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ATTAINABILITY, CONSISTENCY AND THE ROTATION METHOD BY V.M. MILLIONSHCHIKOV

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In the present article we shall investigate the conditions under which a principal possibility exists to construct a control which creates a "good vicinity" for the trivial solution of the equation

$$\dot{x} = v_0(t,x) + u_1 v_1(t,x) + \cdots + u_r v_r(t,x), \quad (t,x) \in \mathbb{R}^{1+n}, \tag{1}$$

where $u \doteq (u_1, \ldots, u_r) \in U$, $0 \in U$. Assume, for example, that it is required to construct a control $u(t) \in U$ such that equation (1) be structurally stable (in the neighborhood of zero). It turns out that the problem is resolvable if the linear equation

$$\dot{x} = A_0(t)x + u_1 A_1(t)x + \cdots + u_r A_r(t)x, \quad (t,x) \in \mathbb{R}^{1+n}, \tag{2}$$

where $A_i(t) = \partial v_i(t,0)/\partial x$, is uniformly locally attainable or uniformly consistent (see below Theorem 1 and Corollary 1). It is shown also that the properties of uniform attainability and consistency are closely related to the so-called "rotation method" of V.M. Millionshchikov, which was applied in investigations of Lyapunov exponents' behavior for small perturbations of control parameters of the equation.

1. Notation and definitions

Let $\mathbb{R}^n$ be a Euclidean space of dimension n, $|x| = \sqrt{x^* x}$ the norm in $\mathbb{R}^n$ ($*$ is the transposition). If the contrary is not indicated, Latin characters stand for the column vectors, Greek characters are used for line vectors (thus, we write ξx for the scalar product of the vectors ξ and x); $\mathcal{O}^n_\varepsilon(x) = \{y \in \mathbb{R}^n : |y - x| \leq \varepsilon\}$, $\mathcal{O}^n_\varepsilon = \mathcal{O}^n_\varepsilon(0)$, $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. For an arbitrary set $D \subset \mathbb{R}^n$ we denote by $\overline{D}$ the closure, by $\operatorname{int} D$ the interiority, by $\mathcal{O}^n_\varepsilon(D)$ an ε-neighborhood, and by $\xi \to c(\xi, D)$ the support function (see [1], Chap. 2, § 12), respectively, of a set D.

We shall identify the space $M_{n,m}$ of linear operators from $\mathbb{R}^m$ to $\mathbb{R}^n$ with the space of matrices of the dimension $n \times m$ (if $n = m$, then we write M_n); $|A| = \max\{|Ax| : |x| = 1\}$ is the norm in $M_{n,m}$. For $Q \in M_{n,m}$, write $\mathcal{B}_\varepsilon(Q) = \{H \in M_{n,m} : |H - Q| \leq \varepsilon\}$, $\mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon(0)$ (we omit the indices m and n in notation $\mathcal{B}_\varepsilon(Q)$ if it is clear which space we refer to).

Let us introduce a dynamical system (Ω, f^t) (Ω is a complete metric separable space, f^t is the flux on Ω, i.e., a one-parameter group of transformations of Ω into itself, continuous with respect to $(t, \omega) \in \mathbb{R} \times \Omega$). We denote by $\gamma(\omega)$ the trajectory of the motion $t \to f^t\omega$, $\gamma_+(\omega)$ stands for a positive semi-trajectory, $\overline{\gamma}(\omega)$ means the closure of $\gamma(\omega)$ in the metric of the space Ω. Let us recall that the set $E \subset \Omega$ is said to be invariant if $f^t E \subset E$ for all $t \in \mathbb{R}$, and E is termed minimal if it is invariant, compact, and $\overline{\gamma}(\omega) = E$ for each $\omega \in E$. If E is minimal, then, for every $\omega \in E$, the equality $\overline{\gamma}(\omega) = \overline{\gamma}_+(\omega)$ takes place. Let us note that each compact invariant set E in Ω contains at least one minimal subset. Further, if $\overline{\gamma}_+(\omega)$ is compact, then the omega-limit set $L^+(\omega)$ of the

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point ω is nonempty (in addition, $L^+(\omega)$ is invariant and compact), and for any compact E the zone of attraction $W^s(E) \doteq \{\omega \in \Omega : L^+(\omega) \subset E\}$ of the set E is defined.

To each pair $(\mathbf{A}, \omega)$, where $\mathbf{A} \doteq (A_0, A_1, \dots, A_r) : \Omega \to M_{n,m}$, $m = n(r+1)$, and to a vector $u = (u_1, \dots, u_r)$ we put into correspondence the equation

$$\dot{x} = A_0(f^t\omega)x + u_1 A_1(f^t\omega)x + \dots + u_r A_r(f^t\omega)x, \quad x \in \mathbb{R}^n. \tag{3}$$

We shall assume that with every $\omega_0 \in \Omega$ the function $t \to |\mathbf{A}(f^t\omega_0)|$ is Lebesgue measurable, bounded on $\mathbb{R}$, and with any $\varepsilon > 0$ and $N > 0$ one can find $\delta > 0$ such that the inequality $\max\limits_{|t| \le N} \int_t^{t+1} |\mathbf{A}(f^s\omega) - \mathbf{A}(f^s\omega_0)| ds < \varepsilon$ holds, when $\rho(\omega, \omega_0) < \delta$ (ρ is the metric in Ω). Equation (3) will be identified with the pair $(\mathbf{A}, \omega)$.

Let an arbitrary set $U \subset \mathbb{R}^r$ be given such that $0 \in U$. The set $\mathcal{U}$, consisting of all measurable functions $u : \mathbb{R} \to U$, will be called the set of admissible controls. Let us fix the equation $(\mathbf{A}, \omega)$. Let $X_u(t, s, \omega)$ be the Cauchy function of equation (3) for a fixed control $u = u(t, \omega)$. Note that the function $\omega \to X_u(t, s, \omega)$ is continuous at every point $\omega_0 \in \Omega$ uniformly with respect to (t, s) on any compact from $\mathbb{R}^2$ and, if the control is stationary with respect to the flux, i.e., $u(t, \omega) = u(f^t\omega)$, then $X_u(t+\tau, s+\tau, \omega) = X_u(t, s, f^\tau\omega)$.

For any $\vartheta > 0$, we construct the attainability set

$$\mathfrak{D}_\vartheta(\omega) = \{X \in M_n : X = X_u(\vartheta, 0, \omega),\ u(\cdot) \in \mathcal{U}\}$$

of equation (3) in the space of matrices M_n from the unitary matrix and the set

$$\mathcal{B}_\varepsilon(I)X_0(\vartheta, 0, \omega) = \{X \in M_n : X = HX_0(\vartheta, 0, \omega),\ H \in \mathcal{B}_\varepsilon(I)\},$$

where I is the unitary matrix.

Lemma 1. *The following inclusions take place*

$$\mathcal{B}_{\nu_1}(X_0(\vartheta, 0, \omega)) \subset \mathcal{B}_\varepsilon(I)X_0(\vartheta, 0, \omega) \subset \mathcal{B}_{\nu_2}(X_0(\vartheta, 0, \omega)), \tag{4}$$

where $\nu_1 = \varepsilon/|X_0(0, \vartheta, \omega)|$, $\nu_2 = \varepsilon|X_0(\vartheta, 0, \omega)|$.

Proof. Let $H = I + G$. Then

$$\mathcal{B}_\varepsilon(I)X_0(\vartheta, 0, \omega) = X_0(\vartheta, 0, \omega) + \mathcal{B}_\varepsilon X_0(\vartheta, 0, \omega),$$

where $\mathcal{B}_\varepsilon X_0(\vartheta, 0, \omega) = \{X \in M_n : X = GX_0(\vartheta, 0, \omega),\ |G| \le \varepsilon\}$. Therefore inclusions (4) are equivalent to the inclusions $\mathcal{B}_{\nu_1} \subset \mathcal{B}_\varepsilon X_0(\vartheta, 0, \omega) \subset \mathcal{B}_{\nu_2}$. If $A \in \mathcal{B}_\varepsilon$, then $|AX_0(\vartheta, 0, \omega)| \le \varepsilon|X_0(\vartheta, 0, \omega)| = \nu_2$; whence follows the second inclusion. If $|A| \le \nu_1$, then $|AX_0(0, \vartheta, \omega)| \le \varepsilon$, therefore

$$A = AX_0(0, \vartheta, \omega)X_0(\vartheta, 0, \omega) \in \mathcal{B}_\varepsilon(I)X_0(\vartheta, 0, \omega).$$

Hence we have the first inclusion. □

Definition 1. Equation (3) is said to be

a) locally attainable if $\vartheta > 0$ and $\varepsilon > 0$ can be found such that the following inclusion holds

$$\mathcal{B}_\varepsilon(I)X_0(\vartheta, 0, \omega) \subset \mathfrak{D}_\vartheta(\omega);$$

b) uniformly locally attainable if $\vartheta > 0$ and $\varepsilon > 0$ can be found such that for all $\omega_0 \in \overline{\gamma}(\omega)$ the following inclusion takes place

$$\mathcal{B}_\varepsilon(I)X_0(\vartheta, 0, \omega_0) \subset \mathfrak{D}_\vartheta(\omega_0).$$

Lemma 2. *Equation* (3) *is locally attainable if and only if* $\vartheta > 0$ *and* $\varepsilon > 0$ *exist such that a control* $(t, H) \to u(t, H)$ *will be found defined on* $[0, \vartheta] \times \mathcal{B}_\varepsilon(I)$ *and ensuring the equality*

$$X_u(\vartheta, 0, \omega) = HX_0(\vartheta, 0, \omega). \tag{5}$$

Equation (3) *is uniformly locally attainable if and only if* $\vartheta > 0$ *and* $\varepsilon > 0$ *exist such that for any continuous function* $H : \overline{\gamma}(\omega) \to \mathcal{B}_\varepsilon(I)$ *a control can be found* $(t, \omega_0) \to u(t, \omega_0)$, *defined on* $[0, \vartheta] \times \overline{\gamma}(\omega)$ *and ensuring the equality*

$$X_u(\vartheta, 0, \omega_0) = H(\omega_0)X_0(\vartheta, 0, \omega_0). \tag{6}$$

Proof. Let for $u = u(t, H)$ equality (5) be fulfilled. Then, for all $H \in \mathcal{B}_\varepsilon(I)$, we have $HX_0(\vartheta, 0, \omega) \in \mathfrak{D}_\vartheta(\omega)$. Consequently, equation (3) is locally attainable.

Obviously, the uniform local attainability implies (6). Let for any continuous $H(\omega_0)$ equality (6) be fulfilled. If $H_0 \in \mathcal{B}_\varepsilon(I)$ can be found such that the matrix $H_0X_0(\vartheta, 0, \omega_0)$ is not contained in $\mathfrak{D}_\vartheta(\omega_0)$, then, for the function $H(\omega_0) \equiv H_0$ and any admissible control equality (6) is not fulfilled. □

Remark 1. The presence of the property of local or uniform attainability makes it possible for us (by virtue of (5) or (6)) to "rotate slightly" the Cauchy function of equation (3) by means of an appropriate control (and by the same token to an influence the solutions' behavior). Not every equation possesses this property, but, obviously, the equation $\dot{x} = A(t)x + U(t)x$, where the matrix $A(t)$ is bounded and the elements of the matrix $U(t)$ are interpreted as controlling functions ($|U(t)| \leq \varepsilon$), is uniformly locally attainable. Thus, (6) is fulfilled and this enabled Millionshchikov (see [2], [3]), and then other investigators (see [4]) to construct the modern theory of the Lyapunov exponents; and this method of "rotations" was named V.M.Millionshchikov's rotation method.

Starting from equation $(\mathbf{A}, \omega)$ we construct the matrix differential equation

$$\dot{Z} = A_0(f^t\omega)Z + (u_1A_1(f^t\omega) + \cdots + u_rA_r(f^t\omega))X_0(t, 0, \omega) \tag{7}$$

with respect to $Z \in M_n$. We denote by $\mathfrak{L}_\vartheta(\omega)$ the set of attainability of equation (7) from the zero in the time ϑ under action of controls $u : [0, \vartheta] \to \mathbb{R}^r$. Thus, $G \in \mathfrak{L}_\vartheta(\omega)$ if and only if $u_G : [0, \vartheta] \to \mathbb{R}^r$ can be found such that equation (7) with $u = u_G(t)$ has a solution $Z(t)$ which satisfies the conditions $Z(0) = 0$, $Z(\vartheta) = G$. By virtue of the linearity of equation (7) the set $\mathfrak{L}_\vartheta(\omega)$ is a linear subspace in M_n.

Definition 2. Equation $(\mathbf{A}, \omega)$ is said to be consistent if $\vartheta > 0$ can be found such that $\mathfrak{L}_\vartheta(\omega) = M_n$. It is termed uniformly consistent if $\vartheta > 0$ and $\beta > 0$ can be found such that for all $\omega_0 \in \overline{\gamma}(\omega)$ the set $\mathfrak{L}_\vartheta(\omega_0)$ coincides with M_n, and the inequality is fulfilled: $|u_G(t, \omega_0)| \leq \beta|G|$, $0 \leq t \leq \vartheta$ (here $u_G(t, \omega_0)$ is a control sending the solution of equation (7) for $\omega = \omega_0$ from the zero to the point G at the moment $t = \vartheta$).

Remark 2. It immediately follows from the definition that, if the equation $(\mathbf{A}, \omega)$ is consistent (uniformly consistent), then for any $\tau \in \mathbb{R}_+$ ($\tau \in \mathbb{R}$) the equation $(\mathbf{A}, f^{-\tau}\omega)$ is consistent (uniformly consistent).

Remark 3. The definitions of consistency and uniform consistency were introduced in [5] for the system $\dot{x} = (A(f^t\omega) + B(f^t\omega)UC^*(f^t\omega))x$ (elements of the matrix U are treated as controlling functions) in connection with the problems on control over Lyapunov exponents; they were studied in [6]–[8] (see the bibliography in [8]).

2. Uniform local attainability

The principal property of uniformly locally attainable equations is contained in Theorem 1 which we shall formulate below. This theorem ascertains that, for an appropriate choice of an admissible control, all the solutions of equation (3) behave as solutions of a model equation with discrete time, which is given beforehand. Thus, there arises a possibility to affect the asymptotic properties of solutions of equation (3), for example, to control the Lyapunov exponents or other analogous characteristics, which cause the asymptotic behavior of the solutions. In some applied problems, this matter is of importance because it enables us to separate motions (by achieving structural stability) and refine the asymptotic of motions at the expense of translation to the left of the exponents. But if the model equation is stationary: $y_{k+1} = Qy_k$ (see (10) in Theorem 1 below), then equation (3) is "almost stationary", i. e., it can be reduced by a Lyapunov transformation to a stationary equation (Corollary 1).

First, the following two simple lemmas will be proven.

Lemma 3. *Let E be an invariant compact set in Ω and for each $\omega \in E$ equation* (3) *be uniformly locally attainable. Then it is uniformly locally attainable on the set E, i. e., the constants $\vartheta > 0$ and $\varepsilon > 0$, which participate in the definition of the local attainability and are common for all $\omega \in E$, can be found.*

Proof. Let $\vartheta(\omega)$ and $\varepsilon(\omega)$ be constants which participate in the definition of the uniform local attainability of equation $(\mathbb{A}, \omega)$. It is required to prove that $\sup_E \theta(\omega) < \infty$, where $\theta(\omega) = \max\{\vartheta(\omega), \varepsilon^{-1}(\omega)\}$. If $\sup_E \theta(\omega) = \infty$, then a converging (by virtue of the compactness of E) sequence $\{\omega_i\}$ can be found such that $\theta(\omega_i) \to \infty$. The latter means that equation $(\mathbb{A}, \widehat{\omega})$, where $\widehat{\omega} = \lim \omega_i$, is not uniformly locally attainable. □

Lemma 4. *Let E be an invariant compact set in Ω, $u : \mathbb{R} \times E \to U$ be an arbitrary admissible control. If for a certain $\tau > 0$ and all $t \in \mathbb{R}_+$ the equality takes place*

$$u(t, \omega) = u(t - \tau, f^\tau \omega), \tag{8}$$

then $X_u(t+\tau, s+\tau, \omega) = X_u(t, s, f^\tau\omega)$, $(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+$. If equality (8) *is fulfilled for any $\tau \in \mathbb{R}$ and for all $t \in \mathbb{R}$, then the control $u(t, \omega)$ is stationary with respect to the flux f^t, i. e., $u(t, \omega) = \widehat{u}(f^t\omega)$ for a certain function $\widehat{u} : E \to U$.*

Proof. We introduce into consideration the matrix equation

$$\dot{X} = A_0(f^t\omega)X + u_1A_1(f^t\omega)X + \dots + u_rA_r(f^t\omega)X, \quad X \in M_n, \tag{9}$$

which corresponds to equation (3). In order to prove the first assertion of Lemma it suffices to substitute a control (satisfying equality (8)) into equation (9) and realize the change of the time $t \to t + \tau$. Then, by virtue of the definition of $u(t, \omega)$, we get the equality $u(t + \tau, \omega) = u(t, f^\tau\omega)$, which is equivalent to (8).

The second assertion follows from (8) with $\tau = t$. □

Theorem 1. *Let E be an invariant compact set in Ω and for each $\omega \in E$ equation* (3) *be uniformly locally attainable. Then $\vartheta > 0$ and $\varepsilon > 0$ exist such that for any function $Q : E \to M_n$, satisfying with all $\omega \in E$ the inequality $|Q(\omega)X_0(0, \vartheta, \omega) - I| \leq \varepsilon$, a control $u : \mathbb{R} \times E \to U$ can be found guaranteeing for the solution $x_u(t, x_0, \omega)$ of equation* (3) *for all $u = u(t, \omega)$, all $x_0 \in \mathbb{R}^n$, and all $k = 0, 1, \dots$ the equalities $x_u(t_k, x_0, \omega) = y_k(x_0, \omega)$. Here $t_k = k\vartheta$, $\{y_k(x_0, \omega)\}_{k=0}^\infty$ is a solution of the problem*

$$y_{k+1} = Q(f^{t_k}\omega)y_k, \quad y_0 = x_0, \quad k = 0, 1, \dots \tag{10}$$

Proof. Let ϑ and ε be as in Lemma 3, and $Q(\omega)$ satisfy the condition of Theorem. For each $\omega \in E$, we construct a control $u^0(\cdot,\omega) : [0,\vartheta] \to U$, defined on the segment $[0,\vartheta]$ and such that (see Lemma 2) for $u = u^0(t, \omega)$ for equation (3) the inequality $X_{u^0}(\vartheta, 0, \omega) = H(\omega)X_0(\vartheta, 0, \omega)$ is fulfilled, where $H(\omega) = Q(\omega)X_0(0,\vartheta,\omega)$. Consequently, for any $x_0 \in \mathbb{R}^n$, the equality $x_{u^0}(\vartheta, x_0, \omega) = Q(\omega)x_0$ takes place or, by virtue of (10), $x_{u^0}(\vartheta, x_0, \omega) = y_1(x_0, \omega)$.

By the same token, for each $k = 0, 1, \dots$, we have constructed the control $u^k = u^0(t, f^{t_k}\omega)$, defined on $[0,\vartheta]$ and ensuring for solution of equation $(\mathrm{A}, f^{t_k}\omega)$ the equality $x_{u^k}(\vartheta, x_0, f^{t_k}\omega) = y_1(x_0, f^{t_k}\omega)$.

Let us "expand" the sequence $\{u^k(t)\}_{k=0}^{\infty}$, $t \in [0,\vartheta]$, onto $\mathbb{R}_+$, by defining a control $u : \mathbb{R}_+ \times E$ on each semi-interval $\Delta_k = [t_k, t_{k+1})$ by the equality $u(t,\omega) = u^0(t - t_k, f^{t_k}\omega)$ with $t \in \Delta_k$. Direct verification gives us that, for each integer $m \geq 0$ for $\tau = t_m$ and all $t \in \mathbb{R}_+$, equality (8) is fulfilled. Therefore, by virtue of Lemma 4, one has the equalities $X_u(t_{k+1}, t_k, \omega) = X_u(\vartheta, 0, f^{t_k}\omega)$.

Let $x_u(t, x_0, \omega)$ be a solution of the equation (A, ω), which corresponds to the control $u(t,\omega)$. Let us show that the sequence $\{x_u(t_k, x_0, \omega)\}_{k=1}^{\infty}$ is a solution of problem (10). Indeed,

$$x_u(t_{k+1}, x_0, \omega) = X_u(t_{k+1}, 0, \omega)x_0 = X_u(t_{k+1}, t_k, \omega)X_u(t_k, 0, \omega)x_0.$$

With regard for the equality $X_u(t_{k+1}, t_k, \omega) = X_u(\vartheta, 0, f^{t_k}\omega)$ we further have

$$x_u(t_{k+1}, x_0, \omega) = X_u(\vartheta, 0, f^{t_k}\omega)x_u(t_k, x_0, \omega) = Q(f^{t_k}\omega)x_u(t_k, x_0, \omega). \quad \square$$

Corollary 1. Let a set $\overline{\gamma}(\omega)$ be compact and equation (3) be uniformly locally attainable. Then $\vartheta > 0$ and $\varepsilon > 0$ exist such that for any real matrix $\Lambda \in M_n$, which satisfies the inequality $|\exp(\vartheta\Lambda)X_0(0,\vartheta,\omega) - I| \leq \varepsilon$, a control $t \to u(t,\omega) \in U$ can be found under which equation (3) is reducible by a Lyapunov transformation to the equation $\dot{y} = \Lambda y$ with a constant matrix Λ.

The control $t \to u(t,\omega)$ possesses the following property: If ω is a periodic point (i.e., $T > 0$ can be found such that $f^{t+T}\omega \equiv f^t\omega$), then for a certain integer $m \geq 1$ the control is periodic with the period equaling mT: $u(t + mT, \omega) \equiv u(t,\omega)$.

Proof. Let $Q(\omega) \equiv \exp(\vartheta\Lambda)$, then the matrix Q satisfies the conditions of Theorem 1. By the matrix Q we construct the control $t \to u(t,\omega)$ (as it has been done in the proof of Theorem 1) and the matrix $\Phi(t) = X_u(t, 0, \omega)\exp(-t\Lambda)$. By virtue of the well-known theorem of N.P. Yerugin (see [9], § 19, p. 119), it suffices to show that the matrix $\Phi(t)$ is a Lyapunov's matrix.

The function $t \to \Phi(t)$ is continuous and bounded on $\mathbb{R}_+$. Indeed, from (10) for all $x_0 \in \mathbb{R}^n$ we have the equality $x_u(t_k, x_0, \omega) = \exp(t_k\Lambda)x_0$. Therefore $X_u(t_k, 0, \omega) = \exp(t_k\Lambda)$. By virtue of the compactness of $\overline{\gamma}(\omega)$, boundedness of $\mathrm{A}(f^t\omega)$, and control $u(t,\omega)$, on $\mathbb{R}_+$ a constant a can be found such that $|X_u(t, s, f^\tau\omega)| \leq a$ for all $(t,s) \in \Delta_k \times \Delta_k$, all $k = 0, 1, \dots$, and all $\tau \in \mathbb{R}_+$. Let $t \in \Delta_k$. Then $\Phi(t) = X_u(t, t_k, \omega)X_u(t_k, 0, \omega)\exp(-t\Lambda) = X_u(t, t_k, \omega)\exp(t_k - t)\Lambda$. Therefore relation $|\Phi(t)| \leq c$ holds for all $t \in \mathbb{R}_+$, moreover, c does not depend on k, In a similar way one can prove the boundedness of $\Phi^{-1}(t)$.

The boundedness of the derivative $\dot{\Phi}(t)$ follows from the equality

$$\dot{\Phi}(t) = (A_0(f^t\omega) + u_1(t,\omega)A_1(f^t\omega) + \cdots + u_r(t,\omega)A_r(f^t\omega))\Phi(t) - \Phi(t)\Lambda$$

and the boundedness of $\Phi(t)$.

Let us prove the second assertion of Corollary 1. Note that, first, it follows directly from the definition of the uniform local attainability and condition $0 \in U$ that if this property takes place for some ϑ and ε, then it takes place for all $\vartheta_1 \geq \vartheta$. Therefore, an integer $m \geq 1$ can be found such that equation (3) is uniformly locally attainable for $\vartheta = mT$. By virtue of (8) we have $u(t + mT, \omega) = u(t, f^{mT}\omega) = u(t,\omega)$. $\square$

3. Uniform consistency and attainability

In this section we shall prove that, in the non-critical case ($0 \in \operatorname{int} U$), the uniform consistency is the uniform local attainability with respect to the first approximation.

By virtue of Lemma 2, the local attainability of equation (3) is equivalent to the resolvability of equation (9) with condition (5), and the uniform local attainability — with condition (6). Let us make in (9) the change $Y(t,\omega) = X_u(t,0,\omega) - X_0(t,0,\omega)$. Then this equation with respect to Y can be written in the form

$$\dot{Y} = A_0(f^t\omega)Y + \Big(u_1A_1(f^t\omega) + \cdots + u_rA_r(f^t\omega)\Big)Y + \\ + \Big(u_1A_1(f^t\omega) + \cdots + u_rA_r(f^t\omega)\Big)X_0(t,0,\omega), \quad Y \in M_n, \quad (11)$$

and condition (5) is divided into two conditions: $Y(0,\omega) = 0$, $Y(\vartheta,\omega) = GX_0(\vartheta,0,\omega)$, where $G = H - I$. We denote by $\mathbb{D}_\vartheta(\omega)$ the set of attainability for the time $\vartheta > 0$ of equation (11) from the point $Y = 0$ under action of the controls $u \in \mathcal{U}$ (note that $0 \in \mathbb{D}_\vartheta(\omega) \subset M_n$). Then the inclusion $0 \in \operatorname{int}\mathbb{D}_\vartheta(\omega)$ with a certain $\vartheta > 0$ is equivalent to the local attainability of equation (3), while the inclusion $\mathcal{B}_\varepsilon \subset \mathbb{D}_\vartheta(\omega_0)$, obeyed for some $\vartheta > 0$, $\varepsilon > 0$ and all $\omega_0 \in \overline{\gamma}(\omega)$, is equivalent to the uniform local attainability.

Theorem 2. *Let $0 \in \operatorname{int} U$ and equation* (3) *be consistent* (*uniformly consistent*). *Then it is locally attainable* (*uniformly locally attainable*).

Proof. Equation (7) serves in the capacity of a linear approximation (in the neighborhood of the point $u = 0$, $Y = 0$) for equation (11), while the consistency property formulated in terms of equation (7) is equivalent to the property of complete attainability of equation (7) (i. e., the set of attainability $\mathfrak{L}_\vartheta(\omega)$ of equation (7) coincides with the space M_n). Let us take advantage of theorem 3 in [10]. To this end we "unroll into a vector" the matrices Y and Z in equations (11) and (7), in theorem 3 in [10] we change the time $t \to \vartheta - t$ and replace the words "local controllability in the small" with the words "local attainability in the small" (the local attainability in the small means local attainability of the "unrolled equation" (11), complemented by the following property: For any $\varepsilon > 0$, $\delta > 0$ can be found such that, if $|G| < \delta$, then the solution $Y(t,\omega)$, which translates the point $Y(\vartheta,\omega) = G$ under an appropriate control, satisfies for all $t \in [0,\vartheta]$ the inequality $|Y(t,\omega)| < \varepsilon$). By virtue of the condition $0 \in \operatorname{int} U$ and theorem 3 cited in [10] the first assertion has been proved.

After similar arguments, the second assertion of Theorem will follow from theorem 1 in [11] (in that paper, the minimality of the set $\overline{\gamma}(\omega)$ was required, but this assumption is superfluous: it suffices to assume that $\overline{\gamma}(\omega)$ is compact). □

4. Consistency criteria

In this section the results of [5], [8] on the consistency of the equation

$$\dot{x} = (A(f^t\omega) + B(f^t\omega)UC^*(f^t\omega))x, \quad x \in \mathbb{R}^n, \quad (12)$$

where U is an $(m \times k)$-matrix of controlling parameters, are transferred to equation (3). For every $i,j,p,s \in \{1,\dots,n\}$, we introduce the notation

$$\gamma_{ijps}(\vartheta,\omega) = \int_0^\vartheta \sum_{l=1}^r e_i^*\widehat{A}_l(t,\omega)e_pe_s^*\widehat{A}_l^*(t,\omega)e_j\,dt,$$

where $e_k \in \mathbb{R}^n$ is the k-th column of the unitary matrix, $\widehat{A}_l(t,\omega) = X_0(0,t,\omega)A_l(f^t\omega)X_0(t,0,\omega)$, $l = 1,\dots,r$. Further, we construct $(n \times n)$-matrices $\Gamma_{ij}(\vartheta,\omega) = \{\gamma_{ijps}(\vartheta,\omega)\}_{p,s=1}^n$, $i,j = 1,\dots,n$, and an $(n^2 \times n^2)$-matrix $\Gamma(\vartheta,\omega) = \{\Gamma_{ij}(\vartheta,\omega)\}_{i,j=1}^n$, which in what follows will be called the consistency matrix (of equation (3) on $[0,\vartheta]$).

Lemma 5. *The consistency matrix possesses the following properties*:

a) $\Gamma(\vartheta,\omega) = \Gamma^*(\vartheta,\omega)$;

b) $\lambda(\vartheta,\omega) \geq 0$, ($\lambda$ *is the least eigenvalue of the matrix* Γ);

c) $\lambda(\vartheta_1,\omega) \geq \lambda(\vartheta,\omega)$ *for* $\vartheta_1 \geq \vartheta$.

Proof. Assertion a) follows from the equality $\gamma_{ijps} = \gamma_{jisp}$. Let us prove assertion b). It suffices to show that, for any $h = \mathrm{col}(h_{11}, \ldots, h_{1n}, \ldots, h_{n1}, \ldots, h_{nn})$, the inequality $h^*\Gamma h \geq 0$ is valid. We have

$$h^*\Gamma h = \sum_{i,j,p,s=1}^{n} h_{ip}h_{js}\gamma_{ijps} = \int_0^{\vartheta} \sum_{l=1}^{r} \sum_{i,j,p,s=1}^{n} h_{ip}h_{js}e_i^*\widehat{A}_l(t,\omega)e_pe_s^*\widehat{A}_l^*(t,\omega)e_j\,dt =$$

$$= \int_0^{\vartheta} \sum_{l=1}^{r} \Big(\sum_{i,p=1}^{n} h_{ip}e_i^*\widehat{A}_l(t,\omega)e_p\Big)\Big(\sum_{j,s=1}^{n} h_{js}e_s^*\widehat{A}_l^*(t,\omega)e_j\Big)\,dt =$$

$$= \int_0^{\vartheta} \sum_{l=1}^{r} \Big(\sum_{i,p=1}^{n} h_{ip}e_i^*\widehat{A}_l(t,\omega)e_p\Big)^2\,dt \geq 0. \quad (13)$$

Assertion c) follows from inequality (13). □

We define the functions $u_{ip}^l : [0,\vartheta] \times \Omega \to \mathbb{R}$ and the vector functions u_{ip} via the equalities

$$u_{ip}^l(t,\omega) = e_i^*\widehat{A}_l(t,\omega)e_p, \quad i,p = 1,\ldots,n, \quad l = 1,\ldots,r, \quad (14)$$

$$u_{ip}(t,\omega) = \mathrm{col}(u_{ip}^1(t,\omega),\ldots,u_{ip}^r(t,\omega)), \quad i,p = 1,\ldots,n. \quad (15)$$

Lemma 6. *The set of vector functions* (15) *is linearly independent on* $[0,\vartheta]$ *if and only if* $\lambda(\vartheta,\omega) > 0$.

Proof. Let functions (15) be linearly independent on $[0,\vartheta]$, but $\lambda(\vartheta,\omega) = 0$. Then a nonzero vector h exists such that $\Gamma(\vartheta,\omega)h = 0$. Consequently, $h^*\Gamma(\vartheta,\omega)h = 0$ and by virtue of (13) for all $l = 1,\ldots,r$ the following equalities are fulfilled

$$\sum_{i,p=1}^{n} h_{ip}e_i^*\widehat{A}_l(t,\omega)e_p = 0, \quad t \in [0,\vartheta]. \quad (16)$$

This is equivalent to the equality $\sum_{i,p=1}^{n} h_{ip}u_{ip}(t,\omega) = 0$, $t \in [0,\vartheta]$, which contradicts the linear independence of functions (15).

Now let us prove the sufficiency of the lemma's conditions. If functions (15) are linearly independent, then a nonzero vector $h \in \mathbb{R}^{n^2}$ can be found such that $\sum_{i,p=1}^{n} h_{ip}u_{ip}(t,\omega) = 0$, $t \in [0,\vartheta]$. Consequently, for all $t \in [0,\vartheta]$ and $l = 1,\ldots,r$, equality (16) is fulfilled. Therefore, by virtue of (13), we have $h^*\Gamma(\vartheta,\omega)h = 0$, i.e., $\lambda(\vartheta,\omega) = 0$. □

Remark 4. The matrix $\Gamma(\vartheta,\omega)$ represents the Gram matrix for the set of vector functions $\{u_{ip}(\cdot,\omega)\}_{i,p=1}^n$ on the segment $[0,\vartheta]$.

Theorem 3. *The following assertions are equivalent:*

a) *equation* (3) *is consistent;*
b) $\vartheta > 0$ *can be found with which the matrix* $\Gamma(\vartheta,\omega)$ *is positive definite* ($\lambda(\vartheta,\omega) > 0$);
c) *the set of vector-functions* $\{u_{ip}(\cdot,\omega)\}_{i,p=1}^n$, *which are defined via equalities* (14), (15), *is linearly independent on* $[0,\vartheta]$ *for a certain* $\vartheta > 0$.

Proof. Conditions b) and c) are equivalent by virtue of Lemma 6. Let us show that b) implies a). Let $\lambda(\vartheta,\omega) > 0$ with a certain $\vartheta > 0$. For any $G \in M_n$, it is required to find a control $u = \mathrm{col}(u_1,\ldots,u_r) : [0,\vartheta] \to \mathbb{R}^r$, under which equation (7) has a solution satisfying the conditions $Z(0) = 0$, $Z(\vartheta) = G$. Thus, the problem of construction of $u(\cdot)$ can be reduced to the problem on resolvability of the equation

$$\int_0^{\vartheta} \sum_{l=1}^{r} \widehat{A}_l(t,\omega)u_l(t)\,dt = Q(\vartheta,\omega), \quad (17)$$

where $Q(\vartheta,\omega) = X_0(0,\vartheta,\omega)G$. A solution $\widehat{u} = \mathrm{col}(\widehat{u}_1,\dots,\widehat{u}_r)$ of equation (17) will be sought in the form $\widehat{u}(t) = \sum_{j,s=1}^{n} h_{js}u_{js}(t)$, where $u_{js}(t)$ are defined via equalities (14), (15). By substituting $\widehat{u}(t)$ into (17) and multiplying (17) by e_i^* from the left and by e_p from the right, we get the following system of n^2 algebraic equations

$$\sum_{j,s=1}^{n} \gamma_{ijps}h_{js} = q_{ip}, \quad i,p = 1,\dots,n, \tag{18}$$

with respect to h_{js}. Here $q_{ip} = e_i^*Q(\vartheta,\omega)e_p$. Since $\lambda(\vartheta,\omega) > 0$, system (18) is resolvable: $h = \Gamma^{-1}q$, where $h = \mathrm{col}(h_{11},\dots,h_{1n},\dots,h_{n1},\dots,h_{nn})$, $q = \mathrm{col}(q_{11},\dots,q_{1n},\dots,q_{n1},\dots,q_{nn})$. Therefore equation (3) is consistent. Note that for the solution $\widehat{u}(t)$ of equation (17) the following estimate exists

$$|\widehat{u}(t)| \le \sum_{j,s=1}^{n} |h_{js}|\,|u_{js}(t)| \le |\Gamma^{-1}(\vartheta,\omega)| \sum_{j,s=1}^{n} |u_{js}(t)|\,|q(\vartheta,\omega)|.$$

Further, the boundedness of $A_l(f^t\omega)$ and $X_0(t,s,\omega)$ for $(t,s)\in[0,\vartheta]\times[0,\vartheta]$ implies the boundedness on $[0,\vartheta]$ of the functions $u_{js}(t)$. Therefore, a constant β (independent of G, but depending on ϑ) can be found such that $|\widehat{u}(t)| \le \beta|G|$, $0 \le t \le \vartheta$.

Let us prove that a) implies c). We choose $\vartheta > 0$ from the definition of consistency. It suffices to show that, if equation (17) is resolvable with respect to $u(t)$ for any Q, then the set of vector functions (15) is linearly independent on $[0,\vartheta]$. Assume the contrary, let a nonzero vector $h = \mathrm{col}(h_{11},\dots,h_{1n},\dots,h_{n1},\dots,h_{nn})$ exist such that $\sum_{i,p=1}^{n} h_{ip}u_{ip}(t,\omega) = 0$, $t\in[0,\vartheta]$. Then $\sum_{i,p=1}^{n} h_{ip}e_i^*\widehat{A}_l(t,\omega)e_p = 0$ for all $l = 1,\dots,r$ and $t\in[0,\vartheta]$. For the matrix $H = \{h_{ip}\}_{i,p=1}^{n}$, a function $\widehat{u} = \mathrm{col}(\widehat{u}_1,\dots,\widehat{u}_r) : [0,\vartheta] \to \mathbb{R}^r$ exists such that

$$\int_0^\vartheta X_0(0,t,\omega)(\widehat{u}_1(t)A_1(f^t\omega) + \dots + \widehat{u}_r(t)A_r(f^t\omega))X_0(t,0,\omega)\,dt = H.$$

Therefore, for all $i,p = 1,\dots,n$,

$$\int_0^\vartheta e_i^*X_0(0,t,\omega)(\widehat{u}_1(t)A_1(f^t\omega) + \dots + \widehat{u}_r(t)A_r(f^t\omega))X_0(t,0,\omega)e_p\,dt = h_{ip}.$$

By multiplying each of these equalities by h_{ip} and summing over $i,p = 1,\dots,n$, we get

$$\sum_{i,p=1}^{n} h_{ip}^2 = \sum_{i,p=1}^{n} h_{ip}\int_0^\vartheta e_i^*X_0(0,t,\omega)(\widehat{u}_1(t)A_1(f^t\omega) + \dots + \widehat{u}_r(t)A_r(f^t\omega))X_0(t,0,\omega)e_p\,dt =$$
$$= \int_0^\vartheta \sum_{l=1}^{r} \widehat{u}_l(t)\Big(\sum_{i,p=1}^{n} h_{ip}e_i^*\widehat{A}_l(t,\omega)e_p\Big)\,dt = 0,$$

which contradicts the condition $h \ne 0$. □

Remark 5. Let a triple $(A_0(\omega), B(\omega), C(\omega))$ be given, where $B = (b_1,\dots,b_m)$, $b_i\in\mathbb{R}^n$, $C = (c_1,\dots,c_k)$, $c_j\in\mathbb{R}^n$. Put $r = mk$ and construct the matrices $A_1 = b_1c_1^*$, $A_2 = b_1c_2^*,\dots, A_k = b_1c_k^*$, $A_{k+1} = b_2c_1^*,\dots, A_{2k} = b_2c_k^*,\dots, A_{r-k+1} = b_mc_1^*,\dots, A_r = b_mc_k^*$. Then the definition of consistency for the equation $\dot{x} = A_0(f^t\omega)x + u_1A_1(f^t\omega)x + \dots + u_rA_r(f^t\omega)x$ with the matrices A_l thus constructed coincides with the definition of consistency introduced in [5] for equation (12).

Theorem 4. *Let a set $\overline{\gamma}(\omega)$ be compact. Then the equation $(\mathbb{A},\omega)$ is uniformly consistent if and only if $\vartheta > 0$ and $\varepsilon > 0$ can be found such that for all $\omega_0 \in \overline{\gamma}(\omega)$ the least eigenvalue $\lambda(\vartheta,\omega_0)$ of consistency matrix $\Gamma(\vartheta,\omega_0)$ satisfies the inequality $\lambda(\vartheta,\omega_0) \ge \varepsilon$.*

Proof. Let the equation $(\mathbf{A},\omega)$ be uniformly consistent. Since the set $\overline{\gamma}(\omega)$ is compact and the function $\omega_0 \to \lambda(\vartheta,\omega_0)$ is continuous, $\omega_1 \in \overline{\gamma}(\omega)$ exists such that $\lambda(\vartheta,\omega_1) = \min\{\lambda(\vartheta,\omega_0) : \omega_0 \in \overline{\gamma}(\omega)\}$. If $\lambda(\vartheta,\omega_1) = 0$, then the equation $(\mathbf{A},\omega_1)$ is not consistent, which contradicts Definition 2.

Let $\lambda(\vartheta,\omega_0) \geq \varepsilon$ for some $\vartheta > 0$, $\varepsilon > 0$ and all $\omega_0 \in \overline{\gamma}(\omega)$. Then, for each $\omega_0 \in \overline{\gamma}(\omega)$, the equation $(\mathbf{A},\omega_0)$ is consistent (Theorem 3). Therefore equation (7) has a solution $Z(\cdot)$ which satisfies the conditions $Z(0) = 0$, $Z(\vartheta) = G$, as $u_G(t,\omega_0) = \sum_{i,p=1}^{n} h_{ip}u_{ip}(t,\omega_0)$, where $u_{ip}(t,\omega_0)$ are defined by equalities (14), (15), while h_{ip} is to be obtained via equations (18). Since the estimate $|h| \leq |\Gamma^{-1}(\vartheta,\omega_0)|\,|q(\vartheta,\omega_0)|$ takes place and $|\Gamma^{-1}(\vartheta,\omega_0)| \leq \varepsilon^{-1}$ for all $\omega_0 \in \overline{\gamma}(\omega)$, we have $|h| \leq \varepsilon^{-1}|q(\vartheta,\omega_0)|$. Further, a constant η can be found such that $|u_{ip}(t,\omega_0)| \leq \eta$ for all $(t,\omega_0) \in [0,\vartheta] \times \overline{\gamma}(\omega)$ and all $i,p = 1,\dots,n$ (this follows from the boundedness of $\mathbf{A}$ on $\overline{\gamma}(\omega)$ and invariance of $\overline{\gamma}(\omega)$ with respect to f^t). Therefore $\beta > 0$ can be found such that $|u_G(t,\omega_0)| \leq \beta|G(\omega_0)|$, $(t,\omega_0) \in [0,\vartheta] \times \overline{\gamma}(\omega)$. □

Theorem 5. *Let $\overline{\gamma}(\omega)$ be minimal. Then the equation $(\mathbf{A},\omega)$ is uniformly consistent if and only if it is consistent.*

The proof is based on arguments in paper [5].

Example 1. The system of equations

$$\dot{x}_1 = u(a_{11}(t)x_1 + a_{12}(t)x_2), \quad \dot{x}_2 = u(a_{21}(t)x_1 + a_{22}(t)x_2), \quad |u| \leq 1,$$

with coefficients almost periodic in the Bohr sense is uniformly consistent (and, therefore, is uniformly locally attainable) if the functions $a_{ij}(t)$ are linearly independent on $\mathbb{R}$ (then, by virtue of almost periodicity they will be linearly independent on $[\tau,\tau+\vartheta]$ for a certain $\vartheta > 0$ and all τ). To prove this fact it suffices to construct by the matrix $A(t) = \{a_{ij}(t)\}$ the closure (in the topology of uniform convergence on the straight line) of the set of shifts $\mathcal{H}(A)$ of the matrix $A(t)$ and give on $\mathcal{H}(A)$ a dynamical system of shifts (see [12], Chap. VI, § 5). Since $\mathcal{H}(A)$ is minimal, by using assertion c) of Theorem 3 and Theorem 5 we get what is desired.

Let us introduce into consideration the mapping vec : $M_n \to \mathbb{R}^{n^2}$, which "unrolls" the matrix $H = \{h_{ij}\}_{i,j=1}^{n}$ along the lines into a column vector $\operatorname{vec} H \doteq \operatorname{col}(h_{11},\dots,h_{1n},\dots,h_{n1},\dots,h_{nn})$. One can easily verify that, for any $C, L, A, M \in M_n$, from $C = LA$ it follows that $\operatorname{vec} C = (L \otimes I_n)\operatorname{vec} A$ and the equality $C = AM$ implies $\operatorname{vec} C = (I_n \otimes M^*)\operatorname{vec} A$ ($\otimes$ is the Kronecker (direct) product of matrices, see [13], p. 235). By using equation $(\mathbf{A},\omega)$, we construct so-called "large equation" (see [6])

$$\dot{z} = F(f^t\omega)z + G(f^t\omega)v, \quad z \in \mathbb{R}^{n^2}, \quad v \in \mathbb{R}^r, \tag{19}$$

where $F(\omega) = A_0(\omega) \otimes I - I \otimes A_0^*(\omega) \in M_{n^2}$, $G(\omega) = (\operatorname{vec} A_1(\omega),\dots,\operatorname{vec} A_r(\omega)) \in M_{n^2\times r}$. We denote by $Z_0(t,s,\omega)$ the Cauchy function of the equation $\dot{z} = F(f^t\omega)z$. We shall identify equation (19) with the triple (F,G,ω).

Let us recall that equation (F,G,ω) is said to be completely controllable (see [14], Chap. 6, § 20) if with a certain $\vartheta > 0$ the controllability matrix

$$W(\vartheta,\omega) = \int_0^{\vartheta} Z_0(0,t,\omega)G(f^t\omega)G^*(f^t\omega)Z_0^*(0,t,\omega)\,dt \tag{20}$$

is definitely positive; it is said to be uniformly completely controllable (see [15]) is $\vartheta > 0$ can be found such that $W(\vartheta,\omega_0)$ is definitely positive uniformly with respect to $\omega_0 \in \overline{\gamma}(\omega)$.

Theorem 6. *The equation $(\mathbf{A},\omega)$ is consistent (uniformly consistent) if and only if the equation (F,G,ω) is completely controllable (uniformly completely controllable).*

Proof. Let $\Gamma(\vartheta,\omega)$ be the consistency matrix of the equation $(\mathbb{A},\omega)$. Since $\widehat{A}_l(t,\omega) = X_0(0,t,\omega)A_l(f^t\omega)X_0(t,0,\omega)$, we have that $\operatorname{vec}\widehat{A}_l(t,\omega) = (X_0(0,t,\omega)\otimes X_0^*(t,0,\omega))\operatorname{vec}A_l(f^t\omega)$. Therefore $\Gamma(\vartheta,\omega) = \int_0^\vartheta\sum_{l=1}^r \operatorname{vec}\widehat{A}_l(t,\omega)\cdot(\operatorname{vec}\widehat{A}_l(t,\omega))^*\,dt$. Further one can easily verify that $Z_0(0,t,\omega)=X_0(0,t,\omega)\otimes X_0^*(t,0,\omega)$. Consequently, $Z_0(0,t,\omega)G(f^t\omega)=(\operatorname{vec}\widehat{A}_1(t,\omega),\dots,\operatorname{vec}\widehat{A}_r(t,\omega))$. Therefore $W(\vartheta,\omega)=\int_0^\vartheta\sum_{l=1}^r \operatorname{vec}\widehat{A}_l(t,\omega)\cdot(\operatorname{vec}\widehat{A}_l(t,\omega))^*\,dt$. Thus, $\Gamma(\vartheta,\omega)=W(\vartheta,\omega)$. □

5. Consistency of stationary equation

Consider equation (3) with a constant matrix $\mathbf{A}(\omega)\equiv\mathbf{A}$ and the equation $(F,\ G,\ \omega)\equiv(F,\ G)$ corresponding to it. For each $\nu=1,\dots,r$, we introduce the matrices $L_0^\nu = A_\nu$, $L_k^\nu = A_0L_{k-1}^\nu - L_{k-1}^\nu A_0$, $k = 1,\dots,n^2-1$.

Theorem 7. *The equation* $\mathbb{A}$ *is consistent if and only if among matrices* L_k^ν, $k=0,\dots,n^2-1$, $\nu=1,\dots,r$, *there exist* n^2 *linearly independent.*

Proof. The equation $\mathbb{A}$ is consistent if and only if the equation (F,G) is completely controllable, which is equivalent to the equality $\operatorname{rank}[G,FG,\dots,F^{n^2-1}G]=n^2$. By applying to the column vectors of matrix G the mapping vec^{-1}, we obtain r matrices $L_0^1,\dots,L_0^r$. We proceed in the same way with the matrices F^kG, $k=1,\dots,n^2-1$. Then at each step we shall get r matrices $L_k^1,\dots,L_k^r$. Therefore the equality $\operatorname{rank}[G,FG,\dots,F^{n^2-1}G]=n^2$ takes place if and only if the linear hull of the matrices L_k^ν, $k=0,\dots,n^2-1$, $\nu=1,\dots,r$, coincides with M_n. □

Corollary 2. Assume that A_0 commutes with the matrices $A_1,\dots,A_r$. Then the equation $\mathbb{A}$ is consistent if and only if the linear hull of the matrices $A_1,\dots,A_r$ coincides with M_n.

Corollary 3. Let the matrices $A_0,\dots,A_r$ be quasi-triangular, i. e., $A_\nu=\{B_{ij}^\nu\}_{i,j=1}^s$, $B_{ii}^\nu\in M_{n_i}$, $\sum_{i=1}^s n_i=n$, $B_{ij}^\nu=0$ for $i>j$. Then the equation $\mathbb{A}=(A_0,\dots,A_r)$ is not consistent.

Proof. For all $\nu=1,\dots,r$, $k=0,\dots,n^2-1$, the matrices L_k^ν have the same form as the matrices A_ν; therefore, their linear hull does not coincide with M_n. □

Example 2. Let us show that the equation $\mathbb{A}=(A_0,A_1)$ is not consistent. It suffices to construct a matrix $W_0\in M_n$, which satisfies the equalities $(\operatorname{vec}W_0)^*F^kb=0$ for all $k=0,\dots,n^2-1$, where $F=A_0\otimes I - I\otimes A_0^*$, $b=\operatorname{vec}A_1$. One can easily verify the following $(\operatorname{vec}W_0)^*F^kb = b^*(\operatorname{vec}W_k)$, $k=0,\dots,n^2-1$, where $W_k=A_0^*W_{k-1}-W_{k-1}A_0^*$. If $\operatorname{Sp}A_1=0$, then with $W_0=I$ the following relation is fulfilled $W_k=0$, $k=1,\dots,n^2-1$ and $b^*(\operatorname{vec}W_0)=0$. If $\operatorname{Sp}A_1\neq0$ and $A_0\neq\mu I$, then we put $W_0=A_0^*-\dfrac{\operatorname{Sp}(A_1A_0)}{\operatorname{Sp}A_1}I$. In this case, the equalities $b^*(\operatorname{vec}W_k)=0$ also hold for all $k=0,\dots,n^2-1$. If $A_0=\mu I$, then $F=0$ and we have no consistency.

Corollary 4. If $\operatorname{Sp}A_\nu=0$ for all $\nu=1,\dots,r$, then the equation $\mathbb{A}$ is not consistent.

Proof. We put $W_0=I$. Then we have $(\operatorname{vec}W_0)^*[G,FG,\dots,F^{n^2-1}G]=0$.

Example 3. Let the degree of the minimal polynomial of the matrix A_0 be equal to m $(m\geq2)$. Then the equation $\mathbb{A}\equiv(A_0,\dots,A_r)$, where $r\leq m-1$, is not consistent. To prove this, it suffices to construct a matrix $W_0\in M_n$, which satisfies the equalities $b_\nu^*(\operatorname{vec}W_k)=0$ for all $\nu=1,\dots,r$, $k=0,\dots,n^2-1$, where $b_\nu=\operatorname{vec}A_\nu$, $W_k=A_0^*W_{k-1}-W_{k-1}A_0^*$. We shall seek W_0 in the form $W_0=\sum_{s=0}^{m-1}\lambda_s(A_0^*)^s$ (the matrices $(A_0^*)^s$, $s=0,\dots,m-1$, are linearly independent). Then we have $W_k=0$ for all $k=1,\dots,n^2-1$. Further, the equalities $b_\nu^*(\operatorname{vec}W_0)=0$ are equivalent to the following ones $\sum_{s=0}^{m-1}\lambda_s\operatorname{Sp}(A_\nu A_0^s)=0$, $\nu=1,\dots,r$, which form a system in r equations with m unknowns $\lambda_0,\dots,\lambda_{m-1}$. Since $r<m$, a nontrivial solution of this system exists, which proves the absence of the consistency.

Corollary 5. If all eigenvalues of the matrix A_0 differ from each other, then the equation $\mathbb{A} \equiv (A_0, \dots, A_r)$, where $r \le n-1$, is not consistent.

Thus, if the minimal polynomial of the matrix A_0 coincides with the characteristic one, then the presence of r controlling parameters, $r < n$, is not sufficient for consistency. Let us show that the quantity of n parameters of the control is sufficient, i. e., matrices $A_1, \dots, A_n$ can be found such that the equation $\mathbb{A} = (A_0, A_1, \dots, A_n)$ will be consistent. Let $\lambda^n + \alpha_1\lambda^{n-1} + \dots + \alpha_n$ be the characteristic polynomial of the matrix A_0. Then the matrix A_0 in a certain basis has the form

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \dots & -\alpha_1 \end{pmatrix}.$$

We construct the matrices $A_1 = e_n e_1^*, \dots, A_n = e_n e_n^*$ and, by the equation $\mathbb{A}$, the equation (F, G). Then

$$F = \begin{pmatrix} -A_0^* & I & 0 & \dots & 0 \\ 0 & -A_0^* & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \\ -\alpha_n I & -\alpha_{n-1} I & -\alpha_{n-2} I & \dots & -\alpha_1 I - A_0^* \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{pmatrix}.$$

Obviously, $\operatorname{rank}[G, FG, \dots, F^{n-1}G] = n^2$. Therefore the equation $\mathbb{A}$ is consistent.

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