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Local Vortices in a Differentially Rotating Flow

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Abstract — A time-dependent local elliptic vortex in a differential two-dimensional incompressible fluid flow is considered. Nonlinear oscillations of vortices of the cyclonic and anticyclonic types are described. It is found that the evolutionary tracks can be both closed and unclosed. The former correspond to the azimuthal oscillations of the principal axis and the latter to the complete rotational state of the elliptic vortex. Steady-state solutions are also obtained; they are represented by ellipses elongated or compressed along the flow. Small oscillations of the vortex equilibrium figures are investigated and a general dispersion relation for arbitrary perturbations is derived. The stability criterion is found.

Keywords: vortex, incompressible fluid, differential flow, cyclones, anticyclone.

Questions concerning the evolution of extensive vortex structures of considerable importance for a series of problems of technical mechanics, geophysics, and astrophysics are currently being intensively studied: in particular, the present state of this topic is described in [1, 2]. It is noteworthy that experiments on the rotation of a thin fluid layer in a vessel reveal many features in common with the phenomenon of circulation in planetary atmospheres [3, 4] and similar phenomena in galactic disks [5]. In particular, the mathematical aspects of these phenomena were described in [6–8]. In this connection, the nonlinear formulation of the problem to some extent makes use of numerical experiments and not entirely rigorous qualitative considerations. In this study we will take a somewhat different path and analyze exactly solvable models which can serve as fairly reliable starting points for testing other (approximate) methods.

Let us consider a two-dimensional ideal fluid flow reducing at infinity to a simple differential flow, i.e., a flow with constant vorticity. Let a relatively autonomous vortex of finite size exist inside the flow. In this environment the vortex will no longer be purely circular, as in the case of the absence of motion at infinity, but more or less compressed (see Fig. 1). As shown for the first time by Chaplygin [9], if the vorticities inside and outside such a local formation are constant (but not equal to one another), there is a solution for an ideal fluid with an elliptic local vortex. However, the steady states themselves (vortex equilibrium figures) and their stability were not studied in [9].

This paper is intended to supplement the above-mentioned study: vortex oscillation equations are derived using a method independent of that developed in [9], their first integral is obtained, the steady states and their stability with respect to arbitrary perturbations are investigated (see Fig. 2), and three vortex rotation regimes are found.

1. FORMULATION OF THE PROBLEM

We introduce the following notation: x, y are Cartesian coordinates, u, v are the corresponding velocity components, p^* is the pressure, ρ is the fluid density (constant), ψ is the stream function in terms of which the velocity components of the incompressible fluid can be expressed in the usual manner:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{1.1}$$

In an ideal fluid the vorticity $\partial v/\partial x - \partial u/\partial y$ is conserved for each particle. The vorticity is equal to $(-\alpha)$ inside the vortex and $(-\beta)$ for the entire outer region. At large distances the last assumption is in

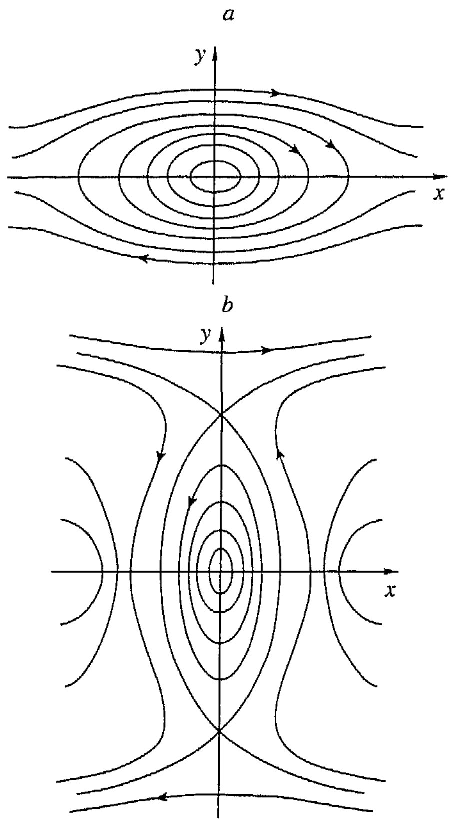

Fig. 1. Streamline pattern: a — $a > b$ (case of an anticyclone); b — $a < b$ (case of a cyclone)

agreement with the asymptotics

$$\psi \approx \frac{\beta y^2}{2}, \qquad u \approx \beta y, \qquad v \approx 0 \tag{1.2}$$

Taking formula (1.1) into account, we find that ψ must satisfy the Poisson equation in one of the forms

$$\Delta\psi = \beta, \qquad \Delta\psi = \alpha \tag{1.3}$$

for the outer medium and the vortex itself, respectively.

We introduce the auxiliary stream function

$$\varphi = \psi - \frac{\beta y^2}{2} \tag{1.4}$$

The function φ can increase at infinity only logarithmically. In the outer region it must satisfy the Laplace equation and in the inner region the modified Poisson equation

$$\Delta\varphi = \alpha - \beta \tag{1.5}$$

We note that the velocity components u and v are assumed to be continuous at the vortex interface. Physically, this is justified at least in most cases of practical importance. In particular, in galaxies a soliton must be, as it were, joined with the surrounding gas mass. However, we cannot entirely ignore other possible applications in which a soliton develops from a certain foreign mass [10]. As distinct from the model considered here, the presence of a foreign mass does not change the circulation in the surrounding medium but creates a jump in the tangential velocity at the boundary of the local vortex. In [1], incidentally, no clear distinction was made between these two cases.

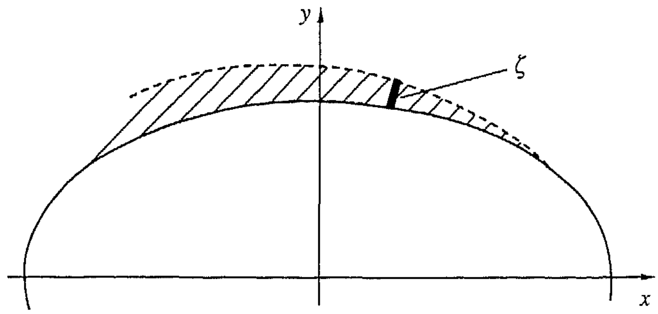

Fig. 2. Perturbation of the ellipse boundary

2. STEADY-STATE VORTEX

Initially, as in [9], we will consider the steady state. Then one of the streamlines simply coincides with the boundary of a soliton, i.e., on its boundary $\psi = \text{const}$. Later, the necessary solution of Eq. (1.5) for an ellipse was thoroughly studied in connection with questions concerning the gravity of a homogeneous elliptic cylinder [11]. If we introduce the notation $A = \sqrt{a^2 + \lambda}$ and $B = \sqrt{b^2 + \lambda}$, then

$$\varphi = \frac{(\alpha - \beta)ab}{2}\left[\ln\frac{A+B}{a+b} + \frac{x^2}{a^2 - b^2}\left(1 - \frac{B}{A}\right) + \frac{y^2}{b^2 - a^2}\left(1 - \frac{A}{B}\right)\right],$$
$$\varphi = \frac{(\alpha - \beta)ab}{2(a+b)}\left(\frac{x^2}{a} + \frac{y^2}{b}\right) \tag{2.1}$$

in the outer and inner regions, respectively, a and b being the semiaxes of the ellipse and λ a positive root of the equation

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \tag{2.2}$$

which exists over the entire outer region. On the boundary of the ellipse itself we have $\lambda = 0$ and the function ψ, together with its derivatives, must be continuous. For (2.1) the asymptotics at infinity are also satisfied automatically. We need only require the satisfaction of the condition $\psi = \text{const}$ on the boundary. Using (1.4) and the second of formulas (2.1), we obtain

$$\frac{(\alpha - \beta)ab}{2(a+b)}\left(\frac{x^2}{a} + \frac{y^2}{b}\right) + \frac{\beta y^2}{2} = \text{const} \tag{2.3}$$

on the ellipse so that the left side of (2.3) must be proportional to the left side for the ellipse. This gives the required equilibrium condition

$$\nu = \frac{\beta}{\alpha} = \frac{a(a-b)}{a^2 + b^2} = \frac{1 - \mu}{1 + \mu^2} \qquad \left(\mu = \frac{b}{a}\right) \tag{2.4}$$

Hence it follows that the vortex is never circular, except in the absence of flow differentiality when $\beta = 0$. We obtain flattened and elongated ellipses ($b < a$ and $b > a$, respectively), when the signs of α and β are the same and different, respectively. We note that in the first case there is a single steady-state solution only when $0 < \nu < 1$ and in the second case there are two solutions when

$$\nu_1 = \frac{\sqrt{2} - 1}{2} < \nu < 0 \tag{2.5}$$

(the left side of (2.5) is the minimum of the fraction $(1 - \mu^2)/(1 + m^2)$), but no solutions exist when $\nu < \nu_1$.

3. EVOLUTION OF ELLIPTIC VORTICES

We will now consider the case in which at the initial instant there is no equilibrium, the vortex is elliptic and its principal axes are arbitrarily inclined to the coordinate axes. When the incompressibility condition is satisfied, the cross-sectional area of the vortex is conserved: $ab = \text{const}$, the equation of the ellipse being taken in a coordinate system inclined to the initial coordinate system at an angle θ. The momentum conservation with respect to x and y can be described by the equations

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= 2\Omega v - \frac{1}{\rho}\frac{\partial p^*}{\partial x} + \Omega^2 x + \frac{\partial V}{\partial x} \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -2\Omega u - \frac{1}{\rho}\frac{\partial p^*}{\partial y} + \Omega^2 y + \frac{\partial V}{\partial y} \end{aligned} \tag{3.1}$$

in which the rotation of the system with an angular velocity Ω and the gravity field with a potential $V(x, y, t)$ are taken into account and t is time.

Substitution of relations (1.1) in (3.1) gives

$$\begin{aligned} \psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy} &= -2\Omega\psi_x - \frac{1}{\rho}\frac{\partial p^*}{\partial x} + \Omega^2 x + \frac{\partial V}{\partial x} \\ -\psi_{xt} - \psi_y\psi_{xx} + \psi_x\psi_{xy} &= -2\Omega\psi_y - \frac{1}{\rho}\frac{\partial p^*}{\partial y} + \Omega^2 y + \frac{\partial V}{\partial y} \end{aligned} \tag{3.2}$$

In (3.2) we replace the derivatives of ψ with respect to the same variable by means of one or other of formulas (1.3). We then obtain

$$-\frac{\partial^2\psi}{\partial y\partial t} = \frac{\partial T}{\partial x}, \qquad \frac{\partial^2\psi}{\partial x\partial t} = \frac{\partial T}{\partial y} \tag{3.3}$$

with the auxiliary function $T(x, y, t)$

$$T = \gamma\psi + \frac{\psi_x^2 + \psi_y^2}{2} + 2\Omega\psi + \frac{p^*}{\rho} - \frac{\Omega^2(x^2 + y^2)}{2} - V \tag{3.4}$$

where $\gamma = \beta$ and $\gamma = \alpha$ in the outer and inner regions, respectively.

From (3.3) it follows that the function T must satisfy the Laplace equation separately in each region. Constructing the difference between the values of the function (3.4) on either side of the boundary of the vortex, we obtain

$$T_e - T_i = (\alpha - \beta)\psi \tag{3.5}$$

where the subscripts e and i denote the values of this function on the outer and inner sides of the discontinuity, respectively. In deriving (3.5) we used the fact that all the other terms on the right side of (3.4), including the pressure p^* are continuous across the boundary. On the other hand, the masses of fluid entering and leaving the layer must be equal. In accordance with definition (1.1) and formulas (3.3), this is equivalent to the continuity of the normal derivative of T across the boundary

$$\frac{\partial T_e}{\partial n} = \frac{\partial T_i}{\partial n} \tag{3.6}$$

We will now construct the simplest harmonic functions in the regions inside and outside the ellipse [12]. Temporarily, we will use a coordinate system x', y' related to the principal axes of the ellipse after redefining λ accordingly. The inner harmonic functions can be constructed as follows:

$$V_{i1} = 1, \qquad V_{i2} = x'^2 - y'^2, \qquad V_{i3} = x'y' \tag{3.7}$$

(in accordance with the sense of the problem, we have no need for the odd functions). The outer functions are as follows:

$$V_{e1} = \frac{A-B}{A+B} + 2\left(\frac{b^2y'^2}{B} - \frac{a^2x'^2}{A}\right)(A+B)^{-2}$$
$$V_{e2} = \frac{x'y'}{\sqrt{AB}(A+B)^2}, \qquad V_{e3} = \ln(A+B) \tag{3.8}$$

We can even directly check that functions (3.8) tend to zero at large distances ($\lambda \to \infty$) or, at least, behave logarithmically (V_{e3}) and must satisfy the Laplace equation. Accordingly, we have T_i and T_e in the form:

$$T_i = c_1 + c_2(x'^2 - y'^2) + c_3x'y'$$
$$T_e = h_1V_{e1}(x', y') + h_2V_{e2}(x', y') + h_3V_{e3}(x', y') \tag{3.9}$$

with the parameters c_1, c_2, c_3, h_1, h_2, and h_3 to be determined. In developing (3.5) it is necessary to substitute the boundary value $\lambda = 0$. This gives

$$h_1\left[\frac{a-b}{a+b} + \frac{2(y'^2 - x'^2)}{(a+b)^2}\right] + \frac{h_2x'y'}{ab(a+b)^2} + h_3\ln(a+b) - c_1 - c_2(x'^2 - y'^2) -$$
$$c_3x'y' - (\alpha-\beta)\left[\frac{(\alpha-\beta)ab}{2(a+b)}\left(\frac{x'^2}{a} + \frac{y'^2}{b}\right) + \frac{\beta}{2}(x'\sin\theta + y'\cos\theta)^2\right] \tag{3.10}$$

Analogously, from (3.6) we obtain

$$2c_2\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right) + c_3\left(\frac{1}{a^2} + \frac{1}{b^2}\right)x'y' =$$
$$h_1\left[\frac{4(bx'^2 - ay'^2)}{a^2b^2(a+b)} - \frac{2(a-b)}{ab(a+b)} + \frac{4}{(a+b)^2}\left(\frac{y'^2}{b^2} - \frac{x'^2}{a^2}\right)\right] +$$
$$h_2\left[\frac{1}{ab(a+b)^2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - \frac{1}{a^2b^3}\right]x'y' + \frac{h_3}{ab} \tag{3.11}$$

From a comparison of the terms with x'^2, y'^2, and $x'y'$ in (3.10) and (3.11), x' and y' on the boundary being related by the standard ellipse equation, we obtain six equations for the coefficients. In what follows it turns out to be sufficient to know only two of them

$$c_2 = \frac{(\alpha-\beta)^2ab(b-a)}{2(a+b)^3} + \frac{\beta(\beta-\alpha)}{2(a+b)^2}(a^2\sin^2\theta - b^2\cos^2\theta) \tag{3.12}$$

$$c_3 = -\frac{2\beta(\alpha-\beta)ab\sin\theta\cos\theta}{(a+b)^2} \tag{3.13}$$

A comparison of (3.3) with (3.9) gives $\partial u/\partial t = -2c_2x' - c_3y'$ and $\partial v/\partial t = 2c_2y' - c_3x'$ (here, u and v are related to the axes x' and y', respectively) and thus after a short time the total velocities can be determined in terms of the stream function

$$\psi + d\psi = \frac{(\alpha-\beta)ab}{2(a+b)}\left(\frac{x'^2}{a} + \frac{y'^2}{b}\right) +$$
$$\frac{\beta}{2}(x'\sin\theta + y'\cos\theta)^2 + \left[\frac{c_3(x'^2 - y'^2)}{2} - 2c_2x'y'\right]dt \tag{3.14}$$

In (3.14) the coordinate system is fixed. If, however, we introduce a coordinate system x' and y' which follows the rotation of the ellipse axes, then in the linear approximation we have

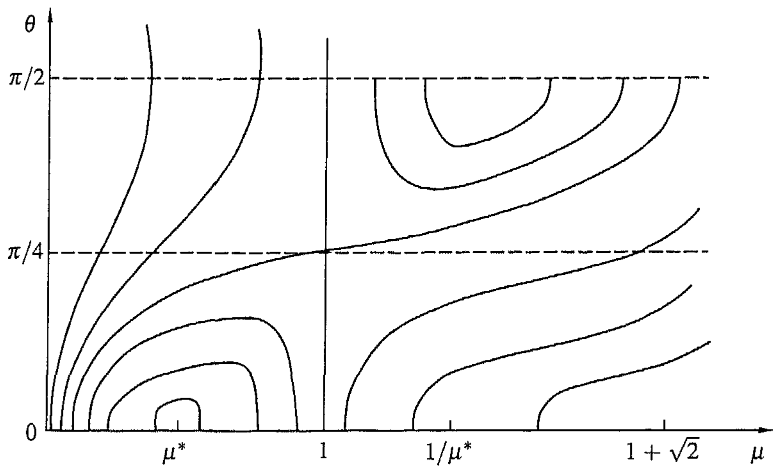

Fig. 3. Evolutionary curves, $\alpha - \beta > 0$ ($\beta = 1$)

$$x' = x'' - y''d\theta, \qquad y' = y'' + x''d\theta \tag{3.15}$$

where $d\theta$ is an additional angle of rotation of the major axis in time dt.

When (3.15) is substituted in (3.14), the substitution $x'\sin\theta + y'\cos\theta \to x''\sin(\theta + d\theta) + y''\cos(\theta + d\theta)$ proceeds with the same accuracy. The term with this combination remains unchanged and after the substitution of (3.15) the remaining terms should coincide with the same expression composing ψ without the increment $d\psi$ but with allowance for the transformation of a and b into $a + da$ and $b + db$. In addition, using the conservation of the soliton area $ab = \text{const}$, we obtain the necessary evolutionary equations for θ and μ

$$\frac{d\theta}{dt} = -\frac{(\alpha - \beta)\mu}{(1 + \mu)^2} - \frac{\beta(\sin^2\theta - \mu^2\cos^2\theta)}{1 - \mu^2}, \quad \frac{d\mu}{dt} = -2\beta\mu\sin\theta\cos\theta \tag{3.16}$$

with account for the explicit expressions (3.12), (3.13) for c_2 and c_3. In the particular case $\beta = 0$ we obtain the Kirchhoff elliptic vortex [13] $d\theta/dt = \Omega\alpha\mu/(1 + \mu)^2$. The nonlinear vibration equations (3.16) were first derived in [9] but without investigating the stability of the steady states.

First of all, we find the following integral of Eqs. (3.16)

$$H = \frac{\alpha - \beta}{\beta}\ln\frac{(1 + \mu)^2}{4\mu} + \mu\cos^2\theta + \frac{1}{\mu}\sin^2\theta \tag{3.17}$$

The system (3.16) can be reduced to canonic form if we take θ and $\mu + 1/\mu$ as the variables. In fact, however, this property was not used.

An analysis of Eqs. (3.16) confirms that steady states exist only if the orientations of the principal soliton axes and the initial coordinate axes coincide. However, we can also consider oscillations of an arbitrary amplitude with respect to the steady state. Without loss of generality, we will take $\beta > 0$. We must then distinguish three cases.

1) If $\alpha - \beta > 0$, then from (2.4) it is clear that $0 < \mu < 1$ for the steady state. We will schematically describe the location of the curves $H = \text{const}$ along which the motion takes place. In particular, when $\theta = 0$ the function H has a single minimum, namely, for the steady state $\mu = \mu^*$. In addition, we note that always $H = 1$ when $\mu = 1$. The behavior of H in the neighborhood of $\mu = 0$, $\theta = 0$ is somewhat more complex. Asymptotically, formula (3.17) can be rewritten in the form:

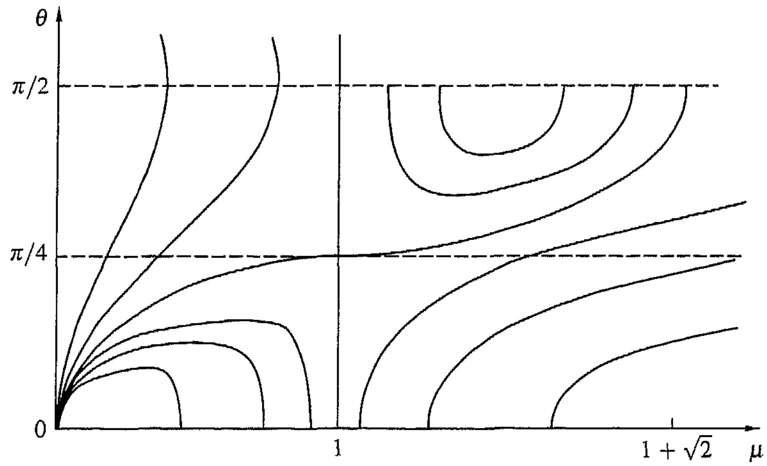

Fig. 4. Evolutionary curves, $0 > \alpha - \beta > -3 - \sqrt{8}$ ($\beta = 1$), example $\alpha = 0$

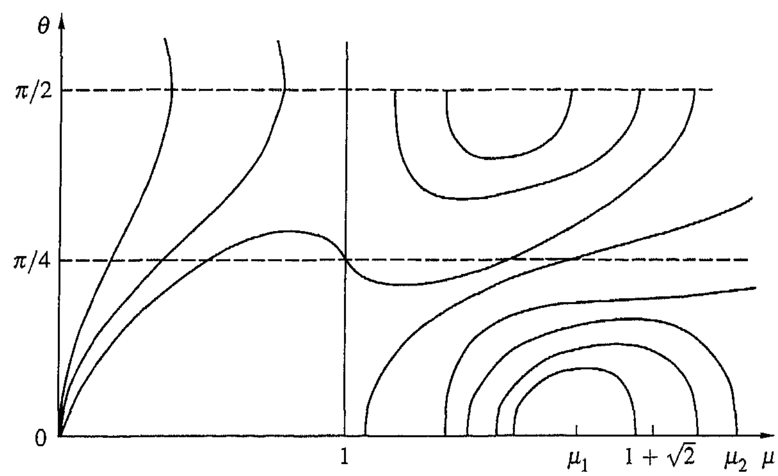

Fig. 5. Evolutionary curves, $\alpha - \beta < -3 - \sqrt{8}$ ($\beta = 1$)

$$\theta^2\mu^{-1} - [(\alpha-\beta)\beta^{-1}]\ln 4\mu \approx H \tag{3.18}$$

From the last equation we can see that the value of μ increases as θ deviates from zero, i.e., after intersecting the μ axis the curves $H = \text{const}$ turn to the right. In addition, we take into account the invariancy of H with respect to the substitution $\mu \to 1/\mu$ and $\theta \to \theta + \pi/2$ and the periodicity of this function with period π, which are geometrically obvious. On the basis of these features we can trace the path of the level curves of H (see Fig. 3). The trajectories $H = \text{const}$ break down into two classes: closed and unclosed. The first means that the value of the angle θ oscillates by no more than $\pi/4$ in either direction and, overall, we obtain oscillations of the major axis of the ellipse about a stable state. The second type of the behavior means that the vortex rotates progressively: the angle θ always increases or decreases. The above-mentioned Kirchhoff vortex ($\beta = 0$) turns out to be a particular case of this type of behavior.

2) In the case $0 > (\alpha - \beta)/\beta > -3 - \sqrt{8}$ finding the extremum of H leads to the equation $\beta\mu^2 + \alpha\mu + \beta - \alpha = 0$ with negative or complex roots which are physically meaningless. From formula (3.18) we can see that in the neighborhood of $\mu = 0$, $\theta = 0$ the curves $H = \text{const}$ turn left and, consequently, arrive at the same point forming a node. All the curves $H = \text{const}$ tend to $\mu = 0$ and $\mu = \infty$ so that the soliton is extended into a thin filament and later disappears (is progressively "crushed"). The location of the curves is shown in Fig. 4.

3) When $(\alpha - \beta)/\beta < -3 - \sqrt{8}$ extrema of H for $\theta = 0$ appear again but this time in the right half of the figure at $\mu > 1$. Testing the sign of the second derivative of H shows that the curves $H = \text{const}$ surround a point with $\mu = \mu_1$ while $\mu = \mu_2$ gives a saddle point (μ_1 and μ_2 are the smaller and larger roots of Eq. (2.4) so that always $\mu_1 < 1 + \sqrt{2}$ and $\mu_2 > 1 + \sqrt{2}$). The location of the curves is shown in Fig. 5. Trajectories of the closed type and trajectories corresponding to vortex crushing are both present.

4. ENERGY RELATIONS

We can arrive at the same qualitative conclusions by determining the kinetic energy of the disturbed state (the potential energy is of no importance since in any potential field it cannot vary in view of the incompressibility). More precisely, we must subtract the energy of the main motion without the vortex from the total kinetic energy taken over some broad domain M. Thus, we determine

$$T = \frac{\rho}{2}\iint\limits_M (u^2 + v^2 - \beta^2 y^2)\,dx\,dy = \frac{\rho}{2}\iint\limits_M \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 - \beta^2 y^2\right] dx\,dy = \frac{\rho}{2}\iint\limits_M \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + 2\beta y\frac{\partial\varphi}{\partial y}\right] dx\,dy$$

The first terms can be transformed in standard fashion

$$\iint\limits_M \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] dx\,dy = \oint\left(\frac{\partial\varphi}{\partial x}dy - \frac{\partial\varphi}{\partial y}dx\right) - \iint\limits_M \varphi\Delta\varphi\,dx\,dy \qquad (4.1)$$

It is natural to choose a strip $|y| < B$ with a fairly large B as the domain M. (For example, a circular domain is physically less convenient due to the entrainment of energy by the flow beyond the bounds of the domain.) The loop integral in (4.1) is of the order of $1/B$ and can be neglected. Thus, in (4.1) we have only the integral over the area of the ellipse E

$$\iint\limits_M \varphi\Delta\varphi\,dx\,dy = (\alpha - \beta)\iint\limits_E \varphi\,dx\,dy = \frac{\pi(\alpha - \beta)^2 a^2 b^2}{2}\left[\ln(a + b) + \frac{1}{4}\right] \qquad (4.2)$$

Formula (4.2) is not affected by rotation of the ellipse. On the other hand, in the remaining integral of $y(\partial\varphi/\partial y)$ this rotation must be taken into account. It is convenient to represent the function $\varphi(x, y)$ as a sum of many similarly constructed functions $\varphi(x, y|x_i, y_i)$ differing in that the Poisson equation (1.5) acts over a small domain with an area ΔS_i about the point x_i, y_i. Then, following formula (2.1), we can find the asymptotics

$$\frac{\partial\varphi_i}{\partial y} \approx \frac{(\alpha - \beta)\Delta S_i (y - y_i)}{2\pi[(x - x_i)^2 + (y - y_i)^2]}, \qquad \int\limits_{-\infty}^{\infty} y\frac{\partial\varphi_i}{\partial y}dx = \frac{(\alpha - \beta)\Delta S_i}{2} y\,\text{sing}(y - y_i)$$

$$\iint\limits_M \frac{\partial\varphi_i}{\partial y}dx\,dy = \frac{\alpha - \beta}{2}(B^2 - y_i^2)\Delta S_i$$

Combining these results for all the small areas, we obtain

$$\iint\limits_M \frac{\partial\varphi}{\partial y}dx\,dy = \frac{\alpha - \beta}{2}\iint\limits_E (B^2 - y^2)\,dx\,dy$$

and then, with allowance for the inclination of the ellipse, we have

$$\frac{2T}{\rho} = -\frac{\pi(\alpha-\beta)^2a^2b^2}{2}\left[\ln(a+b)+\frac{1}{4}\right] + \pi ab\beta(\alpha-\beta)\left(B^2 - \frac{a^2\sin^2\theta + b^2\cos^2\theta}{4}\right)$$

Once a and b have been expressed in terms of μ and the constant quantity ab, this leads, correct to a certain linear transform, to the already mentioned function H (see formula (3.17)). Thus, the conservation of the additional energy for vortex oscillations of any amplitude is confirmed.

5. ARBITRARY SMALL OSCILLATIONS

We will consider the arbitrary oscillations of a vortex which no longer conserve their elliptic shape but restrict our attention to the linear approximation. Let initially $a > b$. We introduce the planar elliptic coordinates ξ, η using the formulas

$$x = c\cosh\xi\cos\eta, \qquad y = c\sinh\xi\sin\eta \tag{5.1}$$

where $c = \sqrt{a^2 - b^2}$ is the focal length and the boundary corresponds to a certain $\xi = \xi_0$ so that $a = c\cosh\xi_0$ and $b = c\sinh\xi_0$.

Elementary harmonic functions, which can be expressed in terms of the elliptic coordinates, are well known [12], namely, the inner and outer functions

$$\begin{aligned} V_{i1} &= \cosh k\xi\cos k\eta, \qquad V_{i2} = \sinh k\xi \sin k\eta \\ V_{e1} &= e^{-k\xi}\cos k\eta, \qquad V_{e2} = e^{-k\xi}\sin k\eta \end{aligned} \tag{5.2}$$

The index k can take the values 0, 1, 2,... so that in each particular case (in the linear approximation) we can get by with the functions with only a single value of k, except for zero, which would simply result in transition to a larger-scale vortex.

The function φ must satisfy the Poisson equation inside the deformed domain and its increment $\delta\varphi$ must satisfy the Laplace equation in both the outer and inner domains so that we can set inside and outside the deformed ellipse, respectively,

$$\delta\varphi_i = p_kV_{1i} + q_kV_{2i}, \qquad \delta\varphi_e = P_kV_{1e} + Q_kV_{2e} \tag{5.3}$$

The functions ψ and φ must be continuous on the boundary of the vortex. On going over to the disturbed boundary, we must add $\zeta(\partial\psi/\partial n)$ to the modified value $\psi + \delta\psi$ of the function ψ on the ellipse, where ζ is the displacement of the boundary reckoned along the normal n. In the linear approximation, with allowance for the continuity of $\partial\psi/\partial n$ in the steady state, the difference between these two sums, inside and outside, can simply be reduced to the difference between the corresponding $\delta\psi = \delta\varphi$, i.e., the single boundary condition has the form $\delta\varphi_i = \delta\varphi_e$ ($\xi = \xi_0$). In expanded form, if we collect the individual terms with $\cos k\eta$ and $\sin k\eta$, this boundary condition means

$$p_k\cosh k\xi_0 = P_ke^{-k\xi_0}, \qquad q_k\sinh k\xi_0 = Q_ke^{-k\xi_0} \tag{5.4}$$

In order to establish a relation between the normal derivatives we turn to Fig. 2. With allowance for the Laplace equation in the outer region, integration of $\Delta\varphi$ over the specified part of the layer between the initial and disturbed boundaries gives

$$\iint \Delta\varphi\, dx dy = \int\limits_{E'} \frac{\partial\varphi}{\partial n} ds - \int\limits_{E} \frac{\partial\varphi}{\partial n} ds$$

(E' is the area of the deformed ellipse) and in the linear approximation we have

$$(\alpha - \beta)\zeta = \left[\frac{\partial \delta\varphi}{\partial n}\right]_i^e \tag{5.5}$$

As follows from (5.1), on the boundary $dx^2 + dy^2 = c^2(\cosh^2\xi - \cos^2\eta)(d\xi^2 + d\eta^2)$ and

$$\frac{\partial}{\partial n} = \frac{1}{c\sqrt{\cosh^2\xi_0 - \cos^2\eta}}\frac{\partial}{\partial \xi} \tag{5.6}$$

We seek the displacement of the boundary in the form:

$$\zeta = \frac{L\cos k\eta + N\sin k\eta}{\sqrt{\cosh^2\xi_0 - \cos^2\eta}} \tag{5.7}$$

Substituting expressions (5.3) and (5.7) in (5.5), with allowance for (5.6) we again find after separating cosines and sines

$$(\beta - \alpha)L = \frac{k}{c}(P_k e^{-k\xi_0} + p_k \sinh k\xi_0), \quad (\beta - \alpha)N = \frac{k}{c}(Q_k e^{-k\xi_0} + q_k \cosh k\xi_0) \tag{5.8}$$

We now turn to the functions T_i and T_e whose perturbations are also harmonic functions. Linearization of (3.5) gives

$$[\delta T]_i^e = (\alpha - \beta)\left(\delta\varphi + \zeta\frac{\partial \psi}{\partial n}\right) \tag{5.9}$$

We seek these perturbations of the auxiliary functions in the form:

$$T_i = fV_{i1} + gV_{i2}, \qquad T_e = FV_{e1} + GV_{e2} \tag{5.10}$$

The procedure for separating cosines and sines, exactly as before, makes it possible to obtain from (5.7) and (5.9)

$$\begin{aligned} Fe^{-k\xi_0} - f\cosh k\xi_0 &= (\alpha - \beta)[p\cosh k\xi_0 - ke^{-\xi_0}(Pe^{-k\xi_0} + p\sinh k\xi_0)] \\ Ge^{-k\xi_0} - g\sinh k\xi_0 &= (\alpha - \beta)[q\sinh k\xi_0 - ke^{-\xi_0}(Qe^{-k\xi_0} + q\cosh k\xi_0)] \end{aligned} \tag{5.11}$$

Due to the conformality of the mapping $x, y \to \xi, \eta$ formulas (3.3) can be rewritten in the form $-\psi_{\eta t} = \partial T/\partial\xi$ and $\psi_{\xi t} = \partial T/\partial\eta$ and analogously for the outer region. The same relation between the increments $\delta\varphi$ and δT still holds and, as a result, the same procedure for separating cosines and sines gives

$$\frac{dq}{dt} = -f, \quad \frac{dp}{dt} = g, \qquad \frac{dQ}{dt} = F, \qquad \frac{dP}{dt} = -G \tag{5.12}$$

Equations (5.4), (5.8), (5.11), and (5.12) are sufficient to determine the evolution of the vortex. Introducing the increment ν, after simple algebra we find

$$\nu^2 = -\beta^2\sigma_1(\xi_0)\sigma_2(\xi_0) \tag{5.13}$$

where

$$\sigma_{1,2}(\xi_0) = \frac{1}{2}e^{2\xi_0}\tanh\xi_0(1 \mp e^{-2k\xi_0}) - ke^{\xi_0}\sinh\xi_0 \tag{5.14}$$

In the trivial case $k = 1$ we are simply concerned with the displacement of the vortex as a whole; therefore $\sigma_2(\xi_0) = 0$ and $\nu = 0$. From (5.14) it follows that σ_2 increases with respect to k so that when $k \geq 2$ we have $\sigma_1 < \sigma_2 < 0$ and $\nu^2 < 0$, i.e., the compressed vortices are always stable, as can be seen with reference to $k = 2$.

When $b > a$ the initial formulas (5.1) and related formulas must be modified. Instead we have $x = c\sinh\xi \sin\eta$, $y = c\cosh\xi \cos\eta$, $c = \sqrt{a^2 - b^2}$, $a = c\sinh\xi_0$, and $b = c\cosh\xi_0$, while formulas (5.2) conserve the initial form. Algebra carried out in accordance with the same plan leads to (5.13) with a somewhat different definition of σ_1 and σ_2

$$\sigma_{1,2}(\xi_0) = \frac{1}{2}e^{2\xi_0}\coth\xi_0(1 \pm e^{-2k\xi_0}) - ke^{\xi_0}\cosh\xi_0$$

Once again, when $k = 1$ we have $\sigma_2 = 0$ identically and further analysis shows that $\sigma_2 < 0$ when $k > 1$. At the same time σ_1 decreases with increase in k. Two cases can be distinguished:

1) $\sigma_1 > 0$ when $k = 2$, then $v^2 > 0$ and we obtain instability with respect to elliptic deformations. Finding the critical ξ_0 and related μ gives a result already known as the critical $\mu = 1 + \sqrt{2}$ in complete agreement with the analysis for elliptic deformations;

2) $\sigma_1 < 0$ when $k = 2$, then we have $\sigma_1 < 0$ for greater k also. Instability manifests itself when even without this the system is unstable to perturbations with $n = 2$.

We note Chaplygin's priority [9] in treating this problem. There are later studies with no reference to [9], chiefly [14]. However, the formulas presented in [14] are obviously erroneous due to the fact that the outer differential flow was taken in invariable form, whereas it was necessary to take its variation into account together with the deformation of the vortex itself. The same error was transferred to [15] in which the question of the stability of the vortex was studied but the results, which differ from ours, are simply invalid due to the inaccuracy mentioned above.

Summary. The nonlinear oscillations, equilibrium, and stability of an elliptic vortex with inhomogeneous vorticity located in a differential incompressible fluid flow are considered. The common rotation of the system at a constant angular velocity Ω physically drops out and has no effect on the results (due to the incompressibility postulate) but in fact it can be present. The results can be applied to gaseous subsystems of fairly oblate galaxies. If we assume that $\Omega > 0$, then in the steady state the motion will exactly correspond to $\beta > 0$ for the deceleration of rotation customary in galaxies as the distance from the center increases. In accordance with the conventional terminology of geophysics the vortices with $\alpha > 0$ and $\alpha < 0$ should be called anticyclonic and cyclonic vortices, respectively. A stability analysis shows that anticyclonic vortices are always stable. This is in agreement with the conclusions based only on quasilinear considerations. On the other hand, when the oblateness is fairly large cyclonic vortices can also be unstable. We note that cyclonic vortices appear starting from a finite value of the difference $\beta - \alpha$ and cannot be considered in the quasilinear approximation. We also point out that the vortex is "material". It consists of trapped particles. Moreover, a capture zone (clear for the cyclonic vortex but smeared for the anticyclonic one) can be observed around the elliptic vortex itself. Similar applications are possible for planetary atmospheres.

REFERENCES

1. P. G. Saffman, *Vortex Dynamics*, Cambridge University Press, Cambridge (1992).
2. A. V. Borisov, I. S. Mamaev, and M. A. Sokolovskii, *Fundamental and Applied Problems of the Theory of Vortices* [in Russian], Institute of Computer Investigations, Moscow, Izhevsk (2003).
3. M. V. Nezlin, "Rossby solitons (Experimental investigations and a laboratory model of natural vortices of the Jupiter's Big Red Spot type)," *Usp. Fiz. Nauk*, **150**, 3 (1986).
4. M. V. Nezlin and E. N. Snezhkin, *Rossby Solitons and Spiral Structures: Astrophysics and Plasma Physics in Shallow Water Experiments* [in Russian], Nauka, Moscow, (1990).
5. V. I. Korchagin and V. I. Petviashvili, "Rossby solitons in a galactic disk," *Pis'ma v Astron. Zhurn.*, **11**, No. 4, 298 (1985).
6. G. G. Sutyrin, "Theory of solitary anticyclones in a rotating fluid," *Dokl. Akad. Nauk SSSR*, **280**, 1101 (1985).
7. G. G. Sutyrin, "Rossby solitary axisymmetric waves of finite amplitude," *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 4, 119 (1985).
8. G. G. Sutyrin and U. G. Yushina, "Formation of a vortex soliton," *Dokl. Akad. Nauk SSSR*, **299**, 580 (1988).

9. S. A. Chaplygin, "On the pulsating cylinder," in: *Collected Works*, Vol. 2 [in Russian], OGIZ, Moscow, Leningrad (1948), P. 138.
10. V. A. Antonov and O. A. Zheleznyak, "On the origin of annular structures in Galaxies," *Astron. Zhurn.*, **65**, No. 3, 461 (1988).
11. B. P. Kondrat'ev, *Theory of Potential and Equilibrium Figures* [in Russian], Institute of Computer Research, Moscow, Izhevsk (2003).
12. V. A. Antonov, E. I. Timoshkova, and K. V. Kholshevnikov, *Introduction to the Theory of the Newtonian Potential* [in Russian], Nauka, Moscow, (1988).
13. H. Lamb, *Hydrodynamics*, Cambridge University Press, New York (1957).
14. S. Kida, "Motion of an elliptic vortex in a uniform shear flow," *J. Phys. Soc. Japan*, **50**, No. 10, 3517 (1981).
15. D. G. Dritschel, "The stability of elliptical vortices in an external straining flow," *J. Fluid Mech.*, *210*, 223 (1990).