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Ellipsoidal self-consistent phase models of stellar systems

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ABSTRACT

This paper presents a review of the dynamics of collisionless, homogeneous gravitating ellipsoids. A combined method for the construction of self-consistent models is applied, which takes into account both the elements of the kinetic method and those of the hydrodynamical description. We study both non-stationary models and equilibrium figures. An intrinsic characteristic of these models is that they are all, in one way or another, degenerate in six-dimensional phase space. We classify the models as five-, four- and three-dimensional phase ellipsoids.

Key words: hydrodynamics – methods: analytical – celestial mechanics, stellar dynamics.

1 INTRODUCTION

Recent advances in astrophysics and stellar dynamics mean that classical theories of fluid equilibrium figures are now unsatisfactory. Thus the development of a new theory of equilibrium figures for gravitating collisionless stellar systems is necessary.

A fundamental problem in stellar dynamics is the construction of self-consistent models for gravitating collisionless systems. In a series of papers, Freeman (1966a,b,c) described some simple stationary models with quadratic potentials. The rotational axis in these models is coincident with one of the axes of symmetry. Hunter (1974, 1975) gave greater precision to this theory.

Freeman's ellipsoidal model has a preferred axis (see Section 2). In six-dimensional phase space, this model (hereafter referred to as SVa) constitutes a five-dimensional phase ellipsoid.

During 1984–92, we constructed five new self-consistent stationary ellipsoidal models with a quadratic potential (Kondratyev 1984, 1992a,b,c,d). These new models have the following properties. (i) In general, they exhibit an 'oblique' rotation, i.e. their rotational axes do not coincide with any principal axis of their moment of inertia ellipsoids. (ii) They all have a phase degeneracy, and in the general case they constitute five-dimensional (classes SVb,c) or four-dimensional (classes SIVa,b,c) phase ellipsoids. (iii) The ellipsoids of classes SVb,c have preferred axes, whereas ellipsoids of classes SIVa,b,c do not usually have a preferred axis. (iv) The motion of an individual star in each of the three directions of rectangular Cartesian coordinates is represented by a superposition of two harmonic oscillations with incommensurable frequencies; the locus of the star traces the surface of a particular type of two-arm figure (the centre of symmetry of this two-arm figure coincides either with some point on the preferred axis (models SVb,c), or with the centre of the ellipsoid (models SIVa,b,c)). (v) The overwhelming majority of stars in the models SVb,c touch the inside of the system's elliptical bounding surface, but, in contrast, the overwhelming majority of stars in the models SIVa,b,c do not touch any part of the bounding surface.

The study of these models has suggested a formulation and a solution of the following problem (Kondratyev 1984, 1989a): do the laws of classical dynamics permit motion in a homogeneous collisionless stellar system, such that at any time the system is ellipsoidal and the mean velocity field is a linear function of position? This mathematical problem is a natural generalization of the famous Dirichlet–Riemann–Chandrasekhar problem for fluid ellipsoids (Chandrasekhar 1969; Kondratyev 1989a).

The steps taken in solving the main problem are contained in Kondratyev & Malkov (1986, 1987a,b) and Kondratyev (1989a,b). Our approach starts from the stellar hydrodynamical equations. We introduce a procedure that leads to a system of differential equations of order 18. This system governs the motion of the collisionless ellipsoid.

It is important to realize that the hydrodynamical description of a stellar system does not determine the sign of the distribution function. The second of the two methods used in the construction of models, the kinetic method, is, however, sensitive to the sign of the distribution function. Here we present some principal elements of the combined method, which overcomes this difficulty and allows the main problem to be solved.

2 ESSENCE OF THE PROBLEM

The collisionless, homogeneous, self-gravitating ellipsoid with bounding surface

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}=1$$

and internal potential

$$\varphi=I-A_1x_1^2-A_2x_2^2-A_3x_3^2$$

consists of stars of equal mass [$I=\varphi(0)=$constant is the central potential]. The coordinate axes $Ox_1x_2x_3$ coincide with the principal axes of the ellipsoid.[1] The ellipsoid in the general case admits the following dynamical description (Kondratyev & Malkov 1986; Kondratyev 1989a,b).

(i) It rotates about its centre with angular velocity $\mathbf{\Omega}(t)$.

(ii) The mean internal velocity field, $\boldsymbol{u}(\boldsymbol{x})$, satisfies the equation

$$\boldsymbol{u}(\boldsymbol{x})=\begin{pmatrix} \dot{a}_1 & a_1\lambda_3 & -a_1\lambda_2 \\ -a_2\lambda_3 & \dot{a}_2 & a_2\lambda_1 \\ a_1\lambda_2 & -a_3\lambda_1 & \dot{a}_3 \end{pmatrix}\begin{pmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{pmatrix}. \quad (1)$$

(iii) The ellipsoid has time-dependent volume and semi-axes $a_i(t)\,(i=1,2,3)$.

(iv) The equations of motion of an individual star are

$$\frac{d^2\boldsymbol{x}}{dt^2}=\nabla\left[\varphi+\frac{1}{2}(\mathbf{\Omega}\times\boldsymbol{x})^2\right]+\boldsymbol{x}\times\mathbf{\Omega}+2(\boldsymbol{v}\times\mathbf{\Omega}). \quad (2)$$

The star at $\boldsymbol{x}$ has the random (peculiar) velocity

$$\boldsymbol{v}'=\boldsymbol{v}-\boldsymbol{u}, \quad (3)$$

in consequence of which the local velocity dispersion tensor is

$$\sigma(\boldsymbol{x},t)=\sigma^{\circ}(t)(1-m^2), \quad (4)$$

where

$$m^2=\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2} \qquad (0\le m\le 1) \quad (5)$$

and $\sigma^{\circ}(t)$ is the value of the tensor at the origin.

(v) The ellipsoid is governed by 15 time-dependent differential equations. Nine of these equations are the elements of $c_{ij}=0\,(i,j=1,2,3)$, where

$$c_{11}=\frac{d^2a_1}{dt^2}-a_1(\lambda_2^2+\lambda_3^2+\Omega_2^2+\Omega_3^2)+2(a_3\lambda_2\Omega_2+a_2\lambda_3\Omega_3)+2A_1a_1-\frac{2\sigma_{11}^{\circ}}{a_1}=0, \quad (6)$$

$$c_{12}=2\frac{d}{dt}(a_1\lambda_3-a_2\Omega_3)-\dot{a}_1\lambda_3+\dot{a}_2\Omega_3+a_1\lambda_1\lambda_2+a_2\Omega_1\Omega_2-2a_3\lambda_1\Omega_2-\frac{2\sigma_{12}^{\circ}}{a_2}=0, \quad (7)$$

$$c_{13}=2\frac{d}{dt}(a_3\Omega_2-a_1\lambda_2)-\dot{a}_3\Omega_2+\dot{a}_1\lambda_2+a_1\lambda_1\lambda_3+a_3\Omega_1\Omega_3-2a_2\lambda_1\Omega_3-\frac{2\sigma_{13}^{\circ}}{a_3}=0, \quad (8)$$

[1]The 'preferred axis' is a technical term that means a line segment interior to the equilibrium homogeneous ellipsoid (fluid or collisionless) along which the centrifugal force is balanced at all points by gravity. In ellipsoids that rotate with angular velocity $\mathbf{\Omega}(0,\Omega_2,\Omega_3)$, the preferred axis exists when the condition in equation (70) (see Section 5.1) applies, and it lies in the (x_2,x_3) plane, with a slope of

$$k=-\frac{2A_2-\Omega_3^2}{\Omega_2\Omega_3}=-\frac{\Omega_2\Omega_3}{2A_3-\Omega_2^2}=\frac{\Omega_2A_2}{\Omega_3A_3}$$

(Kondratyev 1989a).

and similarly for different permutations of the indices (1, 2, 3). There are also six equations for $\sigma^o_{ij}(t)$. In matrix form we have

$$\frac{d\sigma^o}{dt} = \sigma^o \mathbf{H} + \mathbf{H}^T \sigma^o, \tag{9}$$

where the auxiliary matrix **H** is

$$\mathbf{H} = \begin{pmatrix} -\frac{\dot{a}_1}{a_1} & \frac{a_2}{a_1}\lambda_3 - 2\Omega_3 & -\frac{a_3}{a_1}\lambda_2 + 2\Omega_2 \\ -\frac{a_1}{a_2}\lambda_3 + 2\Omega_3 & -\frac{\dot{a}_2}{a_2} & \frac{a_3}{a_2}\lambda_1 - 2\Omega_1 \\ \frac{a_1}{a_3}\lambda_2 - 2\Omega_2 & -\frac{a_1}{a_2}\lambda_3 + 2\Omega_3 & -\frac{\dot{a}_3}{a_3} \end{pmatrix}, \tag{10}$$

and $\mathbf{H}^T$ is the matrix **H** transposed. Thus we have a closed system of differential equations for 15 unknowns, $a_i(t)$, $\Omega_i(t)$, $\lambda_i(t)$ and $\sigma^o_{ij}(t)$ $(i, j = 1, 2, 3)$.

(vi) The dynamical model has five integrals of motion: the entire energy, the total angular momentum and three special phase invariants. The phase invariants exist because, in six-dimensional phase space $(x_1, x_2, x_3, v_1, v_2, v_3)$, the model's motion constitutes an affine transformation of the phase ellipsoid $Q(\boldsymbol{x}, \boldsymbol{v}') = 1$ [see equation (12)]. The third phase invariant,

$$I_3 = 8a_1^2a_2^2a_3^2 \begin{vmatrix} \sigma^o_{11} & \sigma^o_{12} & \sigma^o_{13} \\ \sigma^o_{12} & \sigma^o_{22} & \sigma^o_{23} \\ \sigma^o_{13} & \sigma^o_{23} & \sigma^o_{33} \end{vmatrix}, \tag{11}$$

plays a solid role in the theory. First, I_3 is not an integral of the whole system of 15 equations, but of only six of them (equation 9). Further, since the model must occupy the ellipsoidal in phase space volume $V = (3!/\pi)\sqrt{I_3}$, the existence of I_3 ensures that Liouville's theorem is valid.

To complete our equations we need the expression (Kondratyev 1989a,b)

$$Q(\boldsymbol{x}, \boldsymbol{v}') = m^2 + \frac{1}{8|\sigma^o|} \begin{vmatrix} 2\sigma^o_{11} & 2\sigma^o_{12} & 2\sigma^o_{13} & v'_1 \\ 2\sigma^o_{12} & 2\sigma^o_{22} & 2\sigma^o_{23} & v'_2 \\ 2\sigma^o_{13} & 2\sigma^o_{23} & 2\sigma^o_{33} & v'_3 \\ v'_1 & v'_2 & v'_3 & 0 \end{vmatrix}. \tag{12}$$

Inside the model,

$$m^2 \le Q \le 1, \tag{13}$$

and $Q = 1$ is the equation of the model's bounding phase ellipsoid in phase space.

In velocity space, the model occupies the ellipsoidal region

$$Q' = \frac{1}{2|\sigma^o|(Q - m^2)} \sum_{i,j=1}^{3} \alpha_{ij} v'_i v'_j \le 1. \tag{14}$$

The volume of the ellipsoid (14) is

$$V' = \frac{8\sqrt{2}}{3}\pi\sqrt{|\sigma^o|}(Q - m^2)^{3/2}. \tag{15}$$

The important question here is about the dimensionality of the phase ellipsoid (12) or of the velocity ellipsoid (14). Until now we have believed and have proved that the condition of phase degeneracy

$$I_3 = 0 \tag{16}$$

is valid only for equilibrium figures. From this it follows that all the stationary models have only two parameters. The dimensionality of the non-stationary models was unknown. Direct analysis enables us to resolve this question: equation (16) must hold even in the general case of a non-stationary collisionless ellipsoid.

3 ON THE NECESSITY OF PHASE DEGENERATION

The integral equation for the distribution function $f[Q]$ for our models is

$$\rho\Theta(1-m^2)=\iiint_{Q'\leq 1} f[Q]\,\mathrm{d}v'_1\,\mathrm{d}v'_2\,\mathrm{d}v'_3, \tag{17}$$

where $\rho=$ constant is the model density and $\Theta(\ldots)$ is the Heaviside function. After the substitution

$$\mathrm{d}v'_1\,\mathrm{d}v'_2\,\mathrm{d}v'_3\equiv \mathrm{d}V'(Q)=\frac{\mathrm{d}V'}{\mathrm{d}Q}\mathrm{d}Q=4\sqrt{2}\pi\sqrt{|\sigma^{\circ}|}\sqrt{Q-m^2}\,\mathrm{d}Q, \tag{18}$$

equation (17) reduces to the Volterra integral equation of the first kind:

$$\frac{\rho\Theta(1-m^2)}{4\pi\sqrt{2}\sqrt{|\sigma^{\circ}|}}=\int_{m^2}^{1}\sqrt{Q-m^2}\,f[Q]\,\mathrm{d}Q. \tag{19}$$

Equation (19) has the kernel $\sqrt{Q-m^2}$, which is equal to zero at the lower limit $Q=m^2$. Hence to find a solution of this equation it is necessary to obtain the derivative with respect to m^2. We have

$$\frac{\rho}{2\pi\sqrt{2}\sqrt{|\sigma^{\circ}|}}\delta(1-m^2)=-\int_{1}^{m^2}\frac{f[Q]}{\sqrt{Q-m^2}}\mathrm{d}Q, \tag{20}$$

where $\delta(\ldots)$ is the delta function. Substitution of $S=1-m^2$ and $J=1-Q$ gives the Abel integral equation

$$\frac{\rho}{2\pi\sqrt{2}\sqrt{|\sigma^{\circ}|}}\delta(S)=\int_{0}^{S}\frac{f(J)\,\mathrm{d}J}{\sqrt{S-J}} \tag{21}$$

with the solution

$$f(J)=\frac{\rho}{2\pi^2\sqrt{2}\sqrt{|\sigma^{\circ}|}}\frac{\mathrm{d}}{\mathrm{d}J}(J^{-1/2}). \tag{22}$$

From (22) we have

$$f[Q]=-\frac{\rho}{2\pi^2\sqrt{2}\sqrt{|\sigma^{\circ}|}}(1-Q)^{-3/2}\leq 0. \tag{23}$$

Thus in the general case the distribution function (23) is negative and unphysical. Physical models must be degenerate in phase space.

4 PHASE DEGENERATION AND ITS CONSEQUENCES

The only loophole in the above reasoning lies in the factor $\sqrt{Q-m^2}$ in the integrand of (19). We close this loophole by requiring that

$$|\sigma^{\circ}|=\begin{vmatrix}\sigma^{\circ}_{11} & \sigma^{\circ}_{12} & \sigma^{\circ}_{13}\\ \sigma^{\circ}_{12} & \sigma^{\circ}_{22} & \sigma^{\circ}_{23}\\ \sigma^{\circ}_{13} & \sigma^{\circ}_{23} & \sigma^{\circ}_{33}\end{vmatrix}=0, \tag{24}$$

such that, for instance,

$$\frac{\sigma^{\circ}_{12}}{\sigma^{\circ}_{13}}=\frac{\sigma^{\circ}_{22}}{\sigma^{\circ}_{23}}=\frac{\sigma^{\circ}_{23}}{\sigma^{\circ}_{33}}. \tag{25}$$

Let

$$\mu\equiv\alpha_{22}=\begin{vmatrix}\sigma^{\circ}_{11} & \sigma^{\circ}_{13}\\ \sigma^{\circ}_{13} & \sigma^{\circ}_{33}\end{vmatrix}\neq 0. \tag{26}$$

Then by means of the identities

$$\mu\alpha_{11}-\alpha_{12}^2=\sigma_{33}^{\circ}|\boldsymbol{\sigma}^{\circ}|,$$
$$\mu\alpha_{33}-\alpha_{23}^2=\sigma_{11}^{\circ}|\boldsymbol{\sigma}^{\circ}|, \tag{27}$$
$$\alpha_{12}\alpha_{23}-\mu\alpha_{13}=\sigma_{13}^{\circ}|\boldsymbol{\sigma}^{\circ}|,$$

we transform expression (12) into

$$Q=m^2+\frac{\mu}{2|\boldsymbol{\sigma}^{\circ}|}\left[v_2'+\frac{\alpha_{12}v_1'+\alpha_{23}v_3'}{\mu}\right]^2+\frac{\sigma_{33}^{\circ}v_1'^2+\sigma_{11}^{\circ}v_3'^2-2\sigma_{13}^{\circ}v_1'v_3'}{2\mu}. \tag{28}$$

Here α_{ij} is the cofactor of σ_{ij}° in the matrix $\boldsymbol{\sigma}^{\circ}$.

4.1 A five-dimensional non-stationary model

By virtue of the condition in (24), we should demand in equation (28) that

$$v_2'+\frac{\alpha_{12}v_1'+\alpha_{23}v_3'}{\mu}=0, \tag{29}$$

so that, since $\alpha_{12}=0$ from (25), we have the additional relation

$$\sigma_{33}^{\circ}v_2'-\sigma_{23}^{\circ}v_3'=0. \tag{30}$$

Then from (28), we obtain for Q

$$Q(\boldsymbol{x},\boldsymbol{v}')=m^2+\frac{\sigma_{33}^{\circ}v_1'^2+\sigma_{11}^{\circ}v_3'^2-2\sigma_{13}^{\circ}v_1'v_3'}{2\mu}. \tag{31}$$

Thus Q depends on only two variables, v_1' and v_3'. The plane section (30) of the ellipsoid (14) is an ellipse. Its area, when projected on to the $v_1'Ov_3'$ plane, is

$$S=2\pi\sqrt{\mu}(Q-m^2). \tag{32}$$

Now

$$\mathrm{d}v_1'\,\mathrm{d}v_3'=\frac{\mathrm{d}S}{\mathrm{d}Q}\mathrm{d}Q=2\pi\sqrt{\mu}\,\mathrm{d}Q \tag{33}$$

and equation (19) reduces to

$$\frac{\rho\Theta(1-m^2)}{2\pi\sqrt{\mu}}=\int_{m^2}^{1}f[Q]\,\mathrm{d}Q, \tag{34}$$

where Q is given by (31). Equation (34) has the positive solution

$$f[Q]=\frac{\rho}{2\pi\sqrt{\mu}}\,\delta(1-Q)\geq 0, \tag{35}$$

or, with (30) and (31),

$$f[Q]=\frac{\rho}{2\pi\sqrt{\mu}}\,\delta\left(v_2'-\frac{\sigma_{23}^{\circ}}{\sigma_{33}^{\circ}}v_3'\right)\,\delta\left(1-m^2-\frac{\sigma_{33}^{\circ}v_1'^2+\sigma_{11}^{\circ}v_3'^2-2\sigma_{13}^{\circ}v_1'v_3'}{2\mu}\right). \tag{36}$$

The important question here is which model is described by the distribution function (36)? Notice that at

$$\lambda_i\neq 0,\qquad \Omega_i\neq 0\qquad (i=1,2,3) \tag{37}$$

the differential equations (9), by virtue of the condition in (25), give us additional conditions

$$\frac{\sigma_{11}^{\rm o}}{\sigma_{12}^{\rm o}}=\frac{\sigma_{12}^{\rm o}}{\sigma_{22}^{\rm o}}=\frac{\sigma_{13}^{\rm o}}{\sigma_{23}^{\rm o}}. \tag{38}$$

However, these conditions are incompatible with condition (26). We therefore require that

$$\Omega_i=\lambda_i=0 \qquad (i=1,2,3), \tag{39}$$

that is, the model cannot rotate and there is no internal streaming motion. The equations of motion (6)–(9) then become

$$\frac{d^2a_1}{dt^2}=-2A_1a_1+\frac{2K_1}{a_1^3},$$

$$\frac{d^2a_2}{dt^2}=-2A_2a_2, \tag{40}$$

$$\frac{d^2a_3}{dt^2}=-2A_3a_3+\frac{2K_3}{a_3^3},$$

$$\sigma_{22}^{\rm o}=\sigma_{12}^{\rm o}=\sigma_{13}^{\rm o}=\sigma_{23}^{\rm o}=0, \tag{41}$$

where the integrals of motion are

$$K_1=\sigma_{11}^{\rm o}a_1^2, \qquad K_3=\sigma_{33}^{\rm o}a_3^2. \tag{42}$$

The model has the distribution function

$$f=\frac{\rho}{2\pi\sqrt{\sigma_{11}^{\rm o}\sigma_{33}^{\rm o}}}\,\delta(v_2')\,\delta\left(1-m^2-\frac{v_1'^2}{2\sigma_{11}^{\rm o}}-\frac{v_3'^2}{2\sigma_{33}^{\rm o}}\right). \tag{43}$$

In velocity space the model occupies an elliptical disc, and in phase space it occupies a five-dimensional ellipsoid. The overwhelming majority of stars in this model touch its surface, and the inequality (13) has a formal meaning only.

We come to the conclusion that there are no physical models that occupy six-dimensional phase ellipsoids. This reveals a defect in the equations for non-linear oscillations of a non-rotating collisionless ellipsoid, which were presented previously by other authors (Polyachenko & Friedman 1976, p. 205).

4.2 Doubly degenerate non-stationary models

We now set

$$\mu=0. \tag{44}$$

This condition, together with (25), is equivalent to condition (38). Then formula (31), written as

$$Q=m^2+\frac{\sigma_{33}^{\rm o}}{2\mu}\left(v_1'-\frac{\sigma_{13}^{\rm o}}{\sigma_{33}^{\rm o}}v_3'\right)^2+\frac{v_3'^2}{2\sigma_{33}^{\rm o}}, \tag{45}$$

reduces to

$$Q=m^2+\frac{v_3'^2}{2\sigma_{33}^{\rm o}} \tag{46}$$

and gives a further relation

$$\sigma_{33}^{\rm o}v_1'-\sigma_{13}^{\rm o}v_3'=0. \tag{47}$$

Thus in this case we have two additional relations, (30) and (47), which together impose further degeneracy on the models. Namely, the models now occupy four-dimensional phase ellipsoids, and in velocity space they fill only one-dimensional lines or needles. It should be noticed that this degeneracy implies the existence, at a given point $\boldsymbol{x}$, of linear relations between the components of stellar velocity vectors. Contact with the surface is possible only for stars that lie at the endpoints of the velocity needles (Kondratyev 1992a).

The length of the projection of the velocity needle on to the v_3'-axis is

$$\mathscr{L}=2\sqrt{2\sigma_{33}^{\circ}}\,\sqrt{Q-m^2}. \tag{48}$$

Then, instead of equation (17), we have

$$\frac{\rho\Theta(-m^2)}{\sqrt{2\sigma_{33}^{\circ}}}=\int_{m^2}^{1}\frac{f[Q]}{\sqrt{Q-m^2}}\,\mathrm{d}Q, \tag{49}$$

with solution

$$f[Q]=\frac{\rho}{\pi\sqrt{2\sigma_{33}^{\circ}}}(1-Q)^{-1/2}\geq 0, \tag{50}$$

where Q is given by (46). This solution is physical and may be more explicitly written as

$$f[Q]=\frac{\rho}{\pi\sqrt{2\sigma_{33}^{\circ}}}\;\frac{\delta\left(v_2'-\dfrac{\sigma_{23}^{\circ}}{\sigma_{33}^{\circ}}v_3'\right)\delta\left(v_1'-\dfrac{\sigma_{13}^{\circ}}{\sigma_{33}^{\circ}}v_3'\right)}{\sqrt{1-m^2-(v_3'^2/2\sigma_{33}^{\circ})}}. \tag{51}$$

The form of the distribution function (51) is consistent with the fact that the overwhelming majority of stars in this model do not touch the model's elliptical bounding surface. The inequality (13) now has a physical meaning.

In order that equations (6) to (9), together with the relations (25) and (38), form a closed system, we demand that the ellipsoid rotate about one of its principal axes. For instance, let

$$\lambda_1=\lambda_2=\Omega_1=\Omega_2=0,\qquad \Omega_3=\Omega,\qquad \lambda_3=\lambda. \tag{52}$$

We then require

$$\sigma_{12}^{\circ}=\sigma_{13}^{\circ}=\sigma_{23}^{\circ}=0,\qquad \sigma_{11}^{\circ}=\sigma_{22}^{\circ}=0, \tag{53}$$

which will give us

$$\frac{\mathrm{d}^2a_1}{\mathrm{d}t^2}-a_1(\lambda^2+\Omega^2)+2a_2\lambda\Omega+2A_1a_1=0,$$

$$\frac{\mathrm{d}^2a_2}{\mathrm{d}t^2}-a_2(\lambda^2+\Omega^2)+2a_1\lambda\Omega+2A_2a_2=0,$$

$$\frac{\mathrm{d}^2a_3}{\mathrm{d}t^2}+2A_3a_3-\frac{2K_3}{a_3^3}=0, \tag{54}$$

$$2\frac{\mathrm{d}}{\mathrm{d}t}(a_1\lambda-a_2\Omega)-a_1\dot{\lambda}+a_2\dot{\Omega}=0,$$

$$2\frac{\mathrm{d}}{\mathrm{d}t}(a_1\Omega-a_2\lambda)-a_1\dot{\Omega}+a_2\dot{\lambda}=0.$$

Now, the function (51) is

$$f[Q]=\frac{\rho}{\pi\sqrt{2\sigma_{33}^{\circ}}}\;\frac{\delta(v_1')\,\delta(v_2')}{\sqrt{1-m^2-(v_3'^2/2\sigma_{33}^{\circ})}}. \tag{55}$$

Therefore the model rotates about the x_3-axes, and it is 'cold' in the $(x_1,\,x_2)$ plane. This model occupies a four-dimensional phase ellipsoid.

For the model with $a_1 = a_2$, the angular velocity in the inertial coordinate system is $\Omega_{in} = \Omega - \lambda$. This model is described by the same distribution function, (55), and the equations

$$\frac{d^2 a_1}{dt^2} - \frac{l^2}{4a_1^3} + 2A_1 a_1 = 0,$$

$$\frac{d^2 a_3}{dt^2} - \frac{2K_3}{4a_3^3} + 2A_3 a_3 = 0, \tag{56}$$

where $l = 2\Omega_{in} a_1^2 =$ a constant.

In particular, with $\Omega_{in} = 0$ $(l = 0)$ we have the special case of a non-rotating spheroid.

4.3 Models without velocity dispersion

In the particular case

$$\sigma_{ij}^{o} = 0, \tag{57}$$

we have $Q = m^2$ and

$$f[Q] = \rho\delta(v_1')\,\delta(v_2')\,\delta(v_3'). \tag{58}$$

This is the case of a collisionless ellipsoid with no velocity dispersion. We can obtain its equations of motion from expressions (6) to (8). Equation (9) and both additional conditions (25) and (38) are fulfilled identically. In six-dimensional phase space this model occupies simply the three-dimensional material ellipsoid.

5 THE EQUILIBRIUM FIGURES

Let us consider stationary models. The total angular momentum vector $\boldsymbol{l}$ of a stationary ellipsoid must coincide with both the vector $\boldsymbol{\Omega}$ and the rotational axis, i.e.

$$[\boldsymbol{l} \times \boldsymbol{\Omega}] = 0. \tag{59}$$

Besides this, $\boldsymbol{\Omega}$ and $\boldsymbol{\lambda}$ should both lie in one of the symmetry planes or coincide with one of the principal axes of the ellipsoid. For example, let

$$\Omega_1 = \lambda_1 = 0. \tag{60}$$

Then, from (7) and (8), it follows that

$$\sigma_{12}^{o} = \sigma_{13}^{o} = 0, \tag{61}$$

and the condition of phase degeneracy (25) now gives

$$\tau \equiv \alpha_{11} = \sigma_{22}^{o}\sigma_{33}^{o} - \sigma_{23}^{o2} = 0. \tag{62}$$

The system of equations (6) to (9) reduces to

$$-a_1(\lambda_2^2 + \lambda_3^2 + \Omega_2^2 + \Omega_3^2) + 2(a_3\lambda_2\Omega_2 + a_2\lambda_3\Omega_3) + 2A_1 a_1 = \frac{2\sigma_{11}^{o}}{a_1},$$

$$-a_2(\lambda_3^2 + \Omega_3^2) + 2a_1\lambda_3\Omega_3 + 2A_2 a_2 = \frac{2\sigma_{22}^{o}}{a_2},$$

$$-a_3(\lambda_2^2 + \Omega_2^2) + 2a_1\lambda_2\Omega_2 + 2A_3 a_3 = \frac{2\sigma_{33}^{o}}{a_3},$$

$$a_2\lambda_2\lambda_3 + a_3\Omega_2\Omega_3 - 2a_1\lambda_2\Omega_3 = \frac{2\sigma_{23}^{o}}{a_3}, \tag{63}$$

$$a_2\Omega_2\Omega_3+a_3\lambda_2\lambda_3-2a_1\lambda_3\Omega_2=\frac{2\sigma^{\circ}_{23}}{a_2},$$

$$\left(-\frac{a_2}{a_1}\lambda_2+2\Omega_3\right)\sigma^{\circ}_{11}+\left(\frac{a_1}{a_2}\lambda_3-2\Omega_3\right)\sigma^{\circ}_{22}+\left(-\frac{a_1}{a_3}\lambda_2+2\Omega_2\right)\sigma^{\circ}_{23}=0,$$

$$\left(-\frac{a_3}{a_1}\lambda_2-2\Omega_2\right)\sigma^{\circ}_{11}+\left(-\frac{a_1}{a_3}\lambda_2+2\Omega_2\right)\sigma^{\circ}_{33}+\left(\frac{a_1}{a_2}\lambda_3-2\Omega_3\right)\sigma^{\circ}_{23}=0.$$

This means that, in (62) and (63), we have eight equations for the 10 variables

$$\frac{a_2}{a_1},\frac{a_3}{a_1},\lambda_2,\lambda_3,\Omega_2,\Omega_3,\sigma^{\circ}_{11},\sigma^{\circ}_{22},\sigma^{\circ}_{33},\sigma^{\circ}_{23}. \tag{64}$$

Therefore in the general case all equilibrium ellipsoids with velocity dispersion have two parameters.

There are now two cases: (i) when there is only one additional relation (30) and $\sigma^{\circ}_{11}\neq 0$; (ii) when both (30) and (47) hold, and (47) reduces to $v'_1=0$, i.e. $\sigma^{\circ}_{11}=0$. In the first case we have models SVa,b,c and in the second case models SIVa,b,c (see Section 1).

5.1 The model SVa

This is Freeman's ellipsoid. We have the conditions

$$\lambda_2=\Omega_2=0;\qquad \lambda_3=\lambda,\qquad \Omega_3=\Omega;\qquad \sigma^{\circ}_{23}=\sigma^{\circ}_{22}=0, \tag{65}$$

$$f[Q]=\frac{\rho}{\pi\sqrt{2\sigma^{\circ}_{11}\sigma^{\circ}_{33}}}\,\delta(v'_2)\,\delta\left(1-m^2-\frac{v'^2_1}{2\sigma^{\circ}_{11}}-\frac{v'^2_3}{2\sigma^{\circ}_{33}}\right). \tag{66}$$

In this case we have from (62) and (63)

$$\Omega^2=2A_2,\qquad \lambda=2\frac{a_1}{a_2}\sqrt{2A_2}\geq 0,$$

$$\sigma^{\circ}_{11}=a_1^2\left[A_1+A_2\left(3-4\frac{a_1^2}{a_2^2}\right)\right],\qquad \sigma^{\circ}_{33}=A_3a_3^2. \tag{67}$$

The preferred axis coincides with the x_2-axis and $a_2\geq a_1$. The model has parameters a_1/a_2 and a_3/a_2.

5.2 The models SVb,c

These have an oblique rotation and distribution function

$$f[Q]=\frac{\rho}{\pi\sqrt{2\sigma^{\circ}_{11}\sigma^{\circ}_{33}}}\,\delta\left(v'_2-\frac{\sigma^{\circ}_{23}}{\sigma^{\circ}_{33}}v'_3\right)\,\delta\left(1-m^2-\frac{v'^2_1}{2\sigma^{\circ}_{11}}-\frac{v'^2_3}{2\sigma^{\circ}_{33}}\right). \tag{68}$$

Equations (62) and (63) provide complete information on the models. Nevertheless, by using relation (30) and equations (1) to (3), it is possible to obtain the auxiliary relations

$$\frac{\sigma^{\circ}_{33}}{\sigma^{\circ}_{22}}=\frac{\Omega_2\Omega_3}{2A_2-\Omega_3^2}=\frac{2A_3-\Omega_2^2}{\Omega_2\Omega_3}=\frac{2\Omega_2-(a_3/a_1)\lambda_2}{(a_2/a_1)\lambda_3-2\Omega_3}. \tag{69}$$

In particular, from (69) we have the condition for the existence of a preferred axis:

$$\frac{\Omega_2^2}{2A_3}+\frac{\Omega_3^2}{2A_2}=1. \tag{70}$$

The models have two parameters, a_1/a_2 and $\eta=\Omega_3^2/2A_2$, and in the $(a_1/a_2,\ a_3/a_2)$ plane are situated on the curve that has the implicit equation

$$A_3-A_2=\pi G\rho\,\frac{a_2^2-a_3^2}{a_2^2+a_3^2-2a_1^2}. \tag{71}$$

From (62), (63) and (69) it follows that

$$p=\frac{\lambda_2}{\Omega_2}=2\,\frac{a_1}{a_3}\left[1-\eta\left(\frac{a_2^2-a_3^2}{4a_1^2}\right)\right],$$

$$q=\frac{\lambda_3}{\Omega_3}=2\,\frac{a_1}{a_2}\left[1+(1-\eta)\left(\frac{a_2^2-a_3^2}{4a_1^2}\right)\right]. \tag{72}$$

In addition, we have

$$\sigma_{11}^{\circ}=(A_3a_2^2-A_2a_3^2)\left(\frac{a_2^2-a_3^2}{2a_2a_3}\right)^2(\eta-\eta_1)(\eta-\eta_2)(\eta-\eta_3)\geq 0,$$

$$\sigma_{22}^{\circ}=A_2\left(\frac{a_2^2-a_3^2}{2a_1}\right)^2(1-\eta)(\eta-\eta_1)(\eta-\eta_2)>0,$$

$$\sigma_{33}^{\circ}=A_3\left(\frac{a_2^2-a_3^2}{2a_1}\right)^2\eta(\eta-\eta_1)(\eta-\eta_2)\geq 0, \tag{73}$$

$$\sigma_{23}^{\circ}=\sqrt{A_2A_3}\left(\frac{a_2^2-a_3^2}{2a_1}\right)^2\sqrt{\eta(1-\eta(}(\eta-\eta_1)(\eta-\eta_2)\geq 0,$$

where η_1 and η_2 are solutions of the equation

$$\eta^2-\frac{4a_1^2+a_2^2-a_3^2}{a_2^2-a_3^2}\,\eta+\left(\frac{2a_1a_2}{a_2^2-a_3^2}\right)^2=0 \tag{74}$$

and η_3 is given by

$$\eta_3=\frac{1}{a_2^2-a_3^2}\left[4a_1^2-a_3^2-3a_2^2a_3^2\left(\frac{A_3-A_2}{A_3a_2^2-A_2a_3^2}\right)\right]. \tag{75}$$

The model SVb exists for $\eta_2\leq\eta\leq 1$, and the model SVc for $\eta_3\leq\eta\leq\eta_2$.

At $\eta=\eta_1$ and $\eta=\eta_2$, we have two conjugate one-parameter ellipsoids with no velocity dispersion, which are collisionless analogues of the pressureless Riemann fluid ellipsoid. The model with $\eta=1$ is a special case of Freeman's ellipsoid. Finally, the model with $\eta=\eta_3$ has $\sigma_{11}^{\circ}=0$, and it is a special case of the model SIVc. Fig. 1 shows the domains of existence of the models.

5.3 The models SIVa,b,c

When $\sigma_{11}^{\circ}=0$, the equilibrium figures have needle-shaped velocity ellipsoids. There is now the additional first integral of motion of a star:

$$v_1=\frac{a_1}{a_2}\lambda_3x_2-\frac{a_1}{a_3}\lambda_2x_3, \tag{76}$$

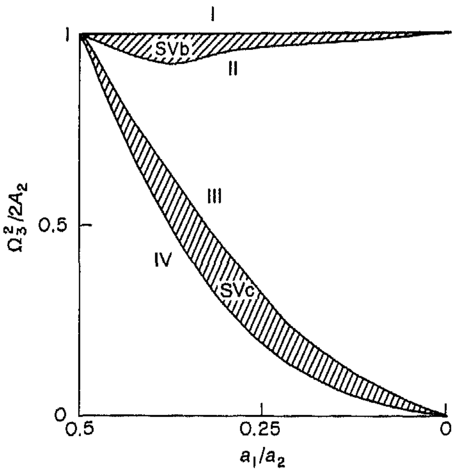

Figure 1. Domains of existence (shaded) of the models SVb,c on the $(a_1/a_2, \eta)$ plane. Here, four one-parameter sequences are shown: I is a special case of Freeman's ellipsoids; II and III are the conjugate ellipsoids with no velocity dispersion (they are analogues of Riemann's pressureless ellipsoids), and IV is a special case of the model SIVc.

which leads to

$$\frac{\sigma_{33}^{o}}{\sigma_{22}^{o}}=\frac{\Omega_2\Omega_3-2(a_1/a_2)\lambda_3\Omega_2+(a_3/a_2)\lambda_2\lambda_3}{2A_2-\Omega_3^2+2(a_1/a_2)\lambda_3\Omega_3-\lambda_3^2}$$
$$=\frac{2A_3-\Omega_2^2+2(a_1/a_3)\lambda_2\Omega_2-\lambda_2^2}{\Omega_2\Omega_3+(a_2/a_3)\lambda_2\lambda_3-2(a_1/a_3)\lambda_2\Omega_3} \tag{77}$$

[cf. (69)]. In the general case these models have no preferred axis.

The distribution function we obtain from (51) may be written as (see also Kondratyev 1992b)

$$f[Q]=\frac{\rho}{\pi\sqrt{2\sigma_{33}^{o}}}\,\delta(v_1')\,\frac{\delta[v_2'-(\sigma_{23}^{o}/\sigma_{33}^{o})v_3']}{\sqrt{1-m^2-(v_3'^2/2\sigma_{33}^{o})}}. \tag{78}$$

Here, we shall not fully describe the models because of their awkwardness, but merely point out some interesting features. The models have two parameters, a_2/a_1 and a_3/a_1. We find, for example, that

$$\Omega_2^2=\frac{2}{(m-n)^2}\{-A_1(n-2)(m-n)+A_2(n-2)[m-(n-2)(1-2m)]+A_3(m-2)[(n-2)(1-2n)-n]\},$$
$$\Omega_3^2=\frac{2}{(m-n)^2}\{A_1(m-2)(m-n)+A_2(n-2)[(m-2)(1-2m)-m]+A_3(m-2)[n-(m-2)(1-2n)]\}, \tag{79}$$

and

$$\sigma^{o}=\frac{N}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & \dfrac{T_2}{T_3}\\ 0 & \dfrac{T_2}{T_3} & \left(\dfrac{T_2}{T_3}\right)^2\end{pmatrix}. \tag{80}$$

Here

$$m=\frac{a_1}{a_3}\frac{\lambda_2}{\Omega_2}\quad\text{and}\quad n=\frac{a_1}{a_2}\frac{\lambda_3}{\Omega_3}. \tag{81}$$

Moreover, from equation (59) it follows that

$$\frac{1-2n}{1-2m}=\frac{a_3^2}{a_2^2}. \tag{82}$$

Also,

$$T_2=\Omega_3(n-2) \qquad \text{and} \qquad T_3=\Omega_2(m-2). \tag{83}$$

To find n or m, we solve the cubic equation

$$C_3n^3+C_2n^2+C_1n+C_0=0, \tag{84}$$

where $C_i(i=0, 1, 2, 3)$ are composite functions from a_2/a_1 and a_3/a_1 (Kondratyev 1992b). This equation has three real-valued roots that give us the models SIVa,b,c (see Figs 2 and 3).

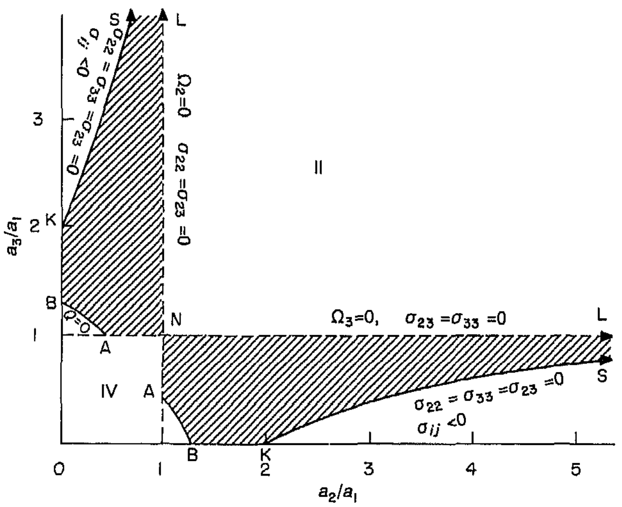

Figure 2. Domains of existence (shaded) of the models SIVa,b on the $(a_2/a_1, a_3/a_1)$ plane. Geometric loci on the line segment AN (the ray NL) represent oblate (prolate) spheroids and BK represents disc-shaped ellipsoids with velocity dispersion. The point A corresponds to a_3/a_1 (or a_2/a_1) = 0.372 03. On the curve AB the discriminant Q of the cubic equation (84) is zero, and both models converge to one. The curve KS with implicit equation (71) represents models SVb,c (and the conjugate ellipsoids with no velocity dispersion). A spherical state of the model is marked by point N.

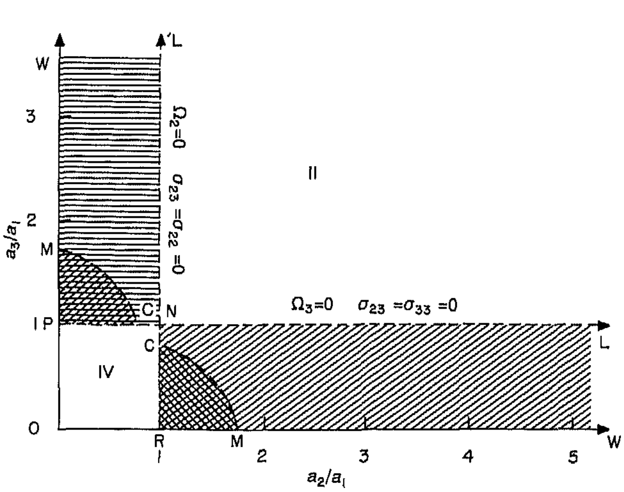

Figure 3. Domains of existence (shaded) of the model SIVc on the $(a_2/a_1, a_3/a_1)$ plane. Geometric loci on three bounds of the domain represent oblate (line segments NR, NP) or prolate (NL) spheroids and lines RW and PW represent disc-shaped ellipsoids. Domains with cross-hatching are occupied by models with $\sigma_{23}^0 \leq 0$. On the curve CM with equation $(a_2^2/a_1^2)+3(a_3^2/a_1^2)=3$, the component $\sigma_{23}^0=0$, and above this curve $\sigma_{23}^0>0$. Models on the curve CM have $m=(a_1/a_3)(\lambda_2/\Omega_2)=2$ and $\sigma_{22}^0=0$.

6 CONCLUSIONS

The study of collisionless models with a quadratic potential was initiated by Freeman and extended by Hunter, Bisnovatyi-Kogan, Antonov, Friedman and Polyachenko among others. However, no coherent theory emerged from these studies.

The present theory results from the discovery of ellipsoids with oblique rotation. These models were first constructed by the kinetic method. A second, alternative line of approach is afforded by the stellar hydrodynamical method. In support of this method was the finding that all third moments from distribution functions for models with a linear mean velocity field as in equation (1) vanish. The hierarchy of stellar hydrodynamical equations is naturally decomposed by this fact, and as a result we have closed system of those equations of first, second and third orders. Both methods are useful and complement each other.

The argument in favour of this theory is that it is associated with Dirichlet's problem for fluid ellipsoids. This problem encompasses both the non-linear vibration of ellipsoids and the complete study of equilibrium figures.

This paper summarizes our investigations during 1984–1992. A unified classification by phase-space structure and a complete scheme of ellipsoidal self-consistent models, including the non-stationary ones, have been developed. We classify all models as five-, four- and three-dimensional phase ellipsoids, and come to the conclusion that there are no physical models occupying six-dimensional ellipsoids. The equations of vibration for all non-stationary models have been obtained. The existence of six classes of equilibrium figures has been proved. These models admit an asymptotic limit transition to disc-shaped ellipsoids (Kondratyev 1992d).

This theory is of some help in the study of non-linear oscillations of galaxies and overcomes limitations of the scalar or tensor virial theorem. The equilibrium figures constitute unique self-consistent models for galaxies with internal streams and precession. These models permit the study of the shapes of elliptical orbits and gas–dust ring stability in precessing galaxies.

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