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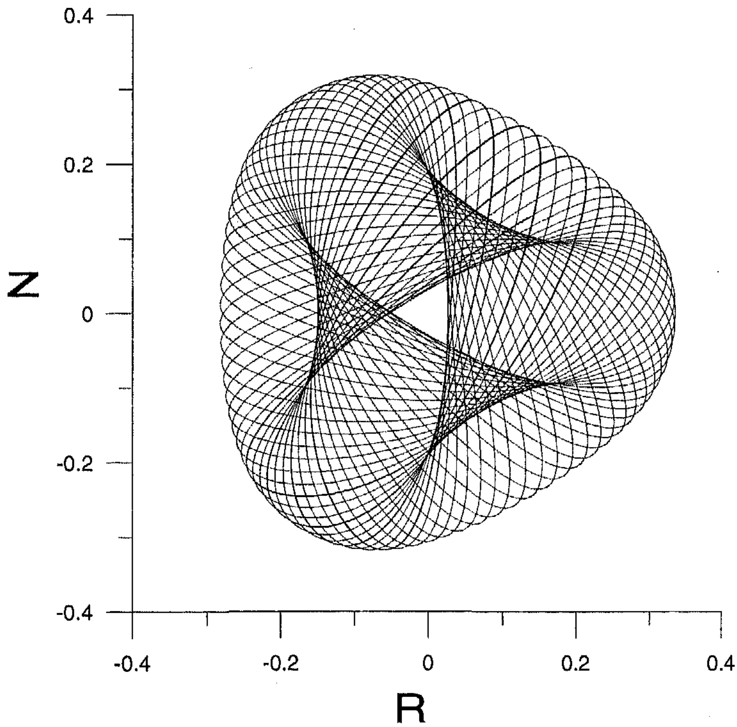
ORDER AND CHAOS IN STELLAR AND PLANETARY SYSTEMS



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Gravitational Potential of Material Wide Ring, Filled by Rosette Orbit

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Abstract. We consider the broad material ring or disk, which is a result of two-dimensional averaging of mass of moving body by rosette orbit. Such disk may be also consists of the identical elliptic orbits uniformly portioned on azimuth angle. The surface density and space Newtonian potential of the disk are obtained.

Even Gauss (see, e.g., Subbotin 1968) used the method of averaging of mass of moving body along its orbit to study of secular perturbations in celestial mechanics. As a result of this averaging, he obtained a one-dimensional material elliptical ring; the linear density of the disk is inversely proportional to velocity of the moving body in given point of the orbit.

Here we consider a broad material ring or disk, which has been obtained as a result of *two-dimensional averaging of mass of moving body by rosette orbit.* Such disk may be consists of the set of identical elliptic orbits uniformly distributed on azimuth angle.

Let come from the energy integral of the two body problem. If the central mass M is very big, the energy integral in the Keplerian problem is equal

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right). \tag{1}$$

The velocity squared of the second body has two components $v^2 = v_r^2 + v_\theta^2$, moreover the azimuth velocity component is found from the law of conservation of the angular moment

$$rv_\theta = \kappa\sqrt{a\left(1-e^2\right)} = \sqrt{GM}\sqrt{a\left(1-e^2\right)}. \tag{2}$$

Then, we find the radial velocity component

$$v_r = \frac{\kappa\sqrt{\left(r_a - r\right)\left(r - r_p\right)}}{r\sqrt{a}}, \quad \left\{\begin{array}{l} r_a = a\left(1+e\right), \\ r_p = a\left(1-e\right). \end{array}\right. \tag{3}$$

In the narrow ring at $(r, r+dr)$ testing body has the response time

$$dt \sim \text{const} \times \frac{r\,dr}{\sqrt{\left(r_a - r\right)\left(r - r_p\right)}}. \tag{4}$$

We shall get surface density function of the ring, if divide dt on $2\pi r\,dr$

$$\sigma\left(r\right) = \frac{\widetilde{C}}{\sqrt{\left(r_a - r\right)\left(r - r_p\right)}}. \tag{5}$$

The total mass of the ring is equal

$$M_{\text{ring}} = 2\pi\widetilde{C}\int\limits_{r_p}^{r_a}\frac{r\,dr}{\sqrt{(r_a-r)(r-r_p)}} = \pi^2\widetilde{C}\,(r_a+r_p)\,. \tag{6}$$

The relation allows us to find the constant $\widetilde{C}$.

By definition, the potential of ring with the surface density (5) is expressed by the double integral on its area

$$\varphi(r,x_3) = G\widetilde{C}\int\limits_{r_p}^{r_a}\frac{r'\,dr'}{\sqrt{(r_a-r')(r'-r_p)}}\int\limits_0^{2\pi}\frac{d\theta'}{\sqrt{r'^2+r^2+x_3^2-2r'x_1\cos\theta}}. \tag{7}$$

The internal integral here is equal

$$2\int\limits_0^{\pi}\frac{d\theta'}{\sqrt{r'^2+r^2+x_3^2-2r'r\cos\theta}} = \frac{4}{\sqrt{r'^2+r^2+x_3^2+2r'r}}K\left(\sqrt{\frac{4r'r}{(r'+r)^2+x_3^2}}\right), \tag{8}$$

where $K(\ldots)$ is the complete elliptic integral of first kind. Now,

$$\varphi(r,x_3) = 4G\widetilde{C}\int\limits_{r_p}^{r_a}\frac{r'\,dr'}{\sqrt{(r_a-r')(r'-r_p)}}\int\limits_0^{\pi/2}\frac{d\gamma}{\sqrt{(r'+r)^2+x_3^2-4r'r\sin^2\gamma}}. \tag{9}$$

Since the algebraic equation

$$r'^2+2r'r\cos 2\gamma+r^2+x_3^2=0 \tag{10}$$

has the complex roots

$$r' = -r\cos 2\gamma \pm i\sqrt{r^2\sin^2 2\gamma+x_3^2}, \tag{11}$$

it is possible to express (9) by the double integral

$$\varphi(r,x_3) = 4G\widetilde{C}\int\limits_0^{\pi/2}d\gamma\int\limits_{r_p}^{r_a}\frac{r'\,dr'}{\sqrt{(r_a-r')(r'-r_p)(r'-A-iB)(r'-A+iB)}}. \tag{12}$$

Here

$$A = -r\cos 2\gamma, \quad B = \sqrt{r^2\sin^2 2\gamma+x_3^2}. \tag{13}$$

By substitution

$$\tan^2\frac{\theta}{2} = \frac{\cos\theta_1}{\cos\theta_2}\frac{r_a-r_p}{r'-r_p}, \tag{14}$$

where

$$\tan\theta_1 = \frac{r_a - A}{B}, \quad \tan\theta_2 = \frac{r_p - A}{B}, \quad \tan\theta_1 \le \tan\theta_2, \quad \theta_1 \le \theta_2, \tag{15}$$

the internal integral in (12) can be reduced to the form

$$I = \mu \int\limits_{\pi}^{0} \frac{r'(\theta)\, d\theta}{\sqrt{1 - \tilde{k}^2 \sin^2\theta}} = \frac{\sqrt{\cos\theta_1 \cos\theta_2}}{B} \int\limits_{0}^{\pi} \frac{r'(\theta)\, d\theta}{\sqrt{1 - \tilde{k}^2 \sin^2\theta}}, \tag{16}$$

with

$$\tilde{k}^2 = \sin^2 \frac{\theta_1 - \theta_2}{2}. \tag{17}$$

Besides, from (14) we find r' as a function from variable θ

$$r'(\theta) = b - 2(r_a - r_p) \frac{\cos\theta_1 \cos\theta_2}{(\cos\theta_1 - \cos\theta_2)^2} \frac{1}{a + \cos\theta}, \tag{18}$$

with

$$a = \frac{\cos\theta_1 + \cos\theta_2}{\cos\theta_1 - \cos\theta_2} < -1, \quad b = \frac{r_a \cos\theta_1 - r_p \cos\theta_2}{\cos\theta_1 - \cos\theta_2}. \tag{19}$$

As far as

$$a^2 - 1 = \frac{4\cos\theta_1 \cos\theta_2}{(\cos\theta 1 - \cos\theta_2)^2}, \tag{20}$$

we have instead of (18)

$$r'(\theta) = b - \frac{(r_a - r_p)}{2} \frac{a^2 - 1}{a + \cos\theta}. \tag{21}$$

With $r'(\theta)$ from (21) the integral (16) reduces to the form

$$\begin{aligned} I &= \frac{\sqrt{\cos\theta_1 \cos\theta_2}}{B} \left\{ 2b \int\limits_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \tilde{k}^2 \sin^2\theta}} - \frac{r_a - r_p}{2} \left(a^2 - 1\right) \times \right. \\ &\quad \left. \left(\int\limits_0^{\pi/2} \frac{d\theta}{(a + \cos\theta)\sqrt{1 - \tilde{k}^2 \sin^2\theta}} + \int\limits_0^{\pi/2} \frac{d\theta}{(a - \cos\theta)\sqrt{1 - \tilde{k}^2 \sin^2\theta}} \right) \right\} \\ &= \frac{\sqrt{\cos\theta_1 \cos\theta_2}}{B} \left\{ 2b \int\limits_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \tilde{k}^2 \sin^2\theta}} - \right. \\ &\quad \left. (r_a - r_p)\left(a^2 - 1\right) a \int\limits_0^{\pi/2} \frac{d\theta}{(a^2 - \cos^2\theta)\sqrt{1 - \tilde{k}^2 \sin^2\theta}} \right\}. \end{aligned} \tag{22}$$

It is possible to express this I through both the complete elliptic integrals of first $K\left(\widetilde{k}\right)$ and third $\Pi\left(\frac{\pi}{2}, -\frac{1}{a^2-1}, \widetilde{k}\right)$ kind

$$I = \frac{\sqrt{\cos\theta_1 \cos\theta_2}}{B}\left\{2bK\left(\widetilde{k}\right) - (r_a - r_p)\, a\, \Pi\right\}. \tag{23}$$

Thus, the ring potential (12) takes the form

$$\varphi(r, x_3) = \frac{4GM}{\pi^2 (r_a + r_p)} \int\limits_0^{\pi/2} \frac{\sqrt{\cos\theta_1 \cos\theta_2}}{\sqrt{r^2 \sin^2 2\gamma + x_3^2}} \left\{2bK\left(\widetilde{k}\right) - a(r_a - r_p)\,\Pi\right\} d\gamma, \tag{24}$$

where a and b from (19). There is the useful relation between a and b

$$a + \nu = \frac{2b}{r_a - r_p}, \quad \nu = \frac{r_a + r_p}{r_a - r_p} > 1. \tag{25}$$

With account of (25) it is possible to exclude the value $2b$ from (24) and then

$$\varphi(r, x_3) = \frac{4GM}{\pi^2 \nu} \int\limits_0^{\pi/2} \frac{\sqrt{\cos\theta_1 \cos\theta_2}}{\sqrt{r^2 \sin^2 2\gamma + x_3^2}} \left\{a\left[K\left(\widetilde{k}\right) - \Pi\right] + \nu K\left(\widetilde{k}\right)\right\} d\gamma. \tag{26}$$

The expression (26) is equal to (24), but it more suitable for clearing up of some limiting cases.

Because of the relations (14) in general case the angles θ_1 and θ_2 depend on γ, this is valid for the function $\widetilde{k} = k(\gamma)$ also.

But if the sampling point is on the symmetry axis Ox_3, then $r = 0$ and

$$\begin{gathered}
A = 0, \quad B = x_3, \quad \tan\theta_1 = \frac{r_a}{x_3}, \quad \tan\theta_2 = \frac{r_p}{x_3}, \\
\cos\theta_1 = \frac{x_3}{\sqrt{r_a^2 + x_3^2}}, \quad \cos\theta_2 = \frac{x_3}{\sqrt{r_p^2 + x_3^2}}, \\
a = \frac{\sqrt{r_a^2 + x_3^2} + \sqrt{r_p^2 + x_3^2}}{\sqrt{r_a^2 + x_3^2} - \sqrt{r_p^2 + x_3^2}}, \quad \widetilde{k}^2 = \frac{1}{2}\left(1 - \frac{r_a r_p + x_3^2}{\sqrt{\left(r_a^2 + x_3^2\right)\left(r_p^2 + x_3^2\right)}}\right).
\end{gathered} \tag{27}$$

In this case the integrand function in (26) does not depend from γ at all, and we have

$$\varphi(x_3) = \frac{2GM}{\pi\nu} \frac{1}{\left[\left(r_a^2 + x_3^2\right)\left(r_p^2 + x_3^2\right)\right]^{1/4}} \left\{a\left[K\left(\widetilde{k}\right) - \Pi\right] + \nu K\left(\widetilde{k}\right)\right\}. \tag{28}$$

In particular, under large x_3, the asymptotic is:

$$\widetilde{k} \to 0, \quad |a| \to \infty, \quad K\left(\widetilde{k}\right) \to \frac{\pi}{2}, \quad \Pi \to \frac{\pi}{2}; \tag{29}$$

then, the member in the square brackets in (28) vanishes and we have the desired result

$$\varphi\left(x_3\right) \approx \frac{GM}{x_3}. \tag{30}$$

The attraction force of the disk on symmetry axis has a maximum at the value x_3/r_a.

References

Subbotin, M. F. 1968, Introduction to theoretical astronomy (Moscow: Nauka)