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CONTROL THEORY

Spectrum Control in Linear Systems with Incomplete Feedback

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Abstract— For a linear stationary control system closed by a linear incomplete feedback, we obtain necessary and sufficient conditions for the solvability of the spectrum control problem in the case of coefficients of special form.

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1. NOTATION AND DEFINITIONS

Let $e_1, \ldots, e_n$ be the canonical basis in the space $\mathbb{R}^n$ [i.e., $e_1 = \operatorname{col}(1, 0, \ldots, 0)$, $\ldots$, $e_n = \operatorname{col}(0, \ldots, 0, 1)$]; let $M_{m,n}$ be the space of real $m \times n$ matrices; let $M_n := M_{n,n}$; let $I = [e_1, \ldots, e_n] \in M_n$ be the identity matrix; let $*$ be the operation of transposition of a vector or a matrix; let $J_0 := I$; let J_1 be the matrix whose first superdiagonal is all ones and whose other entries are zero (i.e., $J_1 := \sum_{i=1}^{n-1} e_i e_{i+1}^* \in M_n$); let $J_k := J_1^k$, $k \in \mathbb{N}$ (thus, $J_k = 0 \in M_n$ for $k \geq n$); let $\chi(A; \lambda)$ be the characteristic polynomial of a matrix A; and let $\operatorname{Sp} A$ be the trace of a matrix A.

Consider the linear stationary control system

$$\dot{x} = Ax + Bu, \qquad (x, u) \in \mathbb{R}^n \times \mathbb{R}^m, \tag{1}$$

$$y = C^* x, \qquad y \in \mathbb{R}^k, \tag{2}$$

specified by a matrix $(A, B, C) \in M_{n,n+m+k}$. Let the control in system (1), (2) be constructed by the principle of linear incomplete feedback in the form $u = Uy$, where $U \in M_{m,k}$ is a constant matrix. The corresponding closed system has the form

$$\dot{x} = (A + BUC^*)x, \qquad x \in \mathbb{R}^n. \tag{3}$$

In the present paper, we consider the spectrum control problem for the matrix $A + BUC^*$ of system (3). This problem can be stated in different ways.

(i) There are given matrices A, B, and C (possibly, complex). For an arbitrary set $\{\mu_j\}_{j=1}^n$ of complex numbers, construct a matrix (possibly, complex) control U such that the eigenvalues $\lambda_j(A + BUC^*)$ of the matrix $A + BUC^*$ coincide with the numbers μ_j.

(ii) There are given real matrices A, B, and C. For an arbitrary set $\{\mu_j\}_{j=1}^n$ of complex numbers closed with respect to complex conjugation, construct a real matrix U such that $\lambda_j(A+BUC^*) = \mu_j$, $j = 1, \ldots, n$.

We assume that A, B, and C are real matrices and solve this problem in the second setting; all obtained assertions and proofs hold for the first setting as well.

Definition 1. We say that the spectrum control problem for the matrix $A + BUC^*$ is solvable if, for any polynomial $p(\lambda) = \lambda^n + \gamma_1 \lambda^{n-1} + \cdots + \gamma_n$ of degree n with real coefficients γ_i, there exists a real matrix $U \in M_{m,k}$ such that the characteristic polynomial $\chi(A + BUC^*; \lambda)$ of the matrix $A + BUC^*$ coincides with $p(\lambda)$.

This problem is also referred to as the *eigenvalue placement problem* [1, p. 159] or the *modal control problem* [2, p. 435]. If $C = I$, then the spectrum control problem is solvable if and only if system (1) is completely controllable [3, p. 320; 4]. This problem with $m < n$ and $k < n$ was studied by numerous authors. A detailed review of known necessary and sufficient conditions for the solvability of this problem can be found in [1, Chap. III, Sec. 7]. In the present paper, we obtain new conditions for the solvability of this problem for system (3) with coefficients of the form (4) (see below), which are necessary and sufficient conditions. These results generalize those in [5, 6].

2. RESULTS

Let the coefficients of system (3) have the following form:

$$\begin{aligned}
&A = \{a_{ij}\}_{i,j=1}^n, \qquad a_{i,i+1} \neq 0, \quad i = 1, \dots, n-1; \qquad a_{ij} = 0, \quad j > i+1;\\
&B = \{b_{ij}\}, \qquad C = \{c_{il}\}, \quad i = 1, \dots, n, \quad j = 1, \dots, m, \quad l = 1, \dots, k;\\
&b_{ij} = 0, \quad i = 1, \dots, p-1, \quad j = 1, \dots, m;\\
&c_{il} = 0, \quad i = p+1, \dots, n, \quad l = 1, \dots, k; \quad p \in \{1, \dots, n\}.
\end{aligned} \tag{4}$$

Let $\lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_n := \chi(A; \lambda)$. Set $\alpha_0 := 1$. We remove the last row from the matrix A and denote the resulting matrix by $Q \in M_{n-1,n}$. Let us construct the matrix $S_1 = \left\| \begin{matrix} e_1^* \\ Q \end{matrix} \right\|$. Then $S_1 \in M_n$ is a lower triangular matrix, and $\det S_1 \neq 0$. For each $l = 2, \dots, n-1$, on the basis of the matrix $S_{l-1} = \{s_{ij}^{l-1}\}_{i,j=1}^n$, we construct the matrix $S_l = \{s_{ij}^l\}_{i,j=1}^n$ as follows: $s_{11}^l := 1$, $s_{1j}^l := s_{j1}^l := 0$, $j = 2, \dots, n$; $s_{ij}^l = s_{i-1,j-1}^{l-1}$, $i, j = 2, \dots, n$. Then the S_l are lower triangular nonsingular matrices for all $l = 1, \dots, n-1$. Let $S = S_{n-1} \cdots S_1$. Then S is a lower triangular nonsingular matrix as well. Let us construct the matrix

$$G := \sum_{i=1}^n \alpha_{i-1} J_{i-1}^*.$$

Theorem 1. *Let system* (3) *with matrices of the form* (4) *be given, and let*

$$\lambda^n + \gamma_1\lambda^{n-1} + \cdots + \gamma_n := \chi(A + BUC^*; \lambda).$$

Then the coefficients γ_i of the characteristic polynomial of the matrix $A + BUC^*$ can be expressed via the coefficients of system (3), (4) *as follows*:

$$\gamma_i = \alpha_i - \mathrm{Sp}(SBUC^*S^{-1}J_{i-1}G), \qquad i = 1, \dots, n. \tag{5}$$

Theorem 2. *The spectrum control problem for system* (3) *with matrices of the form* (4) *is solvable if and only if the matrices*

$$C^*S^{-1}J_0GSB, \quad C^*S^{-1}J_1GSB, \quad \dots, \quad C^*S^{-1}J_{n-1}GSB \tag{6}$$

are linearly independent.

Proof of Theorem 2. We prove Theorem 2 as a corollary of Theorem 1 and explicitly write out a control reducing the characteristic polynomial to a given form. Let

$$p(\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \cdots + \gamma_n$$

be a given polynomial with real coefficients γ_i. The problem is to construct U such that $\chi(A + BUC^*; \lambda) = p(\lambda)$, i.e., such that relations (5) hold:

$$\gamma_i = \alpha_i - \mathrm{Sp}(SBUC^*S^{-1}J_{i-1}G) = \alpha_i - \mathrm{Sp}(UC^*S^{-1}J_{i-1}GSB), \qquad i = 1, \dots, n.$$

It is a system of n linear equations for the mk unknown entries u_{pq}, $p = 1, \dots, m$, $q = 1, \dots, k$, of the matrix U. We introduce the mapping vec : $M_{k,m} \to \mathbb{R}^{km}$, which "expands" the matrix $H = \{h_{ij}\}$, $i = 1, \dots, k$, $j = 1, \dots, m$, by rows into the column vector

$$\mathrm{vec}\, H := \mathrm{col}(h_{11}, \dots, h_{1m}, \dots, h_{k1}, \dots, h_{km}).$$

Obviously, for any matrices $H_1, H_2 \in M_{k,m}$, one has $\operatorname{Sp}(H_1^* H_2) = (\operatorname{vec} H_1)^*(\operatorname{vec} H_2)$. Then the system of linear equations (5) can be rewritten in the vector form

$$\alpha - P^* v = \gamma. \tag{7}$$

Here

$$P = [\operatorname{vec} C^* S^{-1} J_0 GSB, \dots, \operatorname{vec} C^* S^{-1} J_{n-1} GSB] \in M_{km,n}, \qquad \alpha = \operatorname{col}(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n,$$

$$\gamma = \operatorname{col}(\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n, \qquad v = \operatorname{vec} U^* \in \mathbb{R}^{km}.$$

If the matrices (6) are linearly independent, then $\operatorname{rank} P = n$. Then P^*P is nonsingular, so that system (7) is solvable for any γ and, in particular, has the solution $v = P(P^*P)^{-1}(\alpha - \gamma)$. Therefore, the spectrum control problem is solvable, and the corresponding control has the form $U = (\operatorname{vec}^{-1} v)^*$. If the matrices (6) are linearly dependent, then $\operatorname{rank} P < n$ and system (7) is unsolvable whenever $\gamma = \alpha - \beta$, where $\beta \notin \operatorname{Im} P^*$; consequently, the spectrum control problem is unsolvable. The proof of the theorem is complete.

Remark 1. It follows from Theorem 2 that a necessary condition for the solvability of the spectrum control problem for system (3) with matrices (4) is that $mk \geq n$. If the matrices (6) are linearly independent and $mk = n$, then the control U ensuring that $\chi(A + BUC^*; \lambda) = p(\lambda)$ is unique; if the matrices (6) are linearly independent and $mk > n$, then such a control is nonunique.

If only $l < n$ of the matrices (6) are linearly independent, then the spectrum control problem is unsolvable. Nevertheless, the spectrum can be partly controlled; namely, one can ensure that $\chi(A + BUC^*; \lambda) = p(\lambda)$ for any $\gamma = (\gamma_1, \dots, \gamma_n)$ in the l-dimensional linear manifold $\alpha - \operatorname{Im} P^*$ in $\mathbb{R}^n$.

Corollary 1. *If the matrices* (6) *are linearly independent, then system* (3), (4) *is stabilizable in the class of constant matrix controls* U; *i.e., for any* $\varkappa > 0$, *there exists a constant matrix* U *such that the eigenvalues* λ_i *of the matrix* $A + BUC^*$ *satisfy the conditions* $\operatorname{Re} \lambda_i \leq -\varkappa < 0$, $i = 1, \dots, n$.

Remark 2. Theorems 1 and 2 hold for system (3) with coefficients of the form (4). Such systems do not exhaust all possibly systems of the form (3). Nevertheless, the class of such systems is sufficiently large; in particular, it contains the following important class of systems. Consider a plant described by a linear ordinary differential equation of order n with constant coefficients in which a linear combination of m signals and their derivatives of order $\leq n - p$ is supplied to the input and k distinct linear combinations of the plant state z and its derivatives of order $\leq p - 1$ are measured (see [6]). Let us construct a linear incomplete feedback control. By using the standard substitution $z = x_1$, $\dot{z} = x_2$, $\dots$, $z^{(n-1)} = x_n$, we pass from an equation of order n to a system of differential equations. Then the coefficients of the resulting system of equations have exactly the form (4). The corresponding result, Theorem 2 for an ordinary differential equation, was obtained in [6].

3. PROOF OF THEOREM 1

Let $Q' \in M_{n-1}$ be the matrix obtained from S_1 by removing the last row and the last column. We introduce the matrix $A_1 := S_1 A S_1^{-1}$.

Lemma 1. *The matrix* A_1 *has the form* $A_1 = \left\| \begin{array}{c|cc} 0 & & Q' \\ \hline * & * & * \end{array} \right\|$; *here* $0 \in \mathbb{R}^{n-1}$, *and the last row contains numbers such that* $\chi(A; \lambda) = \chi(A_1; \lambda)$.

In this lemma, we claim the following. If we multiply the matrix A by S_1 on the left and by S_1^{-1} on the right, then the rectangular "support block" $Q \in M_{n-1,n}$ of the matrix A is shifted right and down along the diagonal; the last row and the last column of the matrix Q are lost, and the first column (of height $n-1$) consisting of zeros and the first row e_2^* (of length n) are added; the last row of the matrix A changes so as to ensure that the characteristic polynomial is preserved. Lemma 1 is well stated, since the last row of the matrix A_1 can be uniquely reconstructed from its characteristic polynomial. This is a consequence of the following auxiliary assertion.

Lemma 2. *Let* $P = \left\| \begin{matrix} Q \\ \xi \end{matrix} \right\|$ *and* $R = \left\| \begin{matrix} Q \\ \psi \end{matrix} \right\|$ *be matrices such that the characteristic polynomials of these matrices coincide,* $\chi(P;\lambda) = \chi(R;\lambda)$. *Then the last rows of these matrices coincide.*

Proof. Let $\xi = (p_1, \ldots, p_n) \in \mathbb{R}^{n*}$ and $\psi = (r_1, \ldots, r_n) \in \mathbb{R}^{n*}$. Consider the matrix $\lambda I - P$. By Δ_i, $i = 1, \ldots, n$, we denote the principal diagonal minors of this matrix, $\Delta_1 = \lambda - a_{11}$, $\Delta_2 = (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21}$, $\ldots$, $\Delta_n = \det(\lambda I - P)$. Note that, for all $i = 1, \ldots, n$, the degree of the polynomial Δ_i is equal exactly to i, and the leading coefficient of λ^i is equal to unity. We expand $\det(\lambda I - P)$ with respect to the last row; then we obtain

$$\begin{aligned}\chi(P;\lambda) &= (\lambda - p_n)\Delta_{n-1} - (-p_{n-1})(-a_{n-1,n})\Delta_{n-2} + (-p_{n-2})(-a_{n-1,n})(-a_{n-2,n-1})\Delta_{n-3} + \cdots \\ &\quad + (-1)^{n-2}(-p_2)(-a_{n-1,n}) \cdots (-a_{23})\Delta_1 + (-1)^{n-1}(-p_1)(-a_{n-1,n}) \cdots (-a_{12}) \\ &= (\lambda - p_n)\Delta_{n-1} - p_{n-1}a_{n-1,n}\Delta_{n-2} - p_{n-2}a_{n-1,n}a_{n-2,n-1}\Delta_{n-3} - \cdots \\ &\quad - p_2 a_{n-1,n} \cdots a_{23}\Delta_1 - p_1 a_{n-1,n} \cdots a_{12}.\end{aligned}$$

Likewise,

$$\begin{aligned}\chi(R;\lambda) &= (\lambda - r_n)\Delta_{n-1} - r_{n-1}a_{n-1,n}\Delta_{n-2} - r_{n-2}a_{n-1,n}a_{n-2,n-1}\Delta_{n-3} - \cdots \\ &\quad - r_2 a_{n-1,n} \cdots a_{23}\Delta_1 - r_1 a_{n-1,n} \cdots a_{12}.\end{aligned}$$

The characteristic polynomials of the matrices coincide. Let us subtract the first polynomial from the second one; then we obtain

$$\begin{aligned}&(p_n - r_n)\Delta_{n-1} + (p_{n-1} - r_{n-1})a_{n-1,n}\Delta_{n-2} + \cdots \\ &\qquad + (p_2 - r_2)a_{n-1,n} \cdots a_{23}\Delta_1 + (p_1 - r_1)a_{n-1,n} \cdots a_{12} = 0.\end{aligned}$$

The left-hand side is a polynomial of degree $\le n - 1$ in λ, the right-hand side is zero, and consequently, all coefficients of λ^{i-1}, $i = 1, \ldots, n$, are zero. Since the coefficient of λ^{n-1} is zero, and of the polynomials $\Delta_1, \ldots, \Delta_{n-1}$ only the polynomial Δ_{n-1} has degree $n - 1$, i.e., contains the monomial λ^{n-1}, it follows that the coefficient of Δ_{n-1} is zero; consequently, $p_n = r_n$. We proceed in a similar way. Since $a_{k,k+1} \ne 0$ and all polynomials Δ_i have distinct degrees, we have $p_i = r_i$ for all $i = 1, \ldots, n$. The proof of the lemma is complete.

Proof of Lemma 1. Since $S_1 = \left\| \begin{matrix} e_1^* \\ \hline Q \end{matrix} \right\|$, we have $QS_1^{-1} = \| \, 0 \mid I \, \| \in M_{n-1,n}$, $0 \in \mathbb{R}^{n-1}$, $I \in M_{n-1}$. Therefore, $AS_1^{-1} = \left\| \begin{matrix} Q \\ \hline * \end{matrix} \right\| \cdot S_1^{-1} = \left\| \begin{array}{c|cc} 0 & & I \\ \hline * & * & * \end{array} \right\|$. Since $S_1 = \left\| \begin{array}{cc|c} Q' & & 0 \\ \hline * & * & * \end{array} \right\|$, it follows that

$$S_1 A S_1^{-1} = \left\| \begin{array}{cc|c} Q' & & 0 \\ \hline * & * & * \end{array} \right\| \cdot \left\| \begin{array}{c|cc} 0 & & I \\ \hline * & * & * \end{array} \right\| = \left\| \begin{array}{c|cc} 0 & & Q' \\ \hline * & * & * \end{array} \right\|.$$

The proof of the lemma is complete.

The matrix A_1 in Lemma 1 has the form of the matrix A in (4). (All entries lying above the superdiagonal are zero, and the superdiagonal consists of nonzero entries.) In the matrix A_1, we remove the last row and add the first row e_1^*. Then we obtain the matrix S_2. We construct the matrix $A_2 = S_2 A_1 S_2^{-1} = S_2 S_1 A S_1^{-1} S_2^{-1}$ and use Lemma 1. As a result, the "support block" is again shifted right and down along the diagonal, the characteristic polynomial is preserved, the matrix A_2 has the form $A_2 = \left\| \begin{array}{c|cc} 0 & & Q_1' \\ \hline * & * & * \end{array} \right\|$; here $0 \in \mathbb{R}^{n-1}$, and the matrix $Q_1' \in M_{n-1}$ is obtained from the matrix S_2 by removing the last row and the last column. By using Lemma 1 $n - 1$ times, we obtain the matrix $A_{n-1} = S_{n-1} \cdots S_1 A S_1^{-1} \cdots S_{n-1}^{-1} = \left\| \begin{array}{c|cc} 0 & & I \\ \hline * & * & * \end{array} \right\|$, $0 \in \mathbb{R}^{n-1}$,

$I \in M_{n-1}$; moreover, $\chi(A_{n-1};\lambda) = \chi(A;\lambda)$. Consequently, the last row of the matrix A_{n-1} contains the coefficients α_i of the characteristic polynomial of the matrix A, and the matrix A_{n-1} is the companion matrix for the polynomial $\chi(A;\lambda)$. We set $\widetilde{A} = SAS^{-1}$, where $S = S_{n-1}\cdots S_1$, and $\varphi = (-\alpha_n, -\alpha_{n-1}, \ldots, -\alpha_1) \in \mathbb{R}^{n*}$. Then the following assertion holds.

Lemma 3. *The matrix $\widetilde{A}$ has the form*

$$\widetilde{A} = J_1 + e_n\varphi. \tag{8}$$

Now let us construct the matrices $\widetilde{B} := SB$ and $\widetilde{C}^* = C^*S^{-1}$. We have

$$\chi(A + BUC^*;\lambda) = \chi(S(A + BUC^*)S^{-1};\lambda) = \chi(\widetilde{A} + \widetilde{B}U\widetilde{C}^*;\lambda). \tag{9}$$

Further, we use the following lemma, whose proof will be given below.

Lemma 4. *Let $\widetilde{A}$ have the form* (8), *let $D \in M_n$, and let*

$$D = \left\| \begin{array}{c|c} 0 & 0 \\ \hline F & 0 \end{array} \right\|, \qquad F \in M_{n-p+1,p}; \tag{10}$$

here $p \in \{1, \ldots, n\}$. Let $\chi(\widetilde{A} + D;\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \cdots + \gamma_n$. Then $\gamma_i = \alpha_i - \mathrm{Sp}(DJ_{i-1}G)$ for all $i = 1, \ldots, n$.

Since S is a lower triangular matrix, it follows that the matrices $\widetilde{B}$ and $\widetilde{C}$ have the same form as B and C; i.e., the first $p-1$ rows of the matrix $\widetilde{B}$ and the last $n-p$ rows of the matrix $\widetilde{C}$ are zero. This implies that the matrix $\widetilde{B}U\widetilde{C}^*$ has the form (10) of the matrix D, that is, the block form $\left\| \begin{matrix} 0 & 0 \\ F & 0 \end{matrix} \right\|$, where the right upper corner entry of the matrix F lies on the main diagonal. Then it follows from Lemma 4 that if $\chi(\widetilde{A} + \widetilde{B}U\widetilde{C}^*;\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \cdots + \gamma_n$, then the relations

$$\gamma_i = \alpha_i - \mathrm{Sp}(\widetilde{B}U\widetilde{C}^*J_{i-1}G) = \alpha_i - \mathrm{Sp}(SBUC^*S^{-1}J_{i-1}G)$$

hold for all $i = 1, \ldots, n$. By virtue of relation (9), relations (5) are true, which completes the proof of Theorem 1. It remains to prove Lemma 4.

4. PROOF OF LEMMA 4

Let us prove preliminarily an auxiliary assertion.

Lemma 5. *Let*

$$H_1, H_2 \in M_n: \quad H_1 = \left\| \begin{array}{c|c} 0 & 0 \\ \hline H & 0 \\ \hline \xi & 0 \end{array} \right\|, \quad H_2 = \left\| \begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & H & 0 \end{array} \right\|$$

be two given matrices, where $H = \left\| \begin{matrix} h_{p1} & \ldots & h_{pp} \\ \ldots & \ldots & \ldots \\ h_{n-1,1} & \ldots & h_{n-1,p} \end{matrix} \right\| \in M_{n-p,p}$, $\xi = (h_{n1}, \ldots, h_{np}) \in \mathbb{R}^{p*}$, $p \in \{1, \ldots, n-1\}$. *Then*

$$\mathrm{Sp}(H_1J_sG) = \mathrm{Sp}(H_2J_sG) \quad \textit{for all} \quad s = 0, \ldots, n-p-1, \tag{11}$$

$$\mathrm{Sp}(H_1J_sG) = \mathrm{Sp}(H_2J_sG) + \sum_{i=p}^{n} \alpha_{n-i}h_{i,n-s} \quad \textit{for all} \quad s = n-p, \ldots, n-1; \tag{12}$$

i.e.,

$$\begin{gathered}\mathrm{Sp}(H_1J_0G)=\mathrm{Sp}(H_2J_0G),\quad \ldots,\quad \mathrm{Sp}(H_1J_{n-p-1}G)=\mathrm{Sp}(H_2J_{n-p-1}G),\\ \mathrm{Sp}(H_1J_{n-p}G)=\mathrm{Sp}(H_2J_{n-p}G)+(\alpha_0h_{np}+\alpha_1h_{n-1,p}+\cdots+\alpha_{n-p}h_{pp}),\\ \mathrm{Sp}(H_1J_{n-p+1}G)=\mathrm{Sp}(H_2J_{n-p+1}G)+(\alpha_0h_{n,p-1}+\alpha_1h_{n-1,p-1}+\cdots+\alpha_{n-p}h_{p,p-1}),\quad\ldots,\\ \mathrm{Sp}(H_1J_{n-1}G)=\mathrm{Sp}(H_2J_{n-1}G)+(\alpha_0h_{n1}+\alpha_1h_{n-1,1}+\cdots+\alpha_{n-p}h_{p1}).\end{gathered}\tag{13}$$

Proof. We represent the matrix H_1 as the sum $H_1=H'+H''$, where

$$H'=\left\|\begin{array}{c|c}0&0\\\hline H&0\\\hline 0&0\end{array}\right\|,\qquad H''=\left\|\begin{array}{c|c}0&0\\\hline 0&0\\\hline \xi&0\end{array}\right\|.$$

Then $\mathrm{Sp}(H_1J_sG)=\mathrm{Sp}(H'J_sG)+\mathrm{Sp}(H''J_sG)$. First, we find $\mathrm{Sp}(H''J_sG)$. Note that $J_k^*e_i=e_{k+i}$ for $k+i\le n$ and $J_k^*e_i=0\in\mathbb{R}^n$ for $k+i>n$; in addition, $\mathrm{Sp}(e_ie_j^*)=\delta_{ij}$, where δ_{ij} is the Kronecker delta. The matrix H'' can be represented in the form $H''=\sum_{j=1}^p h_{nj}e_ne_j^*$. We have

$$\begin{aligned}\mathrm{Sp}(H''J_sG)=\mathrm{Sp}(GH''J_s)&=\mathrm{Sp}\left(\sum_{k=0}^{n-1}\alpha_kJ_k^*\sum_{j=1}^{p}h_{nj}e_ne_j^*J_s\right)\\ &=\mathrm{Sp}\left(\sum_{j=1}^{p}h_{nj}\left(\sum_{k=0}^{n-1}\alpha_kJ_k^*e_n\right)(e_j^*J_s)\right)=\mathrm{Sp}\left(\sum_{j=1}^{p}h_{nj}\alpha_0e_ne_{j+s}^*\right)=:\varkappa_1.\end{aligned}$$

If $s\le n-p-1$, then $j+s\le n-1$, since $j\le p$. Therefore, $\mathrm{Sp}(e_ne_{j+s}^*)=0$ for any $j\le p$; consequently, $\varkappa_1=0$. If $s\in\{n-p,\ldots,n-1\}$, then $j+s$ coincides with n for $j=n-s$, and $\mathrm{Sp}(e_ne_{j+s}^*)=0$ for the remaining $j\in\{1,\ldots,p\}\backslash\{n-s\}$. Therefore, $\varkappa_1=h_{n,n-s}\alpha_0$. Thus, we have

$$\mathrm{Sp}(H''J_sG)=0\quad\text{for all}\quad s=0,\ldots,n-p-1,\tag{14}$$
$$\mathrm{Sp}(H''J_sG)=h_{n,n-s}\alpha_0\quad\text{for all}\quad s=n-p,\ldots,n-1.\tag{15}$$

Now consider the matrices H_2 and H'. The matrix H' can be represented in the form

$$H'=\sum_{i=p}^{n-1}\sum_{j=1}^{p}h_{ij}P_{ij},$$

where $P_{ij}=e_ie_j^*$ and

$$H_2=\sum_{i=p}^{n-1}\sum_{j=1}^{p}h_{ij}P_{i+1,j+1}.$$

Let us clarify how different $\mathrm{Sp}(P_{ij}J_sG)$ and $\mathrm{Sp}(P_{i+1,j+1}J_sG)$ are for various values of s. We have

$$\varkappa_2:=\mathrm{Sp}(P_{ij}J_sG)=\mathrm{Sp}(GP_{ij}J_s)=\mathrm{Sp}\left(\sum_{k=0}^{n-1}\alpha_kJ_k^*e_ie_j^*J_s\right)=\mathrm{Sp}\left(\sum_{k=0}^{n-1}\alpha_ke_{k+i}e_{j+s}^*\right)=\sum_{k=0}^{n-1}\alpha_k\delta_{k+i,j+s}.$$

If $s\in\{0,\ldots,i-j-1\}$, then $j+s<i\le k+i$; consequently, $\varkappa_2=0$. If $s\in\{i-j,\ldots,n-j\}$, then $0\le s-i+j\le n-i\le n-1$. Then the relation $k+i=j+s$ is true for $k=s-i+j$; therefore, $\varkappa_2=\alpha_{s-i+j}$. If $s\in\{n-j+1,\ldots,n-1\}$, then $j+s>n$; consequently, $e_j^*J_s=0$, whence we obtain $\varkappa_2=0$. Further,

$$\varkappa_3:=\mathrm{Sp}(P_{i+1,j+1}J_sG)=\mathrm{Sp}(GP_{i+1,j+1}J_s)=\mathrm{Sp}\left(\sum_{k=0}^{n-1}\alpha_kJ_k^*e_{i+1}e_{j+1}^*J_s\right)=\sum_{k=0}^{n-1}\alpha_k\delta_{k+i+1,j+s+1}.$$

If $s \in \{0, \dots, i-j-1\}$, then $j+s+1 < i+1 \le k+i+1$; consequently, $\varkappa_3 = 0$. If $s \in \{i-j, \dots, n-j-1\}$, then $0 \le s-i+j \le n-i-1 \le n-1$. In this case, the relation $k+i+1 = j+s+1$ holds for $k = s-i+j$; therefore, $\varkappa_3 = \alpha_{s-i+j}$. If $s \in \{n-j, \dots, n-1\}$, then $j+s+1 > n$; consequently, $e^*_{j+1} J_s = 0$, whence we obtain $\varkappa_3 = 0$. Thus, we have

$$\begin{aligned} &\mathrm{Sp}(P_{ij} J_s G) = \mathrm{Sp}(P_{i+1,j+1} J_s G) = 0, \qquad s \in \{0, \dots, i-j-1\}, \\ &\mathrm{Sp}(P_{ij} J_s G) = \mathrm{Sp}(P_{i+1,j+1} J_s G) = \alpha_{s-i+j}, \qquad s \in \{i-j, \dots, n-j-1\}, \\ &\mathrm{Sp}(P_{ij} J_s G) = \alpha_{n-i}, \qquad \mathrm{Sp}(P_{i+1,j+1} J_s G) = 0, \qquad s = n-j, \\ &\mathrm{Sp}(P_{ij} J_s G) = \mathrm{Sp}(P_{i+1,j+1} J_s G) = 0, \qquad s \in \{n-j+1, \dots, n-1\}. \end{aligned} \tag{16}$$

Therefore, $\mathrm{Sp}(P_{ij} J_s G)$ and $\mathrm{Sp}(P_{i+1,j+1} J_s G)$ are different only for $s = n-j$.

Let $s \in \{0, \dots, n-p-1\}$. Then $s \le n-p-1 \le n-j-1$ for all $j \in \{1, \dots, p\}$. By virtue of the first two rows in (16), we have

$$\sum_{i=p}^{n-1} \sum_{j=1}^{p} h_{ij} \,\mathrm{Sp}(P_{ij} J_s G) = \sum_{i=p}^{n-1} \sum_{j=1}^{p} h_{ij} \,\mathrm{Sp}(P_{i+1,j+1} J_s G);$$

consequently, $\mathrm{Sp}(H' J_s G) = \mathrm{Sp}(H_2 J_s G)$, and, by (14), we obtain (11).

Now let $s \in \{n-p, \dots, n-1\}$. The value $s = n-j$ gets in this interval as j runs from 1 to p. By (16), we have

$$\begin{aligned} &\sum_{j=1}^{p} h_{ij} \,\mathrm{Sp}(P_{ij} J_s G) = \sum_{j=1}^{p} h_{ij} \,\mathrm{Sp}(P_{i+1,j+1} J_s G) + h_{i,n-s} \alpha_{n-i} \\ &\qquad \Longrightarrow \quad \sum_{i=p}^{n-1} \sum_{j=1}^{p} h_{ij} \,\mathrm{Sp}(P_{ij} J_s G) = \sum_{i=p}^{n-1} \sum_{j=1}^{p} h_{ij} \,\mathrm{Sp}(P_{i+1,j+1} J_s G) + \sum_{i=p}^{n-1} h_{i,n-s} \alpha_{n-i}. \end{aligned}$$

Hence it follows that $\mathrm{Sp}(H' J_s G) = \mathrm{Sp}(H_2 J_s G) + \sum_{i=p}^{n-1} h_{i,n-s} \alpha_{n-i}$. By adding the last relation to formula (15), we obtain (12). The proof of Lemma 5 is complete.

Let us proceed with the proof of Lemma 4. Consider the index k of the row in which the right upper corner entry of the left lower block F of the matrix D is located. Let us make the proof by induction on k changing from n to 1. The basis of induction is the following: let $k = n$. Then $D = \left\| \dfrac{0}{\eta} \right\|$, $\eta = (d_{n1}, \dots, d_{nn}) \in \mathbb{R}^{n*}$; consequently,

$$\chi(\widetilde{A} + D; \lambda) = \lambda^n + (\alpha_1 - d_{nn}) \lambda^{n-1} + \dots + (\alpha_n - d_{n1}).$$

Hence we have $\gamma_i = \alpha_i - d_{n,n-i+1}$. Further,

$$\begin{aligned} \mathrm{Sp}(D J_{i-1} G) &= \mathrm{Sp}(G D J_{i-1}) = \mathrm{Sp}\left(\sum_{k=0}^{n-1} \alpha_k J_k^* \sum_{j=1}^{n} d_{nj} e_n e_j^* J_{i-1} \right) \\ &= \mathrm{Sp}\left(\sum_{j=1}^{n} d_{nj} \left(\sum_{k=0}^{n-1} \alpha_k J_k^* e_n \right) (e_j^* J_{i-1}) \right) = \mathrm{Sp}\left(\sum_{j=1}^{n} d_{nj} \alpha_0 e_n e_{j+i-1}^* \right) \\ &= \sum_{j=1}^{n} d_{nj} \delta_{n,j+i-1} = d_{n,n-i+1} \end{aligned}$$

for all $i = 1, \dots, n$.

The proof of the basis is complete. The inductive assumption is the following: let the assertion of the lemma hold for any $k \in \{p+1, \ldots, n\}$. Let us show that it holds for $k = p$ as well. Consider the matrix D in (10). By the inductive assumption, $p < n$. We introduce the following notation:

$$L = \left\| \begin{array}{ccc} d_{p1} & \ldots & d_{pp} \\ \ldots & \ldots & \ldots \\ d_{n-1,1} & \ldots & d_{n-1,p} \end{array} \right\| \in M_{n-p,p}, \qquad \sigma = (d_{n1}, \ldots, d_{np}) \in \mathbb{R}^{p*},$$

$$\psi = (d_{n1}, \ldots, d_{np}, 0, \ldots, 0) \in \mathbb{R}^{n*}.$$

Then $D = \left\| \begin{array}{c|c} 0 & 0 \\ \hline L & 0 \\ \hline \sigma & 0 \end{array} \right\|$. We represent D in the form $D = D' + D''$, where

$$D' = \left\| \begin{array}{c|c} 0 & 0 \\ \hline L & 0 \\ \hline 0 & 0 \end{array} \right\|, \qquad D'' = \left\| \begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \\ \hline \sigma & 0 \end{array} \right\|.$$

Let $\widetilde{D} := J_1^* D = \left\| \begin{array}{cc} 0 & 0 \\ L & 0 \end{array} \right\| \in M_n$ and $\widehat{D} := \widetilde{D} J_1 = \left\| \begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & L & 0 \end{array} \right\| \in M_n$. Further, we set $K := \left\| \begin{array}{c} 0 \\ \hline \varphi \end{array} \right\| \in M_n$, where $\varphi = (-\alpha_n, \ldots, -\alpha_1) \in \mathbb{R}^{n*}$. Then $\widetilde{A} = J_1 + K$. The matrix $\widetilde{A} + D$ is a matrix of the form (4). For the matrix $\widetilde{A} + D$, we construct the matrix T by analogy with the construction of the matrix S_1 for A; namely, from $\widetilde{A} + D$, we remove the last row and add the first row e_1^*. Then $T = I + \widetilde{D} = \left\| \begin{array}{cc} I_p & 0 \\ L & I_{n-p} \end{array} \right\|$, $I_p \in M_p$, $I_{n-p} \in M_{n-p}$. Hence it follows that $T^{-1} = I - \widetilde{D}$. We multiply the matrix T^{-1} by the matrix $(\widetilde{A} + D)$ on the right and the matrix T on the left. We have

$$(\widetilde{A} + D)T^{-1} = (\widetilde{A} + D)(I - \widetilde{D}) = \widetilde{A} + D - \widetilde{A}\widetilde{D} - D\widetilde{D}.$$

The first p rows of the matrix $\widetilde{D}$ are zero, and the last $n - p$ columns of the matrix D are zero; therefore, $D\widetilde{D} = 0$. Further, $\widetilde{A}\widetilde{D} = (J_1 + K)\widetilde{D} = J_1\widetilde{D} + K\widetilde{D}$, but $J_1\widetilde{D} = D'$; consequently, $D - \widetilde{A}\widetilde{D} = D - D' - K\widetilde{D} = D'' - K\widetilde{D}$. Therefore, $(\widetilde{A} + D)T^{-1} = \widetilde{A} + D'' - K\widetilde{D} = \widetilde{A} + K_1$, where $K_1 = \left\| \begin{array}{c} 0 \\ \hline \zeta \end{array} \right\| \in M_n$ and $\zeta = \psi - \varphi\widetilde{D} \in \mathbb{R}^{n*}$. Next, we have

$$T(\widetilde{A} + D)T^{-1} = (I + \widetilde{D})(\widetilde{A} + K_1) = \widetilde{A} + K_1 + \widetilde{D}\widetilde{A} + \widetilde{D}K_1.$$

Since $p < n$, it follows that the last column of the matrix $\widetilde{D}$ is zero; and since the first $n - 1$ rows of the matrix K_1 are zero, we have $\widetilde{D}K_1 = 0$. Further,

$$\widetilde{D}\widetilde{A} = \widetilde{D}(J_1 + K) = \widetilde{D}J_1 + \widetilde{D}K.$$

We have $\widetilde{D}J_1 = \widehat{D}$ and $\widetilde{D}K = 0$, since the first $n - 1$ rows of the matrix K are zero. Consequently, $T(\widetilde{A} + D)T^{-1} = \widetilde{A} + K_1 + \widehat{D}$. Set $\widehat{A} := \widetilde{A} + K_1$. Then

$$T(\widetilde{A} + D)T^{-1} = \widehat{A} + \widehat{D}.$$

Further, $\widehat{A} = J_1 + K + K_1$; i.e., the matrix $\widehat{A}$ has the same form as the matrix $\widetilde{A}$ and differs from it only by the last row. Let $\widehat{\varphi} = (-\widehat{\alpha}_n, \ldots, -\widehat{\alpha}_1) \in \mathbb{R}^{n*}$ be the last row of the matrix $\widehat{A}$. Then $\widehat{\varphi} = \varphi + \zeta = \varphi + \psi - \varphi\widetilde{D}$. By multiplying both sides of the last relation by -1, we obtain $-\widehat{\varphi} = -\varphi - \psi + \varphi\widetilde{D}$. We rewrite this relation in terms of coordinates starting from the last coordinate. We obtain

$$\begin{gathered}
\widehat{\alpha}_1 = \alpha_1, \ \ldots, \ \widehat{\alpha}_{n-p} = \alpha_{n-p}, \\
\widehat{\alpha}_{n-p+1} = \alpha_{n-p+1} - d_{np} - \alpha_1 d_{n-1,p} - \cdots - \alpha_{n-p} d_{pp} = \alpha_{n-p+1} - \sum_{i=p}^{n} \alpha_{n-i} d_{ip}, \quad \ldots, \\
\widehat{\alpha}_n = \alpha_n - d_{n1} - \alpha_1 d_{n-1,1} - \alpha_2 d_{n-2,1} - \cdots - \alpha_{n-p} d_{p1} = \alpha_n - \sum_{i=p}^{n} \alpha_{n-i} d_{i1}.
\end{gathered} \tag{17}$$

Let

$$\lambda^n + \gamma_1 \lambda^{n-1} + \cdots + \gamma_n := \chi(\widetilde{A} + D; \lambda).$$

The matrices $\widetilde{A} + D$ and $\widehat{A} + \widehat{D}$ are similar; therefore,

$$\chi(\widehat{A} + \widehat{D}; \lambda) = \chi(\widetilde{A} + D; \lambda).$$

By the inductive assumption, the assertion of Lemma 4 holds for the matrix $\widehat{A} + \widehat{D}$ since the right upper corner entry of the left lower block $\| \ 0 \ \ L \ \|$ of the matrix $\widehat{D}$ lies on the diagonal in the $(p+1)$st row. Consequently,

$$\gamma_1 = \widehat{\alpha}_1 - \operatorname{Sp}(\widehat{D} J_0 G), \quad \ldots, \quad \gamma_n = \widehat{\alpha}_n - \operatorname{Sp}(\widehat{D} J_{n-1} G). \tag{18}$$

The matrix D has the form of the matrix H_1; and $\widehat{D}$ has the form of the matrix H_2 in Lemma 5. Therefore, by (13), we have the relations

$$\begin{gathered}
\operatorname{Sp}(D J_0 G) = \operatorname{Sp}(\widehat{D} J_0 G), \quad \ldots, \quad \operatorname{Sp}(D J_{n-p-1} G) = \operatorname{Sp}(\widehat{D} J_{n-p-1} G), \\
\operatorname{Sp}(D J_{n-p} G) = \operatorname{Sp}(\widehat{D} J_{n-p} G) + (\alpha_0 d_{np} + \alpha_1 d_{n-1,p} + \cdots + \alpha_{n-p} d_{pp}), \quad \ldots, \\
\operatorname{Sp}(D J_{n-1} G) = \operatorname{Sp}(\widehat{D} J_{n-1} G) + (\alpha_0 d_{n1} + \alpha_1 d_{n-1,1} + \cdots + \alpha_{n-p} d_{p1}).
\end{gathered} \tag{19}$$

By substituting (17) and (19) into (18), we obtain

$$\begin{aligned}
\gamma_1 &= \widehat{\alpha}_1 - \operatorname{Sp}(\widehat{D} J_0 G) = \alpha_1 - \operatorname{Sp}(D J_0 G), \quad \ldots, \\
\gamma_{n-p} &= \widehat{\alpha}_{n-p} - \operatorname{Sp}(\widehat{D} J_{n-p-1} G) = \alpha_{n-p} - \operatorname{Sp}(D J_{n-p-1} G), \\
\gamma_{n-p+1} &= \widehat{\alpha}_{n-p+1} - \operatorname{Sp}(\widehat{D} J_{n-p} G) = \left(\alpha_{n-p+1} - \sum_{i=p}^{n} \alpha_{n-i} d_{ip}\right) - \left(\operatorname{Sp}(D J_{n-p} G) - \sum_{i=p}^{n} \alpha_{n-i} d_{ip}\right) \\
&= \alpha_{n-p+1} - \operatorname{Sp}(D J_{n-p} G), \quad \ldots, \\
\gamma_n &= \widehat{\alpha}_n - \operatorname{Sp}(\widehat{D} J_{n-1} G) = \left(\alpha_n - \sum_{i=p}^{n} \alpha_{n-i} d_{i1}\right) - \left(\operatorname{Sp}(D J_{n-1} G) - \sum_{i=p}^{n} \alpha_{n-i} d_{i1}\right) \\
&= \alpha_n - \operatorname{Sp}(D J_{n-1} G).
\end{aligned}$$

The proof of Lemma 4 is complete.

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