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Part IV

Equilibrium Figures and Gravitational Potential Models

The Theory of Equilibrium Figures and Dynamics of Galaxies

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Abstract. The theory of equilibrium figures arisen at the very origin of celestial mechanics now is an extensive and deeply developed section closely adjoining with astronomy. The theory of equilibrium figures expands an outlook of the researcher and provides basic ideas for studying dynamics and evolution of many astrophysical objects. It is a perfect theory of a model and a prototype for constructing models of galaxies. In practice the theory gives the global and local dynamical description of star systems.

1. Some General Properties of Equilibrium Figures

For liquid figures of relative equilibrium we mention here the Lichtenstein theorem of existence an equatorial symmetry plane, which concerns also to galaxies in similar situations. For liquid and stellar figures with internal flows this plane can be absent as, for example, for some liquid Riemann ellipsoids (Chandrasekhar 1969) or collisionless phase space ellipsoids (Kondratyev 1989).

For astrophysical applications knowledge of the Poincaré limit for the angular velocity of liquid mass with isotropic pressure $\Omega^2 < 2\pi G\bar{\rho}$, improved subsequently by Crudeli ($\Omega^2 < \pi G\rho$), is of great significance. There are some reasons to believe that for all figures of relative equilibrium a much stronger inequality takes place, namely $\omega^2 \equiv \Omega^2/(\pi G\rho) \leq 0.45$, that is valid for the Maclaurin spheroids. It would be important to give the strict proof for the inequality. The criterion can be generalized for stellar systems with anisotropic velocity distribution: $\Omega^2 \leq 2\pi G\rho_{\max}$ or $\bar{\Omega}^2 \leq 2\pi G\rho_{\max}$, where $\bar{\Omega} = L/I$ (Kondratyev 1989). Really, the Poincaré limit is already broken for a special case of Freeman ellipsoids. For stellar systems the last inequality cannot already be strengthened by Crudeli example.

Precession of liquid figures of relative equilibrium is impossible (Appell 1932, Antonov 1973). The precession is also impossible for equilibrium figures with internal flows and anisotropic "pressure" (Kondratyev 1999). For any equilibrium figure with (or without) internal velocity field directions of the vector of angular velocity $\mathbf{\Omega}$, the vector of total angular momentum $\mathbf{L}$, and *the vector of internal flows* $\mathbf{\Lambda}$ should coincide with the direction of *the vector of total circulation* $\mathbf{c}$, as it takes place in the Dirichlet-Riemann-Chandrasekhar problem; vectors $\mathbf{\Omega}$ and $\mathbf{\Lambda}$ should lay in one of the main planes of the figure. The generalization of Poincaré criterion for figures with internal linear velocity field (including non-stationary ones) is possible (Kondratyev 1989).

Liquid figures of relative equilibrium can rotate only around the least axis. Triaxial liquid figures with internal flows can rotate also around the middle axis or to possess an oblique rotation. The equilibrium figures of stellar systems can rotate around any of three axes the inertia ellipsoid or, under certain conditions, to have inclined rotation.

Generally, there is no direct relation between an angular velocity and flattening even for liquid figures of relative equilibrium. Especially, there is no such direct interrelation for figures with internal velocity field: for them also such factor, as a direction of internal flows in relation to rotation of the figure plays a significant role. Even more there is a complex interrelation between flattening and rotation for stellar models and galaxies, for which anisotropy of velocity dispersion can play essential role.

There are some bifurcation point sets for all sequences of equilibrium figures. Till now there is no strict proof of uniqueness of bifurcation points for harmonics above third order for Jacobi ellipsoids. Unfortunately, the proof of uniqueness given by Appell (1932, sect.81) is erroneous (Antonov & Kondratyev). All collisionless figures with ellipsoidal phase distribution function must be sufficiently degenerated in the six-dimensional phase space (Kondratyev 1995).

All gravitating equilibrium figures are of negative heat capacity. From this point an interpretation of known bifurcations (e.g., on the sequence of Maclaurin spheroids) as the physical phase transitions of first-third types (Christodoulou et al. 1995a) is of some interest. The point of view (Lynden-Bell) that the latent reason of all phase transi-

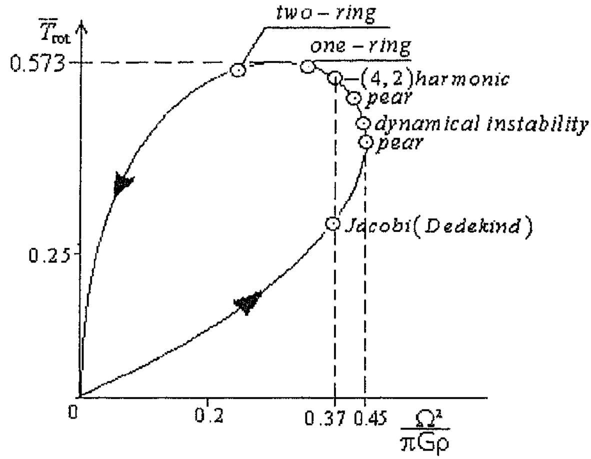

Figure 1.

tions lies in the same negative heat capacity is, however, controversial for systems with many parameters.

2. Classes of Equilibrium Figures

2.1. Figures of relative equilibrium

The Maclaurin spheroids. Role of the "first swallows" have played Newton's "channels" in the homogeneous Earth and the well-known formula for flattening of our planet $\varepsilon = (5/4)(\Omega^2 R^3)/(GM)$. The first stage was completed by researches of Maclaurin, Jacobi, Meyer and Liouville. Two corner stones have appeared in the base of theory of equilibrium figures, Maclaurin spheroids and Jacobi ellipsoids. Surprisingly, but fact: it is impossible to force Maclaurin spheroids to rotate faster, than $\Omega^2/(\pi G\rho) = 0.449$. Flattening of Maclaurin spheroids grows with Ω only until the central pressure decreases up to 0.4823 from spherical value. A specific entropy of the Maclaurin spheroid with maximum Ω ($e = 0.92995$) is minimum for all figures on this sequence: thus, this figure is the most difficult for receiving from collapsing gas cloud. It is curious to connect with this circumstance the result of Gott & Thuan (1975) who found that a maximal flattening which could be received by elliptical galaxy in a course of dissipation-free collapse was close just to E7.

As well, the rotational kinetic energy $T_{rot}(\varepsilon)$ also is a nonmonotone function along the Maclaurin sequence. The valuable information about the Maclaurin spheroids is derived from the dependence T_{rot} on Ω^2. We have bifurcations from the Maclaurin spheroids shown at Figure 1 (except for the Riemann ellipsoids). The valuable information can be received from comparison of the angular momentum distribution for axisymmetric galaxies with one for Maclaurin spheroids.

Jacobi ellipsoids. The dynamical characteristics of these one-parametric ellipsoids are well investigated. Bifurcation of the Jacobi ellipsoids occurs from the critical Maclaurin spheroid at the point of beginning of its secular instability ($\varepsilon = 0.412$) under the condition of preservation its angular momentum. Along the Jacobi sequence the angular momentum L grows, and the circulation $c = 2a_1a_2\Omega_{Jac}$ decreases, so the relation L/c grows. Till now in current literature there is misunderstanding both complex questions about the secular and the dynamical stability of Jacobi ellipsoids (Christodoulou et al. 1995b) (recall here the old argument of principle in discussion between Poincaré and Liapounoff!). The situation is not so harmless, for it touches upon the roots of the theory.

The nonellipsoidal equilibrium figures. The deeply developed theory of nonellipsoidal equilibrium figures also has applications to dynamics of galaxies. Some galaxies show effects of pear-shapeness (NGC 128) and even of higher order harmonics. Numerical accounts of the Japanese authors Eriguchi et al. (1982) have shown that a sequence of pear-shaped figures from the Jacobi ellipsoid comes to an end by the member with singular point. Any division of these pear-shaped figures do not occur!

The Roche model. In some cases the known Roche model with the limiting mass concentration to centre of the figure is useful for studying dynamics and evolution of galaxies. The main characteristic of the Roche model is the relation $\Omega^2 \leq 0.72$. In the limiting case the figure is similar to a lens with flattening $\varepsilon = 0.33$ (the galaxy M 104 "Sombrero"). The generalization of this model with replacement of the central point mass by any other figure (for example, by Jacobi ellipsoid) is possible.

Nonhomogeneous configurations with spheroidal layers. Generally, the equipotential surfaces for inhomogeneous figures cannot be second-order surfaces. However at small flattening of layers a spheroidal stratification take place. The internal structure is described by the Clairaut equation. In case of decreasing density from the centre to periphery, the flattening of layers always grows in this direction. The flattening of layers for equilibrium stellar systems can be different because of influence of "pressure" anisotropy. This theory has no direct applications to galaxies.

The polytropes in rigid or differential rotation. The structure of non-uniform configurations is governed by the equation $\Delta\Phi = -4\pi G\rho(\Phi) + \Delta Q$. For a special case of the polytropic condition $\rho = \theta^n$ the appropriate figures are refereed to as pseudospheroids. Significant contribution to researches of these configurations was brought by J. H. Jeans, S. Chandrasekhar, P. H. Roberts, R. A. James. For galaxies these models have not received yet wide application.

2.2. The figures with internal motion

The quarter of century has passed still and the theory of equilibrium figures has parted, at last, with children's clothes. Dirichlet has brought an important contribution to foundations of this theory. Do the laws of hydrodynamics permit such motion in a homogeneous incompressible gravitating fluid mass, such that for any time the system is ellipsoidal and the velocity field is a linear function of position? Key to this problem is the condition of linearity of internal velocity field. The superposition of gravitation, centrifugal and Coriolis forces gives a wide family of configurations.

In the Dirichlet problem we deal not only with equilibrium figures, but also with nonlinear ellipsoidal oscillations. Earlier we have described four possible cases of oscillations (Kondratyev 1989). The ellipsoid is governed by ten time-dependent differential equations. There are three integrals of motion: the total energy E, the angular momentum $\mathbf{L}$ and the circulation $\mathbf{c}$. Within the framework of the Dirichlet problem it is possible to formulate a problem for evolution of compressible ellipsoid. From the physical point of view there are two different cases: movement of the ellipsoid without pressure $p_0 = 0$ (Rossner 1967) and the case when it is necessary to set a state equation in addition. The oscillations were numerically considered by Fujimoto (1968). The problem of existence of the conjugate and self-conjugate ellipsoids is especially interesting. It is proved, that besides the self-conjugate ellipsoids there also are two other classes of Dirichlet ellipsoids with the symmetrical matrix of equations of motion. These are the non-stationary and stationary non-rotating and momentless ellipsoids (Kondratyev 1989). The ellipsoids with such properties are of the special interest as models of galaxies. For example, the non-rotating figure can be formed during an evolution under the influence of the galaxy neighbours.

The equilibrium figures in the Dirichlet problem are called as Riemann ellipsoids. There are four classes of two-parametric Riemann ellipsoids. S-ellipsoids rotate around a small or middle symmetry axis.

Three classes of Riemann ellipsoids are characterized with an inclined rotation. The last has served as the kinematic prototype for some phase models of galaxies with inclined rotation. The pressureless fluid ellipsoids are identical to some collisionless figures without velocity dispersion (Kondratyev 1995).

Homogeneous ellipsoidal self-consistent phase models of collisionless stellar systems. A fundamental problem in stellar dynamics is constructing self-consistent models for gravitating collisionless systems. The study of ellipsoidal models has suggested a formulation and a solution of the following problem (Kondratyev 1989, 1995): do the laws of classical dynamics permit motion in a homogeneous collisionless stellar system, such that at any time the system is ellipsoidal and the mean velocity field is a linear function of position? This mathematical problem is a natural generalization of the famous Dirichlet-Riemann-Chandrasekhar problem for fluid ellipsoids. The stellar ellipsoid in the general case admits the following dynamical description: (i) it rotates about its centre with an angular velocity $\mathbf{\Omega}(t)$; (ii) it has the mean internal velocity field $\mathbf{u}(\mathbf{x})$ and the local velocity dispersion tensor; (iii) the ellipsoid has time-dependent volume and semi-axes; (iv) the ellipsoid is governed by 15 time-dependent differential equations.

The dynamical model has five integrals of motion: the entire energy, the total angular momentum and three special phase invariants. The phase invariants exist because, there is the affine transformation of the phase ellipsoid in the six-dimensional phase space. There are six classes of collisionless equilibrium figures in this problem (Kondratyev 1995). We classify all models as five -, four- and three-dimensional phase ellipsoids. These models admit transition to disc-shaped ellipsoids. The phase invariants allow to create a unified method of studying stability. These phase models can be used as models of bars in flat galaxies.

Nonhomogeneous self-consistent phase models of collisionless stellar systems. The epoch of construction of self-consistent phase models of stellar systems has come. In view of ignorance of additional integrals of motion and analytical difficulties at finding distribution functions the accent was frequently made on numerical methods (C. Hunter, T. de Zeeuw, D. Merritt and others). But discussion of this wide theme is out of our plan.

Let's emphasize that it is not necessary for a boundary surface of equilibrium star systems to coincide with level surface. Moreover, the level surfaces can not to be at all.

The figures with internal nonlinear velocity field. It seems perspective now to generalize the Dirichlet problem for configurations with an internal nonlinear velocity field (Filippi et al. 1990).

An important task is the study of nonellipsoidal figures, which fork from Riemann ellipsoids. The existence of pear-shaped figures with internal motions is proved (Kondratyev 1990). Some interesting results in calculation of sequences of nonellipsoidal figures are received by Eriguchi & Sugimoto (1981). However, the main results of the subsequent work of these authors (Eriguchi & Hachisu 1985), in which the dumbbell shaped figure with internal motions were calculated by numerical methods are in a root incorrect.

Clement (1967) has generalized the Roche model for triaxial figures with internal flows. The sequence of these non-rotating configurations presented ranges from a sphere to a figure with mild flattening. A further generalization of such model for a case with of inclined rotation is possible.

A new class of resonant spheroidal analytical models of stellar system with a nonellipsoidal phase distribution function has also nonlinear flows of centroids on set of tori (Kondratyev 1997).

3. General Requirements to Optimum Models of Galaxies

A model of an elliptic galaxy (or other model) can be optimum only in relation to set of available observational and theoretical data. It is completely with no prospect to construct such models of galaxies that would grasp and take into account immense data of supervision in all details. However it happens rather difficult to decide what to consider by essential. For this reason it is necessary to develop both analytical and numerical methods of constructing models of galaxies. It can be recommended to apply the tensor virial theorem to an ellipsoidal subsystem in a stratified nonhomogeneous ellipsoid. This approach is based on such characteristics, as a spatial form of a galaxy, a density distribution and structure of equidensity layers. Advantages of such approach are: a) an opportunity to build the models not solving a difficult problem of finding the phase distribution function; b) the mathematical instrument is developed rather deeply; c) the virial method allows beforehand to reject improper models. Lacks: a) the virial method is not self-consistent; b) bypasses of basic question on structure of star system in phase space.

The virial method should cooperate with both method of stellar hydrodynamic equations and the method of constructing phase models. Here there are certain successes: some phase models of spherical stellar systems (H. C. Plummer, I. King, L. P. Ossipkov, and others); the problem of constructing of homogeneous collisionless ellipsoids is studied (K. C. Freeman, B. P. Kondratyev); some phase models of axisymmetric and non-axisymmetric nonhomogeneous configurations (M. Schwarzschild, C. Hunter, T. de Zeeuw, D. Merritt, and others). Our hopes here are connected with cooperation of both analytical methods and numerical results on galaxy simulation.

4. Some Models and Methods

The method of ellipsoidal stratification. The density distribution in inhomogeneous ellipsoid depends only on one parameter $\rho(m)$, where m is implicitly determined by the equation

$$\frac{x_1^2}{a_1^2\alpha_1^2(m)} + \frac{x_2^2}{a_2^2\alpha_2^2(m)} + \frac{x_3^2}{a_3^2\alpha_3^2(m)} = m^2. \tag{1}$$

Generally, flattening of equidensity layers can change from layer to layer. The potentials for these stratified ellipsoids are known (Roberts 1963, Kondratyev 1982). For practical application it is important to consider an intermediate ellipsoidal subsystem. For such subsystem the virial tensor should consist of two members,

$$Z_{ij}(m) = W_{ij}(m) - 2\pi G\delta_{ij} I_{ii}(m) \int\limits_{m^2}^{1} dm^2 \rho(m^2) \frac{dA_i}{dm^2}, \tag{2}$$

and the gravitational energy tensor is known. The tensor virial theorem for the ellipsoidal subsystem allows effectively to describe an equilibrium of E-galaxies, and also bars of spiral galaxies. Application of the tensor virial theorem

$$\begin{aligned}
2T_{11} + W_{11} + \Pi_{11} + I_{11}\Omega^2 + 2\Omega \int\limits_V \rho u_2 x_1\, dV &= 0,\\
2T_{22} + W_{22} + \Pi_{22} + I_{22}\Omega^2 + 2\Omega \int\limits_V \rho u_1 x_2\, dV &= 0,\\
W_{33} + \Pi_{33} &= 0
\end{aligned} \tag{3}$$

to the configuration as whole gives us in inertial frame the rotation energy of configuration T_{rot}. Thus, we obtain the important dynamical parameter $t = T_{rot}/|W|$. In particular, the relation of rotational energy to gravitational energy of non-uniform ellipsoids, consisting from similar layers, in accuracy coincides with the same relation for classical homogeneous equilibrium figures and it is a function of only layer eccentricity only.

Up to 1975 we were sure that we correctly understand dynamics of elliptical galaxies. The profiles of surface brightness of round E-galaxies were excellent adjusted by the model of truncated isothermal sphere. It was believed also, that the flattening of E-galaxies is caused by their rotation, as it has a place for stars and planets. The first signal about abnormality of this picture was received when Bertola & Capaccioli (1976) had found that in NGC 4697 $(v_{rot})_{\max} \approx 85$ km s^{-1}, that is significant less than velocity dispersion $\sigma \approx 310$ km s^{-1}. The rotation has appeared obviously unsufficiently large to support average flattening for this galaxy (Binney 1978). For clearing up of this situation we simulated E-galaxy by the stratified non-uniform ellipsoid and we found for it from photometry data functions $\rho(m)$ and $\varepsilon(m)$. Then we calculated the relation $t_{is} = T_{rot}/|W|$ and, at last, the value $v_{rot}/\sigma = [t_{is}/(0.5 - t_{is})]^{1/2}$. This relation should be compared with the observed one, that will allow to conclude about the average value of anisotropy. The calculation of geometric projection effects gives the bottom limit of anisotropy. At determination of a role of anisotropy in E-galaxies it is necessary to take into account observed profiles of isophote flattening for each galaxy (Kondratyev 1981). However, till now some investigators neglect this factor of internal structure.

The calculation of internal structure allows to divide all E-galaxies into two classes: the class of axisymmetric galaxies of small luminosity, whose rotation is sufficient for support their flattening, and the class of large galaxies with boxy isophotes and, most likely, slow rotation and triaxial form.

The following consequences can be deduced from the virial equations:

1. For rotation around a short axis,
   $a_3 < a_1 \neq a_2$ at $W_{33} > W_{11} \neq W_{22}$;
2. For rotation around a middle axis,
   $a_2 < a_3 < a_1$ at $W_{22} > W_{33} > W_{11}$;
3. For rotation around a long axis (only for collisionless systems),
   $a_3 > a_1 \neq a_2$ at $W_{33} < W_{11} \neq W_{22}$.

Failure of the hypothesis of obligatory rotation of all elliptic galaxies around their small axes provides moreover an effective way for elimination the paradox of inconsistency between flattening and rotation (Kondratyev 1983). To illustrate this effect we shall carry out calculations for the simple model with similar layers. Flattenings of orthogonal sections we shall connect by the formula $\varepsilon_{23} = l\varepsilon_{13}$. In the interval $0 \leq l \leq 1$ the rotation occurs around a short axis; for $l = 1$ $(= 0)$

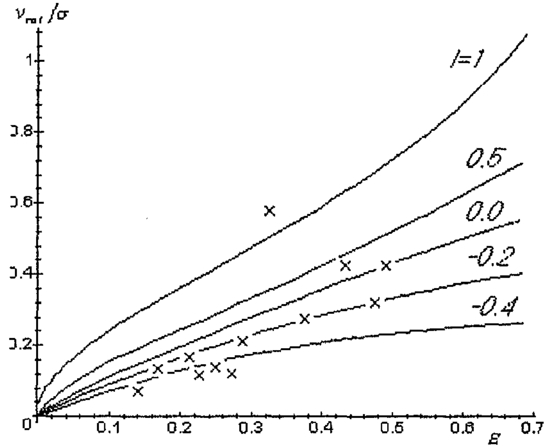

Figure 2.

a spheroid is oblate (prolate), and for any intermediate l it is triaxial. For $l < 0$ an ellipsoid rotates around of its middle axis. So, the rotation around a middle axis also can soften and even remove the paradox of slow rotation for some elliptic galaxies (Figure 2).

On angular momentum distribution in axisymmetric galaxies. Constructing axisymmetric gaseous or stellar models frequently was carried out by determining the angular momentum $L[M(R)]$ inside them as functions of mass contained inside a cylindrical surface with L given on it. Such approach is physically more evident than *a priori* representation an angular velocity distribution inside the configuration. This method is reasonable and from the evolutionary point of view. Crampin & Hoyle (1964) established from rotation curves for eight flat galaxies that the distribution of the specific angular momentum in them coincides with distribution of the specific momentum $l[M(R)]$ in the classical homogeneous Maclaurin spheroids by accuracy up to 1%. The formula for distribution of the angular momentum in Maclaurin spheroid is as follows:

$$L(R) = L_t \left\{ 1 - \frac{5}{2}\left(1 - \frac{M(R)}{M_t}\right) + \frac{3}{2}\left(1 - \frac{M(R)}{M_t}\right)^{5/3} \right\}. \quad (4)$$

Evolutionary aspects. Some features in dynamics of galaxies can be found on the known evolution of classical Maclaurin spheroids and Jacobi ellipsoids under condition of preservation of angular momentum.

At our study of axisymmetric oscillations under the Maclaurin spheroid an interesting effect was found: if the Maclaurin spheroid is oblated stronger, than $\epsilon = 0.9883$ ($e = 0.8475$), what energy we have not added to this configuration, it will not attain during oscillations even to a spherical form, all the more to the form of prolate spheroid. So whether have of a similar potential barrier real flat galaxies?

The opening of the triaxial Jacobi and Riemann ellipsoids has rendered a direct ideological influence on development of our recent representations about dynamics of galaxies. First, just on their example there was idea about an opportunity of existence triaxial E-galaxies, which can rotate "end over end" (Contopoulos 1956, Ogorodnikov 1958). Secondly, the certain meaning should be given to internal currents in triaxial ellipsoids. In absence of viscosity, the angular momentum and circulation should be kept during evolution,

$$L = \frac{M}{5}\Omega\left(a_1^2 + a_2^2\right)\left[1 + \frac{2a_1^2 a_2^2}{\left(a_1^2 + a_2^2\right)^2} f\right], \quad \Phi = \pi a_1 a_2 \Omega\,(2 + f)\,, \quad (5)$$

where $f = \zeta/\Omega$, and ζ is the vorticity of internal flows. The evolution of a configuration will be accompanied with changes of its equatorial flattening and of its value $f = \gamma(n)[C\gamma(n) - 2]/[\gamma(n) - 2C]$, where $n = a_2/a_1$, $\gamma(n) = n + 1/n$, and $C = M\Phi/(5\pi L) = \text{const}$. The calculation gives the following results (Figure 3). The character of evolution of relative internal flows (**RF**) is determined by direction of flows in inertial frame (**IF**). If **IF** have character of negative flows (with $C < 0$), during the evolution the intensity of **RF** can only increase. If **IF** (flows in inertial frame) are direct ones ($0 < C < 1$), during the evolution relative counterflows gradually weaken down to their disappearance and even changes its mark of an orientation. It is interesting, that at rather strong direct flows in inertial frame ($C > 1$) the relative counterflows **RF** at the beginning very quickly increase, suppress solid-state rotation of the figure, and then the figure again begins to rotate as whole and the direct flows are formed inside it already ($f > 0$).

Modeling internal dynamics of spheroidal galaxies (and gas configurations). For non-uniform configurations in weak rotation the following dependence between flattening e and a parameter of rotation m takes place $\varepsilon = (m/4)[(2n + 5)/(n + 1)]$. In a case $n = 0$ (homogeneity) the Newton formula follows from it, and in the other limiting case $n = \infty$ (all mass is at the centre) the Huygens formula $\varepsilon = m/2$.

The isotropic case. The equations in cylindrical coordinates are

$$-\frac{1}{\rho}\frac{\partial p}{\partial R} + \Omega^2 R + \frac{\partial \varphi}{\partial R} = 0, \qquad \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial \varphi}{\partial z} = 0. \qquad (6)$$

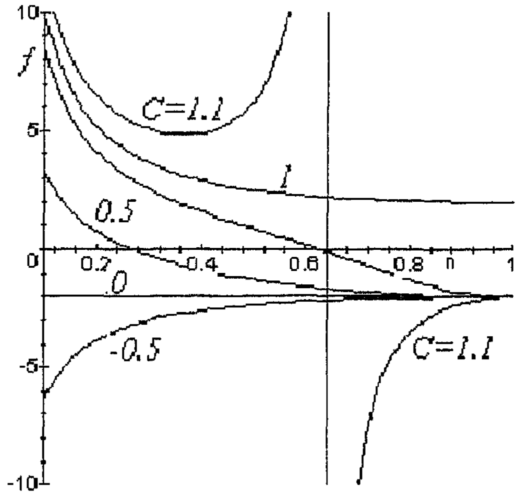

Figure 3.

Combining both equations, we find

$$\Omega^2 R = \frac{R}{z}\frac{c^2}{a^2}\frac{\partial\varphi}{\partial z} - \frac{\partial\varphi}{\partial R} \quad \text{(on boundary)},$$

$$\Omega^2 = \frac{1}{\rho R}\int\limits_{z+}^{z}\left(\frac{\partial\rho}{\partial R}\frac{\partial\varphi}{\partial z} - \frac{\partial\varphi}{\partial R}\frac{\partial\rho}{\partial z}\right) dz \quad \text{(inside spheroid)}.$$

The elementary case with anisotropy. Let $p_{zz} = P/k$, $P = p_R = p_\theta$. Then we have

$$\Omega^2 R = k\frac{R}{z}\frac{c^2}{a^2}\frac{\partial\varphi}{\partial z} - \frac{\partial\varphi}{\partial R} \quad \text{(on boundary)},$$

$$\Omega^2 = \frac{k-1}{R}\frac{\partial\varphi}{\partial R} - \frac{k}{\rho R}\int\limits_{z}^{z^+}\left(\frac{\partial\rho}{\partial R}\frac{\partial\varphi}{\partial z} - \frac{\partial\rho}{\partial z}\frac{\partial\varphi}{\partial R}\right) dz \quad \text{(inside spheroid)}.$$

The general case of variable anisotropy. Use the equations of stellar hydrodynamics

$$\frac{\partial\sigma_R^2}{\partial R} = \frac{\partial\varphi}{\partial R} + \frac{v_{rot}^2 + \sigma_\varphi^2 - \sigma_R^2}{R}, \qquad \frac{\partial}{\partial z}\left(\rho\sigma_z^2\right) = \rho\frac{\partial\varphi}{\partial z}. \tag{7}$$

It follows from the last equation that $\sigma_z^2 = \varphi - \bar{\varphi} + (\rho_s/\rho)(\bar{\varphi} - \varphi_s)$, where

$$\bar{\varphi} = (\rho_s - \rho)^{-1} \int_z^{z_s} \varphi(\partial\varphi/\partial z)\, dz.$$

Let here $k = \sigma_z^2/\sigma_R^2$, $l = \sigma_\varphi^2/\sigma_R^2 - 1$. Then

$$\begin{aligned}\Omega^2 = \quad & \frac{k}{\rho R}\frac{\partial}{\partial R}\left[\rho\left(\varphi - \bar{\varphi}\right) + \rho_s\left(\bar{\varphi} - \varphi_s\right)\right] - \\ & \frac{1}{R}\frac{\partial\varphi}{\partial R} - \frac{kl}{R^2}\left[\varphi - \bar{\varphi} + \frac{\rho_s}{\rho}\left(\bar{\varphi} - \varphi_s\right)\right].\end{aligned}$$

Calculations under the formula above give quite realistic results. In the first approximation and supposing constant anisotropy the equisurfaces of angular velocity were calculated. In galaxies we meet with the baroclinic rotation law. In the second approximation of the given method it is possible to determine an approximate profile of velocity dispersion from the formula for $k = \sigma_R^2/\sigma_z^2$

$$k(R) = \frac{R^l}{\rho\sigma_z^2}\left(\int_\xi^R \rho\frac{\frac{\partial\varphi}{\partial R} + \Omega^2 R}{R^l}\, dR + \mathrm{const}\right). \tag{8}$$

Here $\Omega^2(R, z)$ is known from the first approximation.

5. Other Possible Directions and Applications

It is rather important for theoretical and practical applications to construct triaxial equilibrium figures with nonlinear internal velocity field. Some attempts were done already made, but a complete theory is not developed yet.

The basic part of mass of spiral galaxies is in spheroidal subsystems. It assumes application of methods of the theory to such stellar systems in global aspect.

The study of equilibrium figures of galaxies deformed by an attraction from external bodies.

Use of ring equilibrium figures. There are galaxies even with several rings (NGC 7702), and also the galaxies with "chained" rings (NGC 4324 and NGC 4935).

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