (see [1]) that the function $\psi(n_1, n_2)$ satisfies the discrete Schrödinger equation of the form $H_0\psi = \lambda\psi$, $\lambda \in \mathbb{R}$. Consider now a ferromagnet with periodically arranged impurities. In this case, under some conditions (see [2]), the Schrödinger equation has the form $(H_0 + V(n))\psi = \lambda\psi$, where $V(n) = U(n_1) + U(n_2)$ for a certain periodic function U. Further, in our approach, we have the infinity sum in (3). Therefore instead of boundary conditions (see [1]), we introduce the potential $W(n_1 - n_2)$ describing the interaction between the one-magnon states. So, we obtain the Hamiltonian of the form (1). Note that within this context, the function $\psi(n_1, n_2)$ is symmetric and, consequently, the function $W(n_1)$ should be even. A notion of the quasimomentum (the system momentum) may be rigorously introduced by means of the direct integral decomposition (see section 3).

The spectral properties and the eigenvalues of H with $V = 0$ were investigated in [3, 4] for zero-range interactions. In the continuous case, the eigenvalues of the similar Hamiltonian were studied in [5] also for delta potentials.

The aim of this paper is to investigate the spectrum and the asymptotic behavior of quasi-levels (i.e., eigenvalues and resonances) of the operator H in the cell. We also obtain the simple sufficient condition when a quasi-level is an eigenvalue.

We denote by $\sigma(A)$ and $\sigma_{ess}(A)$ the spectrum and the essential spectrum of the operator A, respectively.

2. Periodic operator

Let $\omega_1 \subset \mathbb{Z}^2$ and let ω_2 be a measurable subset of $\mathbb{R}^m$. We denote by $l^2(\omega_1) \otimes L^2(\omega_2)$ the Hilbert space of all measurable in k functions $\varphi(n, k)$ defined on $\omega_1 \times \omega_2$ such that

$$(\varphi, \varphi) = \sum_{n\in\omega_1} \int_{\omega_2} \varphi(n, k)\overline{\varphi(n, k)}\,\mathrm{d}k < \infty.$$

We apply the direct integral construction (see [6]) to our case. Let us introduce the following unitary operator:

$$U_0 : l^2(\mathbb{Z}^2) \to l^2(\Omega_0) \otimes L^2(\Omega_0^*), \qquad \varphi(n) \mapsto \frac{T}{2\pi} \sum_{m\in\mathbb{Z}^2} \exp[-\mathrm{i}(k, m)T]\varphi(n + Tm). \tag{4}$$

Here $\Omega_0 = [0, 1, \ldots, T - 1]^2$ is the cell of periods and $\Omega_0^* = [-\pi/T, \pi/T)^2$ is the cell in the reciprocal lattice. A vector k is called a *quasimomentum*. We have

$$(U_0\varphi)(n + Tm, k) = \exp[\mathrm{i}(k, m)T](U_0\varphi)(n, k) \tag{5}$$

thus $(U_0\varphi)(n, k)$ is *a Bloch function.*

The direct integral is introduced as

$$\int_{\Omega_0^*}^{\oplus} l^2(\Omega_0)\,\mathrm{d}k = l^2(\Omega_0) \otimes L^2(\Omega_0^*) \cong (L^2(\Omega_0^*))^{T^2}.$$

It is easy to show that $U_0 H_V U_0^{-1} = \{H_V(k)\}_{k\in\Omega_0^*}$ where the operators $H_V(k) = H_0(k) + V(n)$ act on $l^2(\Omega_0)$ similar to the operator H_V (if either $(n_1 \pm 1, n_2)$ or $(n_1, n_2 \pm 1)$ does not belong to Ω_0 then we use (5)). Clearly, $H_V(k)$ is the matrix depending analytically with respect to k.

Consider the eigenvectors of the operator $H_0(k)$ of the form

$$\psi_m(n,k) = \frac{1}{T}\exp\left[\mathrm{i}\left(k + \frac{2\pi m}{T}, n\right)\right], \qquad m \in \Omega_0$$

corresponding to the eigenvalues

$$\lambda_m(k) = 2\left[\cos\left(k_1 + \frac{2\pi m_1}{T}\right) + \cos\left(k_2 + \frac{2\pi m_2}{T}\right)\right].$$

These vectors form the orthogonal basis in $l^2(\Omega_0)$. Therefore the Green function (the matrix of the resolvent $R_0(k,\lambda) = [H_0(k) - \lambda]^{-1}$ of $H_0(k)$) is given by

$$G_0(n-m,k,\lambda) = \frac{1}{T}\sum_{\mu\in\Omega_0} \frac{\exp[\mathrm{i}(k + 2\pi\mu/T, n-m)]}{2[\cos(k_1 + 2\pi\mu_1/T) + \cos(k_2 + 2\pi\mu_2/T)] - \lambda}, \tag{6}$$

where $n, m \in \Omega_0$.

We use the notation $R_V(\lambda) = (H_V - \lambda)^{-1}$ and $R_V(k,\lambda) = [H_V(k) - \lambda]^{-1}$ for the resolvent of operators H_V and $H_V(k)$, respectively. Denote by $G_V(n,m,\lambda)$ and $G_V(n,m,k,\lambda)$ the Green functions of these operators. From the resolvent identity

$$G_V(n,m,k,\lambda) = [1 + R_0(k,\lambda)V]^{-1} G_0(n-m,k,\lambda) \tag{7}$$

it follows that $G_V(n,m,k,\lambda)$ is analytic in (k,λ) in a complex neighborhood of any point $(k_0,\lambda_0) \in \mathbb{R}^2 \times \mathbb{C} \subset \mathbb{C}^2 \times \mathbb{C}$ such that $\lambda_0 \notin \sigma(H_V(k_0))$. (In the case $\lambda_0 \in \sigma[H_0(k_0)]$ we use the change $\lambda \mapsto \lambda + \lambda'$, $V \mapsto V + \lambda'$ such that $\lambda_0 + \lambda' \notin \sigma[H_0(k_0)]$.)

Note that according to (7) and (6) the Green function $G_V(n,m,k,\lambda)$ can be naturally extended in n, m to $\mathbb{Z}^2 \times \mathbb{Z}^2$ and, in addition,

$$G_V(n + T\mu, m, k, \lambda) = G_V(n, m - T\mu, k, \lambda) = \exp[\mathrm{i}(k,\mu)T] G_V(n,m,k,\lambda). \tag{8}$$

Lemma 1. *If $\lambda \notin \sigma(H_V)$, then*

$$G_V(n,m,k,\lambda) = \sum_{\mu\in\mathbb{Z}^2} \exp[-\mathrm{i}(k,\mu)T] G_V(n + T\mu, m, \lambda) \tag{9}$$

and

$$G_V(n,m,\lambda) = \left(\frac{T}{2\pi}\right)^2 \int_{\Omega_0^*} G_V(n,m,k,\lambda)\,\mathrm{d}k. \tag{10}$$

Proof. Using (2) and (8), we get for $n \in \Omega_0$, $\mu \in \mathbb{Z}$

$$\begin{aligned}
\left[U_0^{-1} R_V(k,\lambda) U_0 \varphi\right](n + T\mu) &= \frac{T}{2\pi}\int_{\Omega_0^*}\Bigg(\exp[\mathrm{i}(k,\mu)T] \\
&\quad \times \sum_{m\in\Omega_0} G_V(n,m,k,\lambda) \frac{T}{2\pi}\sum_{\nu\in\mathbb{Z}^2}\exp[-\mathrm{i}(k,\nu)T]\varphi(m + T\nu)\Bigg)\mathrm{d}k \\
&= \left(\frac{T}{2\pi}\right)^2 \sum_{m\in\Omega_0}\sum_{\nu\in\mathbb{Z}^2}\int_{\Omega_0^*} G_V(n + T\mu, m + T\nu, k, \lambda)\varphi(m + T\nu)\,\mathrm{d}k \\
&= \left(\frac{T}{2\pi}\right)^2 \sum_{m\in\Omega}\int_{\Omega_0^*} G_V(n + T\mu, m, k, \lambda)\,\mathrm{d}k\varphi(m).
\end{aligned}$$

Consequently, we have (10). Therefore

$$G_V(n+\mu T, m, k, \lambda) = \left(\frac{T}{2\pi}\right)^2 \int_{\Omega_0^*} \exp[\mathrm{i}(k,\mu)T] G_V(n, m, k, \lambda)\,\mathrm{d}k.$$

This proves (9). □

3. Two-body interaction

Now we pass to the new cell $\Omega = \mathbb{Z}\times\Omega_0$ by means of the unitary operator

$$U : l^2(\mathbb{Z}^2) \to l^2(\Omega)\otimes L^2(\Omega^*) \cong \int_{\Omega^*}^{\oplus} l^2(\Omega)\,\mathrm{d}\kappa,$$

$$\varphi(n) \mapsto \left(\frac{T}{2\pi}\right)^{1/2} \sum_{\mu\in\mathbb{Z}} \exp(-\mathrm{i}\kappa\mu T)\varphi[n+(\mu,\mu)T].$$

Here $\Omega^* = [-\pi/T, \pi/T)$ and $\kappa\in\Omega^*$ is the quasimomentum.

It is easily seen that operators

$$H(\kappa) = H_0(\kappa) + V(n) + W(n_1 - n_2)$$

from the decomposition $UHU^{-1} = \{H(\kappa)\}_{\kappa\in\Omega^*}$ act on $l^2(\Omega)$ analogously to the operator H taking into account (for $H_0(\kappa)$) the Bloch property

$$(U\varphi)[n+(T,T),\kappa] = \exp(\mathrm{i}\kappa T)\varphi(n,\kappa).$$

We use the following notation:

$$\{H_V'(\kappa)\}_{\kappa\in\Omega^*} = UH_VU^{-1}, \qquad \{H'(\kappa)\}_{\kappa\in\Omega^*} = UHU^{-1},$$
$$R_V'(\kappa,\lambda) = [H_V'(\kappa)-\lambda]^{-1}, \qquad R'(\kappa,\lambda) = [H'(\kappa)-\lambda]^{-1}.$$

Denote by $G_V'(n, m, \kappa, \lambda)$ the Green function of the operator $H_V'(\kappa)$.

The following equality can be proved in the same way as formula (9):

$$G_V'(n, m, \kappa, \lambda) = \sum_{\mu\in\mathbb{Z}} \exp(-\mathrm{i}\kappa\mu T) G_V[n+\mu T(1,1), m, \lambda]. \tag{11}$$

Lemma 2. *The spectrum of $H_V'(\kappa)$ can be represented as*

$$\sigma[H_V'(\kappa)] = \bigcup_{k_1+k_2=\kappa} \sigma[H_V(k)]. \tag{12}$$

Proof. Let us choose the cell in the reciprocal lattice for $H_V(k)$ of the form

$$\omega_0^* = \{-\pi/T \leqslant k_1 \leqslant \pi/T;\ -\pi/T \leqslant k_1+k_2 \leqslant \pi/T\}.$$

We have (see (4))

$$\begin{aligned}(U_0\varphi)(n,k) &= \frac{T}{2\pi}\sum_{m_1,m_2\in\mathbb{Z}} \exp[-\mathrm{i}(k_1m_1+k_2m_2)T]\varphi[n+T(m_1,m_2)]\\
&= \frac{T}{2\pi}\sum_{\mu,\nu\in\mathbb{Z}} \exp[-\mathrm{i}(\sigma\mu+\kappa\nu)T]\varphi[n+T(\nu,\nu)+T(\mu,0)]\\
&= \left(\frac{T}{2\pi}\right)^{1/2}\sum_{\mu\in\mathbb{Z}}\exp(-\mathrm{i}\sigma\mu T)\\
&\quad\times\left\{\left(\frac{T}{2\pi}\right)^{1/2}\sum_{\nu\in\mathbb{Z}}\exp(-\mathrm{i}\kappa\nu T)\varphi[n+T(\nu,\nu)+T(\mu,0)]\right\},\end{aligned}$$

where $\sigma = k_1, \mu = m_1 - m_2, \kappa = k_1 + k_2, \nu = m_2$. Thus U_0 is unitarily equivalent to the product of operators $U'U$, where the unitary operator

$$U' : L^2(\Omega \times \Omega^*) \to L^2(\Omega_0 \times \Omega_0^*)$$

is defined by

$$(U'\varphi)(n, \sigma, \kappa) = \left(\frac{T}{2\pi}\right)^{1/2} \sum_{\mu\in\mathbb{Z}} \exp(-\mathrm{i}\sigma\mu T)\varphi[n + T(\mu, 0), \kappa].$$

Hence the operators $H_V(k_1, \kappa - k_1)$ where $k_1 = \sigma \in [-\pi/T, \pi/T)$ form the decomposition of the operators $H'_V(\kappa)$ in the direct integral

$$\int_{[-\pi/T,\pi/T)}^{\oplus} L^2(\Omega_0)\,\mathrm{d}k_1.$$

Denote by $\lambda_n(k_1, \kappa - k_1)$ the nth eigenvalue of $H_V(k_1, \kappa - k_1)$ counted in the increasing order with their multiplicities. It follows from the perturbation theory that $\lambda_n(k_1, \kappa - k_1)$ depends continuously on k_1. From this and [6] (theorem XIII.85) we get (12). □

Lemma 3. *Suppose* $\lambda \notin \sigma(H'_V)$. *Then*

$$G'_V(n, m, \kappa, \lambda) = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} G_V[n, m, (k_1, \kappa - k_1), \lambda]\,\mathrm{d}k_1. \tag{13}$$

Proof. Using (11), (10) and the Bloch property of $G_V(n, m, k, \lambda)$, we have

$$\begin{aligned} G'_V(n, m, \kappa, \lambda) &= \sum_{\mu\in\mathbb{Z}} \exp(-\mathrm{i}\kappa\mu T)\left(\frac{T}{2\pi}\right)^2 \int_{\Omega_0^*} G_V[n + \mu T(1, 1), m, k, \lambda]\,\mathrm{d}k \\ &= \left(\frac{T}{2\pi}\right)^2 \frac{2\pi}{T} \int_{-\pi/T}^{\pi/T} \left[\left(\frac{T}{2\pi}\right)^{1/2} \sum_{\mu\in\mathbb{Z}} \exp[-\mathrm{i}\mu T(\kappa - k_1)]\right. \\ &\quad \times \left.\int_{-\pi/T}^{\pi/T} \left(\frac{T}{2\pi}\right)^{1/2} \exp(\mathrm{i}\mu T k_2) G_V(n, m, k, \lambda)\mathrm{d}k_2\right] \mathrm{d}k_1 \\ &= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} G_V[n, m, (k_1, \kappa - k_1), \lambda]\,\mathrm{d}k_1. \end{aligned}$$

□

Lemma 4. *The function* $W(n_1 - n_2)$, *as a multiplication operator, is the relatively compact perturbation of* $H'_V(\kappa)$.

Proof. The function $G_V[n, m, (k_1, \kappa - k_1), \mathrm{i}]$ depends analytically on k_1, hence its Fourier coefficients

$$\left(\frac{T}{2\pi}\right)^{1/2} \int_{-\pi/T}^{\pi/T} \exp(-\mathrm{i}\mu T k_1) G_V[n, m, (k_1, \kappa - k_1), \mathrm{i}]\,\mathrm{d}k_1$$

exponentially decrease as $|\mu| \to \infty$. Using (13) and (2), we obtain

$$\begin{aligned} &\sum_{n\in\Omega} \sum_{m\in\Omega} |G'_V(n, m, \kappa, \mathrm{i}) W(n_1 - n_2)|^2 \\ &\quad \leqslant C \sum_{\mu\in\mathbb{Z}} \sum_{\nu\in\mathbb{Z}} \sum_{n\in\Omega_0} \sum_{m\in\Omega_0} \left| \int_{-\pi/T}^{\pi/T} G_V[n + (\mu T, 0), m + (\nu T, 0), (k_1, \kappa - k_1), \mathrm{i}]\,\mathrm{d}k_1 \right|^2 \\ &\quad \times \exp(-a'|\mu|) \leqslant C' \sum_{\mu\in\mathbb{Z}} \sum_{\nu\in\mathbb{Z}} \exp(-a''|\mu - \nu|) \exp(-a'|\mu|) < \infty, \end{aligned}$$

where $a', a'' > 0$. Thus $W(n_1 - n_2) R'_V(\kappa, \mathrm{i})$ is the Hilbert–Schmidt operator. □

Theorem 1. *The following relations*

$$\sigma_{\text{ess}}[H'(\kappa)] = \sigma[H'_V(\kappa)] = \bigcup_{k_1+k_2=\kappa} \sigma[H'_V(k)]$$

are valid.

Proof. It follows from lemmas 1, 3 and chapter XIII.4 of [6]. □

4. Quasi-levels

We will treat the case where $\lambda_0 = \lambda_N(k_{10}, \kappa - k_{10})$ is a non-degenerate eigenvalue of $H_V(k_{10}, \kappa - k_{10})$ corresponding to a normalized eigenvector $\psi_N[n, (k_{10}, \kappa - k_{10})]$. It can be assumed that λ_N and ψ_N depend analytically on k_1 in some complex neighborhood of k_{10} (see [6]). In what follows, we suppose that

$$\frac{\partial \lambda_N(k_{10}, \kappa - k_{10})}{\partial k_1} = 0, \qquad \frac{\partial^2 \lambda_N(k_{10}, \kappa - k_{10})}{\partial k_1^2} \neq 0.$$

Further, we assume that the number of points $k_1 \neq k_{10}$ such that $\lambda_M(k_1, \kappa - k_1) = \lambda_0$ for some M is finite and at these points

$$\frac{\partial \lambda_M(k_1, \kappa - k_1)}{\partial k_1} \neq 0.$$

In particular, the above conditions hold if the boundary point of some band of the spectrum of $H_V(\kappa)$ is determined by $\lambda_N(k_1, \kappa - k_1)$.

Set $\xi = k_1 - k_{10}$.

Lemma 5. *(see [7]). Let U be a sufficiently small complex neighborhood of λ_0. Then for any $\lambda \in U$ there exist two solutions $\xi_j = \xi_j(\lambda)$, $j = 1, 2$ of the equation*

$$\lambda_N(k_{10} + \xi, \kappa - k_{10} - \xi) = \lambda$$

such that $\xi_1(\lambda_0) = \xi_2(\lambda_0)$ and $\xi_1(\lambda) \neq \xi_2(\lambda)$ if $\lambda \neq \lambda_0$. Moreover, there exists the function $\xi_2 = \mu(\xi_1)$ analytically depending on $\lambda \in U$ such that $\mu'(0) = -1$.

Assume that $\xi_1(\lambda) > 0$ if $\lambda > \lambda_0$. In the following, we often use the parameter $\xi_1 = \xi_1(\lambda)$ instead of λ. Respectively, we use the notation $G'_V(n, m, \kappa, \xi_1)$ instead of $G'_V(n, m, \kappa, \lambda)$, etc.

Let U be a sufficiently small complex neighborhood of zero. Then the functions $\xi_j(\lambda)$, $j = 1, 2$, generate the analytic covering $\mathcal{V}$ over U [8] of two sheets. These functions form the unique analytic function ξ defined on $\mathcal{V}$ and ξ is the analytic continuation of ξ_1 (or ξ_2). Further, $\operatorname{sgn}(\operatorname{Im}\xi)$ corresponds to a certain sheet of $\mathcal{V}$.

The following lemmas 6 and 7 give the different representations of the Green function $G'_V(n, m, \kappa, \xi_1)$.

In the following lemma 6, we extend analytically the function $G'_V(n, m, \kappa, \xi_1)$ to $U \setminus \{0\}$ in ξ_1 where U is a neighborhood of zero. We set $\sqrt{W} = \sqrt{|W|}\operatorname{sgn} W$ (only for W).

Lemma 6. *Let U be a sufficiently small complex neighborhood of zero. Then, for $\xi_1 \in U \setminus \{0\}$, we have*

$$G'_V(n, m, \kappa, \xi_1) = \frac{\mathrm{i}\psi_N[n, (k_{10}, \kappa - k_{10})]\overline{\psi_N[m, (k_{10}, \kappa - k_{10})]}}{\xi_1 \partial^2 \lambda_N(k_{10}, \kappa - k_{10})/\partial k_1^2} + g(n, m, \kappa, \xi_1),$$

where $\sqrt{|W(n)|}g(n, m, \kappa, \xi_1)\sqrt{W(m)}$ is the $l^2(\Omega \times \Omega)$-valued analytic function in ξ_1.

For a proof, see the similar result (lemma 2) in [7] for the one-particle periodic (non-discrete) Schrödinger operator.

Corollary 1. *The operator-valued function* $\sqrt{|W|}R'_V(\kappa, \xi_1)\sqrt{W}$ *extends to a complex neighborhood of zero as a meromorphic function with respect to the parameter* ξ_1. *Moreover, this function takes its values in the set of Hilbert–Schmidt operators (see the proof of lemma 4).*

Remark 1. From the resolvent identity

$$[1 + \sqrt{|W|}R'_V(\kappa, \xi_1)\sqrt{W}]^{-1} = 1 - \sqrt{|W|}R'(\kappa, \xi_1)\sqrt{W} \tag{14}$$

and Fredholm theorems [6, 9] we deduce that the operator-valued function $\sqrt{|W|}R'(\kappa, \xi_1)\sqrt{W}$ is meromorphic in ξ_1 in a neighborhood of zero and takes its values in the set of Hilbert–Schmidt operators.

Remark 2. Suppose that $V(n) = U(n_1) + U(n_2)$. Then we have $\lambda_N(k_1, \kappa - k_1) = \lambda_{N_1}(k_1) + \lambda_{N_2}(\kappa - k_{10})$ where $\lambda_{N_1}(k_1)$ $(\lambda_{N_2}(\kappa - k_{10}))$ are eigenvalues of the one-dimensional Schrödinger operator $h_0(k_1) + U(n_1)$ $(h_0(\kappa - k_1) + U(n_2)$, respectively) in the cell $[0, T - 1]$. Here $h_0(k)(\psi)(n) = \psi(n + 1) + \psi(n - 1)$.

We put $k_{1j} = k_{10} + \xi_j$, $j = 1, 2$.

Lemma 7. *Let* $\xi_1 \neq 0$ *be a sufficiently small complex number. Then the following equality holds:*

$$G'_V(n, m, \kappa, \xi_1) = \frac{\mathrm{i}\psi_N[n, (k_{11}, \kappa - k_{11})]\overline{\psi_N[m, (k_{11}, \kappa - k_{11})]}}{\partial\lambda_N(k_{11}, \kappa - k_{11})/\partial k_1}\vartheta(n_1 - m_1)$$
$$- \frac{\mathrm{i}\psi_N[n, (k_{12}, \kappa - k_{12})]\overline{\psi_N[m, (k_{12}, \kappa - k_{12})]}}{\partial\lambda_N(k_{12}, \kappa - k_{12})/\partial k_1}\vartheta(m_1 - n_1) + \gamma(n, m, \kappa, \xi_1).$$

Here $\vartheta(t)$ *is the Heaviside function and* γ *satisfies the bound:*

$$|\gamma(n, m, \kappa, \xi_1)| \leqslant C\exp(-\sigma|n_1 - m_1|), \qquad \sigma > 0.$$

The proof of the analogous result (for the periodic continuous Schrödinger operator) is given in [10].

We say that the pole of the operator-valued function $\sqrt{|W|}R'(\kappa, \xi_1)\sqrt{W}$ with respect to ξ_1 (and also the corresponding value $\lambda = \lambda_N(k_{10} + \xi_1, \kappa - k_{10} - \xi_1)$) is the *quasi-level* of $H'(\kappa)$.

By virtue of (14) and analytic Fredholm theorem [9], a sufficiently small $\xi_1 \neq 0$ is a quasi-level if and only if there exists a nontrivial solution of the equation

$$\varphi = -\sqrt{|W|}R'_V(\kappa, \xi_1)\sqrt{W}\varphi \tag{15}$$

in $l^2(\Omega)$.

Thus a quasi-level is an eigenvalue or a resonance.

Let $\xi_1 \neq 0$ be a quasi-level. The number

$$\dim\ker[1 + \sqrt{|W|}R'_V(\kappa, \xi_1)\sqrt{W}]$$

is called the *multiplicity* of ξ_1.

Let $\varepsilon > 0$ be a (small) parameter. Now we introduce the operator $H'_\varepsilon(\kappa) = H'_0(\kappa) + \varepsilon W$ where $H'_0(\kappa)$ is taken from the decomposition $UH_0U^{-1} = \{H'_0(\kappa)\}_{\kappa\in\Omega^*}$.

Theorem 2. *Suppose that*

$$W_N = \sum_{n\in\Omega} W(n_1 - n_2)|\psi_N[n, (k_{10}, \kappa - k_{10})]|^2 \neq 0. \tag{16}$$

Then we have the following.

(*a*) *For all sufficiently small* $\varepsilon > 0$ *there exists a unique quasi-level* $\lambda = \lambda_N(k_{11}, \kappa - k_{11})$ *of* $H'(\kappa)$ *of the multiplicity one.*

(*b*) *The following formula holds:*

$$\lambda = \lambda_0 - \frac{\varepsilon^2 W_N^2}{2\partial^2\lambda_N(k_{10}, \kappa - k_{10})/\partial k_1^2} + O(\varepsilon^3). \tag{17}$$

(*c*) *In addition to that, if*

$$\partial^2\lambda_N(k_{10}, \kappa - k_{10})/\partial k_1^2 \cdot W_N < 0,$$

then the quasi-level is the eigenvalue.

Proof. Using lemma 6, we rewrite (15) in the form

$$\varphi(n) = -\frac{\mathrm{i}\varepsilon\varphi_N[n, (k_{10}, \kappa - k_{10})]}{\xi_1\partial^2\lambda_N(k_{10}, \kappa - k_{10})/\partial k_1^2}\sum_{m\in\Omega}\overline{\varphi_N'[m, (k_{10}, \kappa - k_{10})]}\varphi(m) + \varepsilon A(\xi_1)\varphi(n), \tag{18}$$

where $\varphi_N = \sqrt{|W|}\psi_N$, $\varphi_N' = \sqrt{W}\psi_N$, and $A(\xi_1)$ is the operator with the matrix $-\sqrt{|W|}g\sqrt{W}$. Set $f = [1 - \varepsilon A(\xi_1)]\varphi$ for a sufficiently small ε. Then, by (18), $f = C\varphi_N$ where $C = \text{const}$. Hence equation (15) has a nontrivial solution for $\xi_1 \neq 0$ if and only if there exists a solution of the algebraic equation

$$\xi_1 = -\frac{\mathrm{i}\varepsilon\{[1 - \varepsilon A(\xi_1)]^{-1}\varphi_N, \varphi_N'\}}{\partial^2\lambda_N(k_{10}, \kappa - k_{10})/\partial k_1^2}. \tag{19}$$

It follows from lemma 6 that the operator-valued function $A(\xi_1)$ is analytic in a neighborhood of zero. By virtue of the Rouche theorem, there exists a unique solution (quasi-level) ξ_1 of (19). Using (19) and the expansion of $[1 - \varepsilon A(\xi_1)]^{-1}$ in the Taylor series, we obtain the following formula:

$$\xi_1 = \frac{\varepsilon}{\mathrm{i}\partial^2\lambda_N(k_{10}, \kappa - k_{10})/\partial k_1^2}\left(\sum_{n\in\Omega} W(n_1 - n_2)|\psi_N[n, (k_{10}, \kappa - k_{10})]|^2\right) + O(\varepsilon^2)$$
$$= \frac{\varepsilon W_N}{\mathrm{i}\partial^2\lambda_N(k_{10}, \kappa - k_{10})/\partial k_1^2} + O(\varepsilon^2). \tag{20}$$

By (20) and (16), we have $\xi_1 \neq 0$. Further, from the equality $\varphi = C(1 - \varepsilon A(\xi_1))^{-1}\varphi_N$ it follows that the quasi-level multiplicity is equal to unity.

Now we prove the last statement of the theorem. Suppose that $\varphi \neq 0$ belongs to $l^2(\Omega)$ and satisfies (15). Then the function

$$\psi = -\varepsilon R_V'(\kappa, \xi_1)\sqrt{W}\varphi = -\varepsilon R_V'(\kappa, \xi_1)W\psi \tag{21}$$

satisfies the equation $H_\varepsilon'(\kappa)\psi = \lambda\psi$ where $\lambda = \lambda_N(k_{10} + \xi_1, \kappa - k_{10} - \xi_1)$. (Obviously, $\varphi = \sqrt{|W|}\psi$). Therefore it will suffice to prove that $\psi \in l^2(\Omega)$. By lemma 7, we have

$$\begin{aligned}\psi(n) = &-\frac{\mathrm{i}\varepsilon\psi_N[n, (k_{11}, \kappa - k_{11})]}{\mathrm{i}\partial\lambda_N(k_{11}, \kappa - k_{11})/\partial k_1}\\ &\times\sum_{m\in\Omega\cap\{m_1<n_1\}}\overline{\psi_N[m, (k_{11}, \kappa - k_{11})]}\sqrt{W(m_1 - m_2)}\varphi(m)\\ &+\frac{\mathrm{i}\varepsilon\psi_N[n, (k_{12}, \kappa - k_{12})]}{\mathrm{i}\partial\lambda_N(k_{12}, \kappa - k_{12})/\partial k_1}\\ &\times\sum_{m\in\Omega\cap\{m_1\geqslant n_1\}}\overline{\psi_N[m, (k_{12}, \kappa - k_{12})]}\sqrt{W(m_1 - m_2)}\varphi(m)\\ &-\varepsilon\sum_{m\in\Omega}\gamma(n, m, \xi_1)\sqrt{W(m_1 - m_2)}\varphi(m).\end{aligned} \tag{22}$$

Suppose $n_1 \geqslant 0$. Using lemma 7, the Cauchy inequality and (2), we obtain

$$\left|\sum_{m\in\Omega} \gamma(n,m,\xi_1)\sqrt{W(m_1-m_2)}\varphi(m)\right|^2$$
$$\leqslant C\sum_{m\in\Omega}|\gamma(n,m,\xi_1)|^2|W(m_1-m_2)| \leqslant C_1\sum_{m_1\in\mathbb{Z}}\exp(-2\sigma|n_1-m_1|-a|m_1|)$$
$$= C_1\left(\sum_{m_1\geqslant n_1}\exp[-2\sigma(m_1-n_1)-am_1] + \sum_{0\leqslant m_1<n_1}\exp[-2\sigma(n_1-m_1)-am_1]\right.$$
$$+\sum_{m_1<0}\exp[-2\sigma(n_1-m_1)+am_1] = C_1(\exp(2\sigma n_1)\frac{\exp[-(2\sigma+a)n_1]}{1-\exp[-(2\sigma+a)]}$$
$$\left.+\exp(-2\sigma n_1)\frac{1+\exp[(2\sigma-a)n_1]}{1-\exp(2\sigma-a)}+\exp(-2\sigma n_1)\frac{\exp[-(2\sigma+a)]}{1-\exp[-(2\sigma+a)]}\right). \tag{23}$$

This expression decreases exponentially as $n_1 \to \infty$. Evidently, the analogous result is true for $n_1 \leqslant 0$.

We note that from the equality

$$\psi_N(n,k) = -\sum_{m\in\Omega_0} G_V[n-m,k,\lambda_N(k)]V(m)\psi_N(m,k) \tag{24}$$

and (6) it follows that

$$|\psi_N(n,k)| \leqslant \exp(|\mathrm{Im}\, k_1||n_1| + |\mathrm{Im}\, k_2||n_2|) \tag{25}$$

for k belonging to some complex neighborhood of $k_0 \in \Omega_0^*$.

Let $n_1 \geqslant 0$ and let $|\xi_1| \leqslant \delta < a/2$ where a is taken from (2). From (25) and (2) we get

$$\left|\sum_{m\in\Omega\cap\{m_1\geqslant n_1\}} \overline{\psi_N[m,(k_{1j},\kappa-k_{1j})]}\sqrt{W(m_1-m_2)}\varphi(m)\right|$$
$$\leqslant C\left[\sum_{m_1=n_1}^{\infty}\exp[(2\delta-a')m_1]\right]^{1/2} = C\frac{\exp[(2\delta-a')n_1]}{1-\exp(2\delta-a)}, \qquad j=1,2. \tag{26}$$

By virtue of (22), (23), and (26) we obtain

$$\psi(n) = -\frac{\mathrm{i}\varepsilon\psi_N[n,(k_{11},\kappa-k_{11})]}{\mathrm{i}\partial\lambda_N(k_{11},\kappa-k_{11})/\partial k_1}$$
$$\times\sum_{m\in\Omega}\overline{\psi_N[m,(k_{11},\kappa-k_{11})]}\sqrt{W(m_1-m_2)}\varphi(m)+\eta_+(n) \tag{27}$$

for $n_1 \geqslant 0$ where $\eta_+ \in l^2(\Omega\cap\{n_1\geqslant 0\})$. Similarly,

$$\psi(n) = -\frac{\mathrm{i}\varepsilon\psi_N[n,(k_{12},\kappa-k_{12})]}{\mathrm{i}\partial\lambda_N(k_{12},\kappa-k_{12})/\partial k_1}$$
$$\times\sum_{m\in\Omega}\overline{\psi_N[m,(k_{12},\kappa-k_{12})]}\sqrt{W(m_1-m_2)}\varphi(m)+\eta_-(n) \tag{28}$$

for $n_1 \leqslant 0$ and $\eta_- \in l^2(\Omega\cap\{n_1\leqslant 0\})$.

Using (6), we rewrite (24) as

$$\psi_N[n, (k_1, \kappa - k_1)] = -\frac{1}{T} \exp\{i[(k_1, \kappa - k_1), n]\}$$
$$\times \sum_{\mu\in\Omega_0} \sum_{m\in\Omega_0} \frac{\exp[i(2\pi\mu/T, n)] \exp[-i((k_1, \kappa - k_1) + 2\pi\mu/T, m)]}{2[\cos(k_1 + 2\pi\mu_1/T) + \cos(\kappa - k_1 + 2\pi\mu_2/T)] - \lambda_N(k_1, \kappa - k_1)}$$
$$\times V(m)\psi_N[m, (k_1, \kappa - k_1)].$$

Therefore, $|\psi_N|$ decreases exponentially (increases exponentially) as $n_1 \to \infty$ or $n_1 \to -\infty$ in the case $\operatorname{Im} n_1 k_1 > 0$ (in the case $\operatorname{Im} n_1 k_1 < 0$, respectively). Now the last statement of the theorem is the consequence of (20), (27), (28), lemma 5 and the equalities $\operatorname{Im} k_{1j} = \operatorname{Im} \xi_{1j}$, $j = 1, 2$. □

Remark 3. Under the conditions of theorem 2, the eigenfunction $\psi(n)$ of the operator $H'(\kappa)$ satisfies the estimate

$$|\psi(n)| \leqslant Ce^{-\alpha|n|}, \qquad \alpha > 0$$

for $n \in \mathbb{Z}$.

Remark 4. In the case $V = 0$,

$$G'_V(n, m, \kappa, \lambda) = -\frac{\exp[ik(n-m)/2]}{\sqrt{\lambda^2 - 16\cos^2(\kappa/2)}} \left[g\left(\frac{\lambda}{4\cos(\kappa/2)}\right)\right]^{|n-m|},$$

where $g(w) = w - \sqrt{w^2 - 1}$. (This function is inverse of $w = \frac{1}{2}(z + 1/z)$.) The formula (17) can be rewritten as

$$\lambda = \pm\left(4\cos(\kappa/2) + \frac{\varepsilon^2 W_0}{8\cos(\kappa/2)}\right) + O(\varepsilon^3),$$

where $W_0 = \sum_{n_1\in\mathbb{Z}} W(n_1)$. (Here $\pm 4\cos(\kappa/2)$ are the boundary points of the spectrum of $H'_0(\kappa)$.)

Remark 5. Suppose that

$$\partial^2\lambda_N(k_{10}, \kappa - k_{10})/\partial k_1^2 \cdot W_N > 0.$$

Then $\operatorname{Im} \xi_1 < 0$ and ξ_1 is the resonance. In addition, if

$$\sum_{m\in\Omega} \overline{\psi_N[m, (k_{1j}, \kappa - k_{1j})]\sqrt{W(m_1 - m_2)}}\varphi(m) \neq 0, \qquad j = 1, 2,$$

then the solution $\psi(n)$ of (21) (the metastable state) increases exponentially as $|n_1| \to \infty$ (see the proof of theorem 2).

5. Concluding remarks

Let $H'_\varepsilon(\kappa)$ be the Hamiltonian of the pairs of the interacting one-magnon states in a ferromagnet with periodically placed impurities; here κ is a lattice quasimomentum and ε is a coupling constant for the magnon–magnon interaction. Let us consider the periodic discrete Schrödinger operator $H_V(k)$ (the Hamiltonian without the interaction) where k is a periodic quasimomentum. Let $\lambda_{n_0}(k)$ be an eigenvalue of $H_V(k)$ such that $(k_{10}, \kappa - k_{10})$ is a non-degenerate stationary point of $\lambda_{n_0}(k)$ with respect to k_1. Then for any sufficiently small ε there exist the unique quasi-levels (the eigenvalue or the resonance) of $H'_\varepsilon(\kappa)$ in some neighborhood of $\lambda_0 = \lambda_{n_0}(k_{10}, \kappa - k_{10})$. We obtain the asymptotic formula for this quasi-levels as $\varepsilon \to 0$. We also find the simple condition when a quasi-level is an eigenvalue.

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