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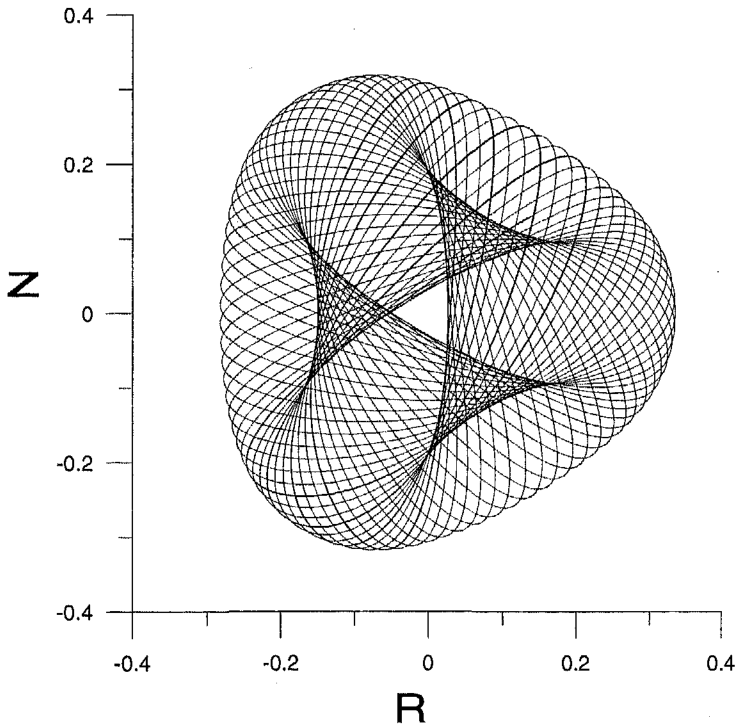
ORDER AND CHAOS IN STELLAR AND PLANETARY SYSTEMS



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On the Limiting Angular Velocity of the Rotation of the Stellar Systems

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Abstract. We proved: for any rotating stationary inhomogeneous gravitating stellar system of arbitrary form with continuous density distribution is always fulfilled the important inequality $\Omega^2 \leq 2\pi G\rho_{\max}$.

In 1900 H. Poincare (see, e.g., Poincare 2000) proved the existence of important limit of angular velocity for fluid gravitating figures

$$\Omega^2 < 2\pi G\,\bar{\rho}, \tag{1}$$

where $\bar{\rho}$ is the average density of configuration. Fulfilment of the inequality guarantees a direction of total gravity inside the rotating mass and nonnegativity of the hydrostatic pressure.

For stellar systems the similar inequality have been derived by K. F. Ogorodnikov (1958) and L. P. Osipkov (1999). They came from the virial theorem. In particular, the main Osipkov's inequality (20) includes three structured parameters and, in addition, flattening of model. Thereby, its result is in strong degrees model dependent. Here we consider this problem, leaning on equations of motion themselves a stars. Is it taken into account *possible anisotropy of velocity dispersion* in stellar systems.

We shall consider in the inertial cartesian system $Oxyz$ a configuration of N gravitating stars with mass m_i. The internal potential of the system is well known

$$\varphi(\mathbf{r}) = G\sum_j \frac{m_j}{|r_j - r|}. \tag{2}$$

Let us define the function

$$S = \sum_i m_i(x_i\dot{x}_i + y_i\dot{y}_i), \tag{3}$$

which represents a moving of the system particles in planes Oxy. For stationary configuration

$$\frac{dS}{dt} = \sum_i m_i\left[\dot{x}_i^2 + \dot{y}_i^2 + x_i\ddot{x}_i + y_i\ddot{y}_i\right] = 0 \tag{4}$$

or, with account of the motion equations of star,

$$\frac{dS}{dt} = \sum_i m_i\left[\dot{x}_i^2 + \dot{y}_i^2 + x_i\frac{\partial\varphi}{\partial x_i} + y_i\frac{\partial\varphi}{\partial y_i}\right] = 0. \tag{5}$$

Now we introduce the angular velocity of the rotation of the configuration Ω and reject peculiar chaotic star velocities; then from (5) one has the following inequality

$$\Omega^2 \sum_i m_i \left(x_i^2 + y_i^2\right) + \sum_i m_i \left[x_i \frac{\partial \varphi}{\partial x_i} + y_i \frac{\partial \varphi}{\partial y_i}\right] \leq 0. \tag{6}$$

In the left part of the formula (6) we transform the second sum, using the expression (2) for the potential. Obviously, the force components on unit of the mass in testing point r_i are

$$\begin{aligned} \frac{\partial \varphi}{\partial x_i} &= -G \sum_j \frac{m_j (x_i - x_j)}{|r_j - r_i|^3}, \\ \frac{\partial \varphi}{\partial y_i} &= -G \sum_j \frac{m_j (y_i - y_j)}{|r_j - r_i|^3}, \end{aligned} \tag{7}$$

so

$$\sum_{i=1}^{N} m_i \left[x_i \frac{\partial \varphi}{\partial x_i} + y_i \frac{\partial \varphi}{\partial y_i}\right] = -G \sum_i \sum_j \frac{m_i m_j \left[x_i \left(x_i - x_j\right) + y_i \left(y_i - y_j\right)\right]}{|r_i - r_j|^3}. \tag{8}$$

By virtue of symmetry, the indexes i and j here we may interchange

$$\sum_{i=1}^{N} m_i \left[x_i \frac{\partial \varphi}{\partial x_i} + y_i \frac{\partial \varphi}{\partial y_i}\right] = -G \sum_j \sum_i \frac{m_i m_j \left[x_j \left(x_j - x_i\right) + y_j \left(y_j - y_i\right)\right]}{|r_i - r_j|^3} \tag{9}$$

and take the half-sum of the expressions (8) and (9). Then

$$\sum_{i=1}^{N} m_i \left[x_i \frac{\partial \varphi}{\partial x_i} + y_i \frac{\partial \varphi}{\partial y_i}\right] = -\frac{1}{2} G \sum_i \sum_j \frac{m_i m_j \left[\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2\right]}{|r_i - r_j|^3}. \tag{10}$$

As a result, we shall get the following inequality

$$\Omega^2 I \leq \frac{G}{2} \sum_i \sum_j \frac{m_i m_j \left[\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2\right]}{|r_i - r_j|^3}, \tag{11}$$

where I is the inertia moment of the system with respect to rotational axis.

Before this we had discrete masses. Under continuous distribution of the matter in stellar configuration instead of (11) we have

$$\Omega^2 I \leq \frac{G}{2} \int\limits_{V_1} \int\limits_{V} \frac{\rho \rho_1 \left[(x - x_1)^2 + (y - y_1)^2\right] dx\, dy\, dz\, dx_1\, dy_1\, dz_1}{\left[(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2\right]^{3/2}}. \tag{12}$$

Write here the notations $R_1 = \sqrt{x_1^2 + y_1^2}$, $R = \sqrt{x^2 + y^2}$ and take into consideration, that R_1 always less R—then each pair of material points is taken into account only once, consequently,

$$
\begin{gathered}
\Omega^2 I \le G \int \int_{R_1<R} \frac{\rho\rho_1[(x-x_1)^2+(y-y_1)^2]}{[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2]^{3/2}}\, dV\, dV_1 < \\
2G\rho_{\max} \int_{-\infty}^{+\infty} \left(\int \frac{\rho[(x-x_1)^2+(y-y_1)^2]}{[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2]^{3/2}}\, dV\, dx_1\, dy_1 \right) dz_1 = \\
2\pi G\rho_{\max} \int \rho R^2\, dV\, dx_1\, dy_1 = 2\pi G\rho_{\max} I.
\end{gathered} \tag{13}
$$

Thus, for any rotating stationary inhomogeneous gravitating stellar system with arbitrary figure is always fulfilled the important inequality

$$\Omega^2 \le 2\pi G\rho_{\max} \tag{14}$$

($\rho_{\max}$ is density maximum), which extends the known Poincare's inequality (1) on equilibrium figures with possible anisotropy of velocity dispersion.

The Poincare's inequality (1) is realized only for figures with isotropic pressure and in right its part is average density. In the inequality (14) enters namely maximal density $\rho_{\max}$.

If system has an axial symmetry and differential rotation, it be possible to define some average angular velocity through the angular moment L

$$\bar{\Omega} = \frac{L}{I}, \tag{15}$$

and for it the similar criterion is

$$\bar{\Omega}^2 \le 2\pi G\rho_{\max}. \tag{16}$$

This inequality is valid for stellar systems with continuous density distribution.

As well known, for figures with isotropic pressure the Poincare's inequality was improved by Crudeli $\bar{\Omega}^2 < \pi G\bar{\rho}$. However, for stellar systems because of velocity dispersion anisotropy the Crudeli argument do not pass; so the inequality (14) it is impossible to improve, rejecting factor 2 in right its part. For instance, for the uniform collisionless Freeman ellipsoid (Kondratyev 2003), which has at most fast rotation $\Omega^2 = 2A_1$, in prolate spheroid limit will be fulfilled the relation $A_1 = \pi G\rho$, and here we have just that case of the formula (14) $\Omega^2 = 2\pi G\rho$, which fall outside Crudeli limit, but does not break the limit (14).

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