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SOME PRINCIPAL QUESTIONS OF THE THEORY OF EQUILIBRIUM FIGURES

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The objective of this paper is to describe several problems unifying objective of which is to obtain an improved understanding of the theory of equilibrium figures. We will consider three basic questions. Ones begin with a demonstration that on the Jacobi sequence the bifurcation point for the pear-shaped equilibrium figures must coincides with the corresponding neutral point. Our method is original and independent from Cartan method. In the second section we have proved an impossibility of the quasiprecession for large class of equilibrium figures with (or without) internal flows. At last, the new formula for the angular velocity $\Omega/\sqrt{\pi G\rho}$ of rotating, self-gravitating homogeneous equilibrium figures

$$\Omega^2 = 1 + \eta - \sqrt{(1+\eta)^2 - (6W_i - W_t)/(\pi G\rho I_3)}$$

has been derived, where η is the normalized total gravitational potential energy, W_i is the «internal» potential energy of the figure.

ДЕЯКІ ПРИНЦИПОВІ ПИТАННЯ ТЕОРІЇ РІВНОВАЖНИХ ФІГУР, Кондратьев Б. П. — Описуються деякі задачі, мета яких — досягнути кращого розуміння теорії рівноважних фігур. Доводиться, що на послідовності Якобі точка біфуркації для грушевидної рівноважної фігури повинна збігатися з відповідною нейтральною точкою. Наш метод оригінальний і незалежний від методу Картана. Доведена неможливість квазіпрецесії для широкого класу рівноважних фігур з внутрішніми течіями або без них. Виведена нова формула для кутової швидкості обертання $\Omega/\sqrt{\pi G\rho}$ і самогравітуючих однорідних рівноважних фігур

$$\Omega^2 = 1 + \eta - \sqrt{(1+\eta)^2 - (6W_i - W_t)/(\pi G\rho I_3)}$$

де η — нормалізована загальна гравітаційна потенціальна енергія, W_i — «внутрішня» потенціальна енергія фігури.

НЕКОТОРЫЕ ПРИНЦИПИАЛЬНЫЕ ВОПРОСЫ ТЕОРИИ РАВНОВЕСНЫХ ФИГУР, Кондратьев Б. П. — Описаны некоторые задачи, объединяющая цель которых — достичь лучшего понимания теории равновесных фигур. Доказано, что на последовательности Якоби точка бифуркации для грушевидной равновесной фигуры должна совпадать с соответствующей нейтральной точкой. Наш метод оригинален и независим от метода Картана. Доказана невозможность квазипрецессии для широкого класса равновесных фигур с внутренними течениями или без них. Наконец, выведена новая формула для угловой скорости $\Omega/\sqrt{\pi G\rho}$ вращения самогравитирующих однородных равновесных фигур

$$\Omega^2 = 1 + \eta - \sqrt{(1+\eta)^2 - (6W_i - W_t)/(\pi G\rho I_3)}$$

где η — нормализованная общая гравитационная потенциальная энергия, W_i — «внутренняя» потенциальная энергия фигуры.

1. REMARKS TO THE STABILITY PROBLEM OF JACOBI ELLIPSOIDS

1a. Since the works of H. Poincare and A. M. Liapounov [8, 9] it has been known that on the Jacobi sequence is the critical ellipsoid

$$\frac{a_2}{a_1} = 0.432232; \qquad \frac{a_3}{a_1} = 0.345069; \qquad \frac{\Omega^2}{\pi G\rho} = 0.284030, \tag{1.1}$$

which becomes neutral relative to third-harmonic mode of oscillation. Liapounov proved that in this first bifurcation point and beyond the Jacobi ellipsoids become seqular instability. For judgment about the secular instability (SI) it is necessary to investigate a small variation of energy E for arbitrary deformation of the figure with further conditions V = const, ρ = const and the angular moment L = const:

$$\delta E = \frac{\rho}{2}\int g\tau^2 ds - \frac{G\rho^2}{2}\int\int\frac{\tau\tau'}{D}ds ds', \tag{1.2}$$

where $g(x)$ is the total gravity on the surface, $\tau(x)$ is the thickness of the deforming layer. The first term expresses the perturbation of the energy at fixed field of force, the second term gives the effect of self-gravitation of the layer. Testing with using Lame functions gives

$$\begin{aligned}&\delta E > 0 \text{ until the critical point (1.1);}\\ &\delta E < 0 \text{ beyond the point (1.1).}\end{aligned} \tag{1.3}$$

This fact ascertained gives rise to the conclusion about SI for the supercritical Jacobi ellipsoid.

1b. In 1924 Cartan [4] showed that the point (1.1) is also a point of beginning of dynamical instability (DI) of the Jacobi ellipsoids. As to DI for the supercritical ellipsoids that the one follows from general laws of mechanics: if a single frequency for pear-shaped mode of oscillation reaches to zero, then the dynamical stability is changing by DI. The critical mode in this case is only one as far as interchange by places of obtuse edge and pointedness simply means for the pear a change of sign of the amplitude. It is characteristic that for the Maclaurin spheroids, conversely, asymmetric perturbations appear by pairs: τ is proportional to $\cos k\theta$ or $\sin k\theta$.

Further, the gyroscopic forces nothing changes in the case with one unstable mode (for the pear-shaped figures), although can change the result for conjugate birth of there (for the Maclaurin spheroids).

1c. More recently the group of investigators [6] advocated that these SI and DI points cannot coincide on the Jacobi sequence. These authors belief even that the Jacobi ellipsoids suffer neither a secular nor a dynamical instability from third-harmonic perturbations (ibid, p. 506). See below the point 1d.

But the exponents of this point of view are not entirely consistent. We represent new analytical arguments in behalf of coincidence of the points SI and DI on the Jacobi sequence.

Consider a fluid gravitating mass rotating with constant angular velocity Ω around the fixed x_3-axis. Small oscillations with the frequency σ about the equilibrium state are governed by 3 well-known linearized equations

$$i\sigma u_1 = \frac{\partial \Phi}{\partial x_1} + 2\Omega u_2, \qquad i\sigma u_2 = \frac{\partial \Phi}{\partial x_2} - 2\Omega u_1, \qquad i\sigma u_3 = \frac{\partial \Phi}{\partial x_3}, \tag{1.4}$$

where

$$\Phi = \delta\left[\varphi - \frac{p}{\rho} + \frac{\Omega^2}{2}(x_1^2 + x_2^2)\right].$$

Here, ρ is the density, $\varphi(\mathrm{x})$ is the potential, $p(\mathrm{x})$ is the pressure, $u_i(\mathrm{x})$ are the components of the velocity field. From (1.4) we find

$$\begin{aligned} u_1 &= \frac{1}{4\Omega^2 - \sigma^2}\left(i\sigma \frac{\partial \Phi}{\partial x_1} + 2\Omega \frac{\partial \Phi}{\partial x_2}\right), \\ u_2 &= \frac{1}{4\Omega^2 - \sigma^2}\left(i\sigma \frac{\partial \Phi}{\partial x_2} - 2\Omega \frac{\partial \Phi}{\partial x_1}\right), \\ u_3 &= \frac{1}{i\sigma}\frac{\partial \Phi}{\partial x_3}. \end{aligned} \tag{1.5}$$

Then the equation of continuity $\mathrm{div}\,u = 0$ gives us

$$\sigma^2 \Delta\Phi = 4\Omega^2 \frac{\partial^2 \Phi}{\partial x_3^2}. \tag{1.6}$$

But Laplacian Φ can be obtain straight from (1.4) also

$$\Delta\Phi = 2\Omega\zeta, \tag{1.7}$$

where $\zeta = \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2}$.

Thus, the hydrodynamical equations yield

$$\Delta\Phi = 2\Omega\zeta = \frac{4\Omega^2}{\sigma}\frac{\partial^2 \Phi}{\partial x_3^2}. \tag{1.8}$$

In particular, for $\sigma = 0$ (*this is the neutral point*) we have $\dfrac{\partial^2 \Phi}{\partial x_3^2} = 0$ and, consequently,

$$u_1 = \frac{1}{2\Omega}\frac{\partial \Phi}{\partial x_2}, \qquad u_2 = -\frac{1}{2\Omega}\frac{\partial \Phi}{\partial x_1}. \tag{1.9}$$

In this case the equation of continuity gives

$$\frac{\partial u_3}{\partial x_3} = 0, \qquad \text{i. e.} \quad u_3 = u_3(x_1, x_2). \tag{1.10}$$

Further, the boundary condition for the triaxial ellipsoid with semiaxes a_i

$$\frac{x_1}{a_1^2}u_1 + \frac{x_2}{a_2^2}u_2 + \frac{x_3}{a_3^2}u_3 = 0 \tag{1.11}$$

at $\sigma = 0$, with regard to the formulas (1.9), yields

$$\frac{x_1}{a_1^2}\frac{\partial\Phi}{\partial x_2}-\frac{x_2}{a_2^2}\frac{\partial\Phi}{\partial x_1}+2\Omega\frac{x_3}{a_3^2}u_3=0. \tag{1.12}$$

This boundary condition must be fulfilled both on upper ($x_3 > 0$) and lower ($x_3 < 0$) part of surface of the ellipsoid. Therefore in (1.12) it is required to equate to zero separately both the even (with respect to x_3)

$$\frac{x_1}{a_1^2}\frac{\partial\Phi}{\partial x_2}-\frac{x_2}{a_2^2}\frac{\partial\Phi}{\partial x_1}=0, \tag{1.13}$$

and the add side

$$2\Omega\frac{x_3}{a_3^2}u_3=0. \tag{1.14}$$

From the equations (1.13) and (1.14) we find, that in the neutral point must be

$$u_3=0, \qquad \Phi=\Phi\left(\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}\right). \tag{1.15}$$

But far not any disturbing layer can be satisfied to the last condition in (1.15). For example, let the Jacobi ellipsoid is perturbing by the third-harmonic perturbations

$$\tau/l=a_0x_1^3+a_1x_1x_2^2+a_2x_1x_3^2+bx_1. \tag{1.16}$$

Then, as it is well known from the potential theory, the variation

$$\delta\varphi=A_0x_1^3+A_1x_1x_2^2+A_2x_2x_3^2+Bx_1 \tag{1.17}$$

is the polynom of third order too. Besides, in virtue of

$$\delta p=-\frac{\partial p}{\partial n}, \tag{1.18}$$

where

$$p=p_0\left(1-\frac{x_1^2}{a_1^2}-\frac{x_2^2}{a_2^2}-\frac{x_3^2}{a_3^2}\right),$$

we easily find and the variation

$$\delta\Phi=\tilde{A}_0x_1^3+\tilde{A}_1x_1x_2^2+\tilde{A}_2x_1a_3^2\left(1-\frac{x_1^2}{a_1^2}-\frac{x_2^2}{a_2^2}\right)+\tilde{B}x_1. \tag{1.19}$$

The point is that the variation Φ *from (1.19) cannot be represented in the desired form (1.15).*

Thus, there is not purely the neutral point on the Jacobi sequence. But there is on the sequence only the bifurcation point for the pear-shaped figures under consideration with the necessary and sufficient condition

$$\Phi=0. \tag{1.20}$$

As far as the bifurcation point is, by definition, the neutral point too, we conclude that the bifurcation point on the Jacobi sequence for the pear-shaped equilibrium figures must coincides with the neutral point (1.1). Q. E. D.

1d. It is important to stress (in spite of [6]) that the conclusion about instability of the supercritical Jacobi ellipsoids not at all depends from stability or instability of the pear-shaped figures. Really, a stability of the Jacobi ellipsoids is investigating by the linear theory, but the analysis of stability of the pear-shaped equilibrium figures is essentially the non-linear one by parameter of deformation.

2. ON IMPOSSIBILITY OF QUASIPRECESSION OF BOTH LIQUID AND COLLISIONLESS EQUILIBRIUM FIGURES WITH (OR WITHOUT) INTERNAL FLOWS

2a. Recent advances in astrophysics and stellar dynamics mean that several basic questions about rotation of nonrigid bodies remain to be answered. Here we shall discuss possible conditions of stationary rotation for wide class of equilibrium figures.

Consider an isolated configuration (rigid body, mass of fluid or collisionless stellar system).

Definition 1: Quasiprecession is the dynamical regime of nonstationary rotation $\Omega=\Omega(t)$ of the system around fixed point or fixed axis (the latter in individual cases for nonrigid bodies). The vector Ω is determined as the angular velocity of own coordinate system of the configuration.

For example, in state of the quasiprecession is a rigid body rotating around fixed point. In general case this rotation is governed by 3 Euler equations. Here the vector Ω changes both own length and direction. In particular, the axially symmetric rigid body has the regular precession state, when both the total angular velocity $\Omega(t)$ and the spin angular velocity ω_s preserve own length and do rotary with the same Ω_{pr} on circular cones around angular moment L.

2b. A. Poincare, P. Appel [3] and V. A. Antonov [1] previously presented convincing evidence that any gravitating rotating liquid mass in relative equilibrium state (i. e. without internal velocity field in the own coordinate system) can has only uniform stationary rotation around fixed axis. In other words, the figure of relative equilibrium cannot be in this quasiprecession state and has

$$\frac{d}{dt}\Omega = 0. \tag{2.1}$$

2c. We greatly extended the class of nonprecessing configurations. Let the coordinate axes $Ox_1x_2x_3$ coincide with the principal axes of the body, which in general case admits the following dynamical description. It has the density distribution $\rho(\mathbf{x})$, the gravitational internal potential $\varphi(\mathbf{x})$ and the mean internal velocity field $\mathbf{u}(\mathbf{x})$.

Presence of the internal flows introduces considerable novelty elements in the theory. Really, classical one-parametric Maclaurin spheroids and Jacobi ellipsoids are only special case in family of two-parametric Rieman ellipsoids [5]. Fluid Rieman ellipsoids, just as their collisionless partners [7], exhibit, in general, an «oblique» rotation, i. e. their rotational axes do not coincide with any principal axis of their moment of inertia ellipsoids. The equilibrium figures with «oblique» rotation have not equatorial symmetry plane (it is call to mind that the existence of such plane for fluid equilibrium figures without internal flows is postulated by known Liechtenstein's theorem). It is important to underline as far as the directions of the vectors Ω and L for any such equilibrium figures are coincide since we cannot classify the equilibrium figures with «oblique» rotation as quasiprecessing configurations.

Besides, the local stress tensor

$$p_{ij}(\mathbf{x}) = \rho(\mathbf{x})\sigma_{ij}(\mathbf{x}). \tag{2.2}$$

For stellar systems the local velocity dispersion tensor σ_{ij} has six independent components. In special case of classical fluid with hydrostatic pressure the internal stresses are isotropic and the ones are described by the spherical tensor

$$p_{ij}(\mathbf{x}) = \begin{pmatrix} p(\mathbf{x}) & 0 & 0 \\ 0 & p(\mathbf{x}) & 0 \\ 0 & 0 & p(\mathbf{x}) \end{pmatrix}. \tag{2.3}$$

So that all these characteristics do not depend from t. Thus, we assume that the configuration under consideration really is an equilibrium figure with stationary internal dynamical structure.

It is required to prove that such equilibrium figure cannot be in quasiprecession state and for it the condition (2.1) is fulfilled.

Proof by contradiction.

Assuming the contrary, the equations of motions for this configuration in the rotating coordinate axes are [8]

$$(\mathbf{u}\nabla)\mathbf{u} = -\frac{1}{\rho}\,\mathrm{div}(\rho\sigma_{ij}) + \mathrm{grad}\,[\varphi + 0.5(\Omega\times\mathbf{x})^2] + (\mathbf{x}\times\dot{\Omega}) + 2(\mathbf{u}\times\Omega). \tag{2.4}$$

Applying to the equation (2.4) the operator rot, in virtue of the identities

$$\begin{gathered} \mathrm{rot}(\Omega\times\mathbf{x}) = 3\Omega - (\Omega\nabla)\mathbf{x}, \\ \mathrm{rot}(u\times\Omega) = (\Omega\nabla)\mathbf{u} - \Omega\,\mathrm{div}\mathbf{u}, \\ \mathrm{rot\ grad} = 0, \end{gathered} \tag{2.5}$$

as a result we obtain the equation

$$\Psi(\mathbf{x}) = \dot{\Omega} + (\Omega\nabla)\mathbf{u} - \Omega\,\mathrm{div}\mathbf{u} \tag{2.6}$$

(where in the incompressible case $\mathrm{div}\mathbf{u} = 0$). Here by the vector function of coordinate $\psi(\mathbf{x})$ we denote

$$\Psi(\mathbf{x}) = \frac{1}{2}\,\mathrm{rot}\left[(\mathbf{u}\nabla)\mathbf{u} + \frac{1}{\rho}\,\mathrm{div}(\rho\sigma_{ij})\right]. \tag{2.7}$$

From the equation (2.6) we have

$$\begin{gathered} \dot{\Omega}_1 - \Omega_1\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) + \Omega_2\frac{\partial u_1}{\partial x_2} + \Omega_3\frac{\partial u_1}{\partial x_3} = \Psi_1(\mathbf{x}), \\ \dot{\Omega}_2 + \Omega_1\frac{\partial u_2}{\partial x_1} - \Omega_2\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) + \Omega_3\frac{\partial u_2}{\partial x_3} = \Psi_2(\mathbf{x}), \\ \dot{\Omega}_3 + \Omega_1\frac{\partial u_3}{\partial x_3} + \Omega_2\frac{\partial u_3}{\partial x_2} - \Omega_3\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = \Psi_3(\mathbf{x}). \end{gathered} \tag{2.8}$$

This is a closed system of 3 linear time-dependent equations with constant coefficients for 3 unknown $\Omega_1(t)$, $\Omega_2(t)$, $\Omega_3(t)$. The system (2.8) is solved by standard method. It is sufficient here to restrict oneself to following qualitative interesting features. In case a characteristic equation of the system (2.8) have real roots only s_i then the most rapidly increasing are $\Omega_0 e^{S_i t}$ or $\Omega_0 t e^{S_i t}$ (Ω_0 is the constant vector). If we substitute these solutions in (2.4), then verify, that in the right side appear the terms as $\Omega_0^2 e^{2S_i t}$ or $\Omega_0^2 t^2 e^{2S_i t}$. It is easily now to see that the left side (2.4) is time-independent one at all, and for compensating of the mentioned terms we must require $\Omega = 0$. The some conclusion about stationary rotation we get as well in other case when the characteristic equation of the system (2.8) has two complex roots. Q. E. D.

3. A NEW FORMULA FOR ANGULAR VELOCITY OF ROTATING, SELF-GRAVITATING, LIQUID OR GASEOUS EQUILIBRIUM FIGURES

Consider a gravitating body with the volume V and density distribution $\rho(\mathrm{x})$, which has the potential $\varphi(\mathrm{x})$.

Definition 2. The internal and external gravitational (potential) energy of the body is respectively

$$W_i = -\frac{1}{8\pi G}\int_V (\mathrm{grad}\varphi)^2 dV, \tag{3.1}$$

$$W_e = -\frac{1}{8\pi G}\int_{R_e - V} (\mathrm{grad}\varphi)^2 dV = \frac{1}{8\pi G}\oint_S \varphi\frac{\partial\varphi}{\partial n} ds, \tag{3.2}$$

where $\mathbf{n}$ is the external unit normal to the surface S of this body.

The total potential energy is

$$W_t = W_i + W_e = -\frac{1}{8\pi G}\int_{R_e} (\mathrm{grad}\varphi)^2 dV = -\frac{1}{2}\int_V \rho\varphi dV. \tag{3.3}$$

For example, for a homogeneous sphere with radius R and mass M we find

$$W_i = -\frac{1}{10}\frac{M^2 G}{R}, \qquad W_e = -\frac{1}{2}\frac{M^2 G}{R}, \tag{3.4}$$

where

$$\frac{W_i}{W_e} = \frac{1}{5}, \qquad \text{i. e.}\ \ 6W_i = W_t = -\frac{3}{5}\frac{M^2 G}{R}. \tag{3.5}$$

Let now consider a rotating homogeneous liquid equilibrium figure. From the hydrodynamical equation

$$\mathrm{grad}\,\frac{p}{\rho} = \mathrm{grad}\left(\varphi + \frac{1}{2}\Omega^2(x_1^2 + x_2^2)\right) \tag{3.6}$$

it follows that

$$(\mathrm{grad}\varphi)^2 = \left(\mathrm{grad}\,\frac{p}{\rho}\right)^2 + \Omega^4 r^2 - 2\Omega^2 \mathrm{r}\,\mathrm{grad}\,\frac{p}{\rho} \tag{3.7}$$

($\mathrm{r} = x_1\mathrm{i}_1 + x_2\mathrm{i}_2$). After integrating (3.7) by the volume V we obtain with regard to (3.1)

$$-8\pi G W_i = \int_V \left(\mathrm{grad}\,\frac{p}{\rho}\right)^2 dV + \frac{\Omega^4}{\rho} I_3 - 2\Omega^2\int \mathrm{r}\,\mathrm{grad}\,\frac{p}{\rho}\, dV, \tag{3.8}$$

where I_3 is the moment of inertia of the body about x_3-axis. Since the pressure on the free surface S of the equilibrium figure must be zero

$$P_s = 0, \tag{3.9}$$

then, as easily to see,

$$\int_V \left(\mathrm{grad}\,\frac{p}{\rho}\right)^2 dV = -\int_V \frac{p}{\rho}\Delta\left(\frac{p}{\rho}\right) dV = \frac{4\Pi G\rho - 2\Omega^2}{\rho}\cdot\Pi, \tag{3.10}$$

where we denote the interior energy

$$\Pi = \int_V p dV. \tag{3.11}$$

Next integrating by parts the third integral in (3.8) and again taking account of (3.9), we find

$$2\Omega^2\int_V \mathrm{r}\,\mathrm{grad}\frac{p}{\rho} dV = -\frac{4\Omega^2}{\rho}\cdot\Pi. \tag{3.12}$$

Thus, the equation (3.8) can be written in the form

$$-8\pi G W_i = \left(\frac{2\Omega^2}{\rho} + 4\pi G\right)\Pi + \frac{\Omega^4 I_3}{\rho}. \tag{3.13}$$

On the other hand, from the virial equations of the second order [7] we have

$$-3\Pi = 2T_{rot} + W_t = \Omega^2 I_3 + W_t \tag{3.14}$$

Inserting Π from (3.14) into (3.13), after simple transformations we obtain the biquadratic equation for the normalized value $\Omega^2 = \Omega^2/(2\pi G\rho)$

$$\Omega^4 - 2\Omega^2(1+\eta) + \frac{6W_i - W_t}{\pi G\rho I_3} = 0, \tag{3.15}$$

where we denote $\eta = W_t/(2\pi G\rho I_3)$. Its roots are

$$\Omega^2 = 1 + \eta \pm \sqrt{(1+\eta)^2 - (6W_i - W_t)/(\pi G\rho I_3)}. \tag{3.16}$$

It is easily to verify on the example of the nonrotating sphere with $\Omega = 0$ and W_i, W_t from (3.4), (3.5), that in (3.16) we must choose the sign «–».

Thus, we have derived new expression for the angular velocity in terms of W_i and W_t rather than the separate components of the potential energy tensor W_{ij} and of the moment inertia tensor I_{ij} [5]. We test the validity of the formula

$$\Omega^2 = 1 + \eta - \sqrt{(1+\eta)^2 - (6W_i - W_t)/(\pi G\rho I_3)} \tag{3.17}$$

on the classical Maclaurin spheroids and Jacobi ellipsoids.

4. CONCLUSIONS

The first section includes those subsection in which we argue against with the paper [6]. The touched upon questions are too much meaning for the all theory of equilibrium figures and we once more turn our attention to this classical problem.

In the second sections we prove an impossibility of free precession for liquid and collisionless equilibrium figures. These results can help correctly interpret some observational date in dynamics of rotating cosmic bodies.

The formula for angular velocity can be useful in computational search of new exotic nonellipsoidal equilibrium figure, which can fork from the classical Maclaurin spheroids and Jacobi ellipsoids. Besides, the formula would be necessary too for clearing up of some general characteristics of rotating equilibrium mass. The question is about existence proof of new extremum for angular velocity

$$\frac{\Omega^2}{2\pi G\rho} \leq 0.225, \tag{4.1}$$

which is considerably lower than well-known Crudeli limit. In connection with this certain hint is given in [2].

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