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CONTROL THEORY

Lyapunov Reducibility and Stabilization of Nonstationary Systems with an Observer

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Abstract—For a linear nonstationary control system with an observer, we assume that the coefficients are locally Lebesgue integrable and integrally bounded on $\mathbb{R}$ and construct a linear feedback such that the closed-loop plant–controller system is Lyapunov reducible to the special triangular form corresponding to an independent shift of the diagonal coefficients in the original system and in the system of asymptotic estimation of the state by an arbitrary pregiven quantity. For a periodic system, we prove that the constructed controls and Lyapunov transformation are periodic. We obtain corollaries on the uniform stabilization and global controllability of the central and singular exponents of the system.

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Let $\mathbb{R}^n$ be the Euclidean space of dimension n with norm $|x| = \sqrt{x^*x}$, let $*$ stand for the operation of transposition; and let M_{mn} be the space of real $m \times n$ matrices with norm

$$|A| = \max_{|x|\le 1} |Ax|.$$

Consider the linear nonstationary control system with an observer,

$$\dot{x} = A(t)x + B(t)u, \qquad (t,x,u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m, \tag{1}$$

$$y = C^*(t)x, \qquad y \in \mathbb{R}^k. \tag{2}$$

We assume that the norms of the matrix functions A, B, and C are locally Lebesgue integrable and integrally bounded [1, p. 252] in the norm for A and in the squared norm for B and C; i.e.,

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} |A(s)|\,ds < \infty, \qquad \sup_{t\in\mathbb{R}} \int_t^{t+1} |B(s)|^2\,ds < \infty, \qquad \sup_{t\in\mathbb{R}} \int_t^{t+1} |C(s)|^2\,ds < \infty. \tag{3}$$

Consider the problem of constructing a stabilizing feedback for system (1), (2). For the stationary system (1), (2), this problem can be solved by constructing a system of asymptotic estimation of the state [2, Th. 7.7]. Here we obtain a similar result for nonstationary systems. On the basis of system (1), (2) and the output y, we construct the system

$$\dot{\widehat{x}} = A(t)\widehat{x} + V(t)(y(t) - C^*(t)\widehat{x}) + B(t)u, \qquad \widehat{x} \in \mathbb{R}^n. \tag{4}$$

Here $\widehat{x}(t)$ is an estimate of the state of system (1), (2). Let the control law u have the form

$$u = U(t)\widehat{x}. \tag{5}$$

By substituting the control (5) into system (1), (2), (4), we obtain the closed-loop $2n$-dimensional system

$$\begin{pmatrix} \dot{x} \\ \dot{\widehat{x}} \end{pmatrix} = \begin{pmatrix} A(t) & B(t)U(t) \\ V(t)C^*(t) & A(t) + B(t)U(t) - V(t)C^*(t) \end{pmatrix} \begin{pmatrix} x \\ \widehat{x} \end{pmatrix}. \tag{6}$$

Let $\widetilde{x} = x - \widehat{x}$. By using the nonsingular change of variables $\begin{pmatrix} x \\ \widetilde{x} \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} x \\ \widehat{x} \end{pmatrix}$, where $I \in M_{nn}$ is the identity matrix, we reduce system (6) to the form

$$\begin{pmatrix} \dot{x} \\ \dot{\widetilde{x}} \end{pmatrix} = \begin{pmatrix} A(t) + B(t)U(t) & -B(t)U(t) \\ 0 & A(t) - V(t)C^*(t) \end{pmatrix} \begin{pmatrix} x \\ \widetilde{x} \end{pmatrix}. \tag{7}$$

If system (1), (2) is stationary and has the property of total controllability and total observability, then one can choose constant controls U and V such that the characteristic polynomials $\chi(A+BU)$ and $\chi(A - VC^*)$ are stable [2, Th. 7.7]. This implies the stabilization of system (7) and hence of the closed-loop system (6).

In the present paper, we construct a stabilizing control for the nonstationary system (1), (2), in particular, for a system with periodic coefficients. In systems (7) and (6), the matrices $U(\cdot)$ and $V(\cdot)$ play the role of controls. We treat an admissible control as an arbitrary function $U(\cdot)$ [or $V(\cdot)$] with values in M_{mn} (respectively, in M_{nk}) such that its squared norm is a function locally Lebesgue integrable and integrally bounded on $\mathbb{R}$; i.e., $U(\cdot)$ is an admissible control if $|U| \in L_2^{\mathrm{loc}}(\mathbb{R})$ and $\sup_{t\in\mathbb{R}} \int_t^{t+1} |U(s)|^2\, ds < \infty$.

Let $X(t, s)$ be the Cauchy matrix of the system

$$\dot{x} = A(t)x, \tag{8}$$

that is, the solution of the Cauchy matrix problem $\dot{X} = A(t)X$, $X(s) = I$. One can readily show (with the use of the Gronwall–Bellman lemma and the Ostrogradsky–Liouville formula) that the first condition in (3) implies the following property of the Cauchy matrix: for any $\varkappa > 0$, there exist numbers $c_1, c_2 > 0$ such that the inequalities

$$|X(t, s)| \leq c_1, \qquad \det X(t, s) \geq c_2 > 0 \tag{9}$$

hold for all $\tau \in \mathbb{R}$ and for arbitrary $t, s \in [\tau, \tau + \varkappa]$.

Let us construct the Kalman matrix $W(t, \tau) = \int_\tau^t X(\tau, s)B(s)B^*(s)X^*(\tau, s)\, ds$ of system (1). The matrix $W(t, \tau)$ is symmetric and nonnegative definite for any $t \geq \tau$ and satisfies the inequality $W(t_1, \tau) \geq W(t, \tau)$ in the sense of quadratic forms for any $t_1 \geq t$.

Recall [3] that system (1) is said to be (a) *completely controllable* (CC) on the interval $[t_0, t_0 + \vartheta]$ if the Kalman matrix $W(t_0 + \vartheta, t_0)$ is positive definite; (b) *ϑ-uniformly completely controllable* (ϑ-UCC) if there exists an $\alpha > 0$ such that $W(t + \vartheta, t) \geq \alpha I$ in the sense of quadratic forms for all $t \in \mathbb{R}$; (c) *uniformly completely controllable* (UCC) if there exists a $\vartheta > 0$ such that system (1) is ϑ-UCC.

It follows from the second condition in (3) and the property (9) that if system (1) is ϑ-UCC, then there exist $\alpha, \delta > 0$ (depending on ϑ) such that the inequality

$$0 < \alpha I \leq W(t + \vartheta, t) \leq \delta I \tag{10}$$

holds for all $t \in \mathbb{R}$.

Next, consider the property of *complete observability* [4, p. 304] of system (1), (2) with zero input signal; i.e.,

$$\dot{x} = A(t)x, \qquad y = C^*(t)x \tag{11}$$

for $u = 0$. By virtue of the duality principle [4, p. 304], system (11) is *ϑ-uniformly completely observable* (ϑ-UCO) if and only if the system

$$\dot{x} = -A^*(t)x + C(t)v, \qquad (t, x, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \tag{12}$$

is ϑ-UCC, and system (11) is *uniformly completely observable* (UCO) if and only if system (12) is UCC.

If system (1) is ϑ-UCC, then it is ϑ_1-UCC for any $\vartheta_1 \geq \vartheta$. Therefore, if system (1) is UCC and system (11) is UCO, then, without loss of generality, one can assume that there exists a common value $\vartheta > 0$ such that system (1) is ϑ-UCC and system (11) is ϑ-UCO.

Consider the system

$$\dot{x} = (A(t) + B(t)U(t))x, \tag{13}$$

whose matrix is a diagonal block of the matrix of system (7). By $X_U(t,s)$ we denote the Cauchy matrix of system (13). We say that a λ-*transformation* can be applied to system (1) if there exists a $T > 0$ such that, for each $\lambda \in \mathbb{R}$, one can indicate an admissible control $U(\cdot)$ ensuring that the Cauchy matrix $X_U(t,s)$ of system (13) satisfies the relation

$$X_U((k+1)T, kT) = e^{\lambda T} X((k+1)T, kT) \tag{14}$$

for all $k \in \mathbb{Z}$.

This definition was given in [5]. It is related to the definition of a λ-transformation of system (8) (see [1, p. 249]), which adds a perturbation λI to the matrix $A(t)$ of system (8). The Cauchy matrix $Z(t,s)$ of the perturbed system

$$\dot{z} = (A(t) + \lambda I)z \tag{15}$$

satisfies the relation

$$Z(t,s) = e^{\lambda(t-s)} X(t,s) \tag{16}$$

for all $t, s \in \mathbb{R}$. It follows from (14) that the Cauchy matrix of system (13) satisfies the relation $X_U(kT, lT) = e^{\lambda(k-l)T} X(kT, lT)$ for all $k, l \in \mathbb{Z}$; this relation is similar to (16) but holds on the set of points $\{kT,\ k \in \mathbb{Z}\} \subset \mathbb{R}$. It is also known [1, p. 249] that if $\lambda_1 \le \cdots \le \lambda_n$ is the complete spectrum of Lyapunov exponents of system (8), then $\lambda_1 + \lambda \le \cdots \le \lambda_n + \lambda$ is the complete spectrum of Lyapunov exponents of system (15).

A transformation $x = L(t)z$ of system (8) is called a *Lyapunov transformation* [1, p. 247] if $L : \mathbb{R} \to M_{nn}$ is an absolutely continuous function such that $|L(t)|$ and $|L^{-1}(t)|$ are bounded and $|\dot{L}(t)|$ is integrally bounded on $\mathbb{R}$. System (8) is said to be *reducible to the system* $\dot{z} = Q(t)z$ if there exists a Lyapunov transformation $x = L(t)z$ relating these systems, i.e., if the relation

$$Q(t) = L^{-1}(t)A(t)L(t) - L^{-1}(t)\dot{L}(t)$$

holds for almost all $t \in \mathbb{R}$. The systems with matrices $A(t)$ and $Q(t)$ are said to be *asymptotically equivalent*. Obviously, if the matrix $A(t)$ is integrally bounded, then so is the matrix $Q(t)$, and vice versa. System (8) is said to be *reducible* if it can be reduced to a system with a constant matrix.

Theorem 1. *If system* (1) *is UCC, then a λ-transformation can be applied to it.*

Proof. We use the idea of proof of a similar theorem in [5]. Let system (1) be ϑ-UCC. Let us fix an arbitrary $\lambda \in \mathbb{R}$. We shall construct an admissible control $U : \mathbb{R} \to M_{mn}$ such that relation (14) holds for $T = 2\vartheta$. For an arbitrary $k \in \mathbb{Z}$, on the interval $\Delta_k = [kT, (k+1)T]$, we construct a function $U_1(t)$, $t \in \Delta_k$, solving the matrix control problem

$$\dot{Y} = A(t)Y + B(t)U_1(t), \tag{17}$$

$$Y(kT) = I, \qquad Y((k+1)T) = e^{\lambda T} X((k+1)T, kT). \tag{18}$$

The solution of Eq. (17) with the initial condition $Y(kT) = I$ has the form

$$Y(t) = X(t, kT)\left(I + \int\limits_{kT}^{t} X(kT, s)B(s)U_1(s)\,ds\right). \tag{19}$$

The second condition in (18) is satisfied if and only if

$$I + \int\limits_{kT}^{(k+1)T} X(kT, s)B(s)U_1(s)\,ds = e^{\lambda T} I. \tag{20}$$

Suppose that there exist numbers $\beta_1 > 0$ and $\beta_2 > 0$ independent of $k \in \mathbb{Z}$ such that the control $U_1(t)$, $t \in \Delta_k$, ensuring the validity of relation (20) satisfies the inequality

$$\|U_1\|^2_{L_2(\Delta_k)} := \int\limits_{kT}^{(k+1)T} |U_1(s)|^2\, ds \le \beta_1,$$

and the solution $Y(t)$ of problem (17), (18) corresponding to this control and given by (19) satisfies the inequality $\det Y(t) \ge \beta_2$ for all $t \in \Delta_k$. Then $|Y^{-1}(t)|$ is bounded on Δ_k uniformly with respect to $k \in \mathbb{Z}$. Set $U(t) := U_1(t)Y^{-1}(t)$, $t \in \Delta_k$. Then $|U| \in L_2(\Delta_k)$ and $\sup_{k\in\mathbb{Z}} \|U\|_{L_2(\Delta_k)} < \infty$ [i.e., $U(t)$, $t \in \mathbb{R}$, is an admissible control], and $Y(\cdot)$ satisfies the equation

$$\dot{Y} = (A(t) + B(t)U(t))Y \tag{21}$$

and the boundary conditions (18). It follows from relation (21) and the first boundary condition that $Y(t) = X_U(t, kT)$, and the second boundary condition implies relation (14).

By virtue of property (9), there exist numbers $\beta_3, \beta_4 > 0$ such that the inequalities

$$0 < \beta_3 \le \det X(t, kT) \le \beta_4$$

hold for all $k \in \mathbb{Z}$ and for arbitrary $t \in \Delta_k$. Therefore, the problem under consideration can be reduced to finding a matrix function $U_1(\cdot)$ satisfying the conditions

$$|U_1| \in L_2(\Delta_k), \qquad \sup_{k\in\mathbb{Z}} \|U_1\|_{L_2(\Delta_k)} < \infty$$

and such that relation (20) holds and the condition

$$\det\left(I + \int\limits_{kT}^{t} X(kT, s)B(s)U_1(s)\, ds\right) \ge \gamma$$

is satisfied for all $t \in \Delta_k$ and for some number $\gamma > 0$ independent of $k \in \mathbb{Z}$ and t. Hence we have the inequality $\det Y(t) \ge \beta_2$.

We seek $U_1(t)$ in the form $U_1(t) = B^*(t)X^*(kT, t)H$. Then it follows from (20) that

$$I + W((k+1)T, kT)H = e^{\lambda T} I,$$

where $W(t, \tau)$ is the Kalman matrix. System (1) is ϑ-UCC; consequently, it is T-UCC. Therefore, the matrix $W((k+1)T, kT)$ is invertible, and

$$H = (e^{\lambda T} - 1)W^{-1}((k+1)T, kT).$$

Let us show that the matrix

$$R(t) = I + \int\limits_{kT}^{t} X(kT, s)B(s)B^*(s)X^*(kT, s)\, ds\, (e^{\lambda T} - 1)W^{-1}((k+1)T, kT)$$

is invertible for all $t \in \Delta_k$ and $\det R(t) \ge \gamma > 0$ for all $t \in \Delta_k$, where γ is independent of $k \in \mathbb{Z}$. Consider the matrix

$$Q(t) := R(t)W((k+1)T, kT) = W((k+1)T, kT) + (e^{\lambda T} - 1)W(t, kT).$$

By virtue of the property (10), $0 < \gamma_1 \le \det W((k+1)T, kT) \le \gamma_2$, where γ_1 and γ_2 are independent of $k \in \mathbb{Z}$. Therefore, it suffices to show that $\det Q(t) \ge \gamma_3 > 0$ for all $t \in \Delta_k$, where γ_3 is independent of $k \in \mathbb{Z}$. We have

$$\begin{aligned}
Q(t) &= \int_{kT}^{(k+1)T} X(kT,s)B(s)B^*(s)X^*(kT,s)\,ds - \int_{kT}^{t} X(kT,s)B(s)B^*(s)X^*(kT,s)\,ds \\
&\quad + e^{\lambda T}\int_{kT}^{t} X(kT,s)B(s)B^*(s)X^*(kT,s)\,ds \\
&= \int_{t}^{(k+1)T} X(kT,s)B(s)B^*(s)X^*(kT,s)\,ds + e^{\lambda T}\int_{kT}^{t} X(kT,s)B(s)B^*(s)X^*(kT,s)\,ds \\
&= X(kT,t)\int_{t}^{(k+1)T} X(t,s)B(s)B^*(s)X^*(t,s)\,ds\,X^*(kT,t) \\
&\quad + e^{\lambda T}\int_{kT}^{t} X(kT,s)B(s)B^*(s)X^*(kT,s)\,ds \\
&= X(kT,t)W((k+1)T,t)X^*(kT,t) + e^{\lambda T}W(t,kT).
\end{aligned}$$

The inequality $W((k+1)T,t) \geq W(t+\vartheta,t) \geq \alpha I$ holds for all $t \in [kT, kT+\vartheta]$ by virtue of the ϑ-uniform complete controllability. This, together with property (9), implies the existence of an $\alpha_1 > 0$ (independent of k) such that $X(kT,t)W((k+1)T,t)X^*(kT,t) \geq \alpha_1 I$. The matrix $e^{\lambda T}W(t,kT)$ is positive semidefinite; consequently, $Q(t) \geq \alpha_1 I > 0$, $t \in [kT, kT+\vartheta]$.

If $t \in [kT+\vartheta, (k+1)T]$, then $W(t,kT) \geq W(kT+\vartheta,kT) \geq \alpha I$. Hence it follows that $e^{\lambda T}W(t,kT) \geq \alpha_2 I > 0$, where $\alpha_2 = \alpha e^{\lambda T}$. The matrix $X(kT,t)W((k+1)T,t)X^*(kT,t)$ is positive semidefinite; consequently, $Q(t) \geq \alpha_2 I > 0$, $t \in [kT+\vartheta, (k+1)T]$. Therefore, the inequality $Q(t) \geq \alpha_3 I > 0$ holds for all $t \in \Delta_k$, where $\alpha_3 = \min\{\alpha_1, \alpha_2\}$ is independent of $k \in \mathbb{Z}$. Consequently, $\det Q(t) \geq \gamma_3 > 0$, $t \in \Delta_k$, where γ_3 is independent of $k \in \mathbb{Z}$.

Let us show that the constructed control U_1 satisfies the desired conditions. We have

$$U_1(t) = B^*(t)X^*(kT,t)W^{-1}((k+1)T,kT)(e^{\lambda T}-1), \qquad t \in \Delta_k.$$

Since system (1) is ϑ-UCC, it follows that $W^{-1}((k+1)T,kT)$ is bounded in norm uniformly for all $k \in \mathbb{Z}$. By virtue of property (9), the matrix $X^*(kT,t)$, $t \in \Delta_k$, is bounded in norm uniformly for all $k \in \mathbb{Z}$. By virtue of the second condition in (3), the function $B^*(\cdot)$ also satisfies the second condition in (3). Therefore, $|U_1| \in L_2(\Delta_k)$ and $\sup_{k\in\mathbb{Z}} \|U_1\|_{L_2(\Delta_k)} < \infty$. The proof of the theorem is complete.

Remark 1. If a λ-transformation can be applied to system (1), then, for any $\lambda \in \mathbb{R}$, there exists an admissible control $U(\cdot)$ ensuring relation (14) for all $k \in \mathbb{Z}$. Then the Cauchy matrix $X_U(t,s)$ of system (13) and the Cauchy matrix $Z(t,s)$ of system (15) coincide for all $t, s \in \{kT,\ k \in \mathbb{Z}\}$. This (see [6]) implies the asymptotic equivalence of systems (13) and (15).

Corollary 1. *Let system* (1) *be UCC. Then for each* $\lambda \in \mathbb{R}$, *there exists an admissible control* $U(\cdot)$ *for which system* (13) *can be reduced to system* (15).

Remark 2. Theorem 1 was stated and proved in [5] under the assumption that $A(\cdot)$ is a bounded continuous function and $B(\cdot)$ is bounded and uniformly continuous; admissible controls were chosen to be bounded piecewise continuous functions. The proof of Theorem 1 given in [5] can readily be generalized to the case in which the function $A(\cdot)$ is bounded and piecewise continuous and the function $B(\cdot)$ is bounded and piecewise uniformly continuous. Here we present a simpler proof of this theorem under the less restrictive conditions (3); the class of admissible controls is wider here. In the case of a matrix $B(\cdot)$ essentially bounded on $\mathbb{R}$, one can substantially restrict the set of admissible controls: a control $U(\cdot)$ providing the validity of relation (14) can be chosen to be measurable and bounded on $\mathbb{R}$. A number of corollaries on the global controllability of central and singular exponents [1, p. 116] were obtained in [5]; these results supplement Tonkov's earlier-obtained results [7] on the stabilization of system (13). The proofs of these corollaries were carried out on the basis of the following fact. The complete spectrum of Lyapunov exponents and central and singular exponents are Lyapunov invariants. A Lyapunov transformation does not change these characteristics. Under the passage from system (8) to system (15), these characteristics are shifted by λ. Therefore, if system (1) is UCC, then, by using admissible controls, one can make Lyapunov invariants of system (13) coinciding with Lyapunov invariants of system (15). In particular, one can globally control the upper or lower central exponent or the singular exponent of system (13); one can globally shift the complete spectrum of Lyapunov exponents of system (13) and stabilize system (13).

Remark 3. In [8] it was assumed that the function $A(\cdot)$ is bounded and piecewise continuous and the function $B(\cdot)$ is bounded and piecewise uniformly continuous, and it was shown that if system (1) is UCC, then system (13) can be globally scalarized [8]. From this, corollaries on the global controllability of central, singular, and exponential exponents and the complete spectrum of Lyapunov exponents were obtained.

Theorem 2. *Let $A(\cdot)$ and $B(\cdot)$ be ω-periodic functions, and let system* (1) *be completely controllable. Then there exists a $T > 0$ that is a multiple of ω and ensures that, for each $\lambda \in \mathbb{R}$, there exists a T-periodic admissible control $U(\cdot)$ such that the Cauchy matrix $X_U(t,s)$ of the T-periodic system* (13) *satisfies relation* (14) *for all $k \in \mathbb{Z}$. In addition, there exists a T-periodic Lyapunov transformation $x = L(t)z$ reducing the T-periodic system* (13) *to the T-periodic system* (15).

Proof. If the ω-periodic system (1) is completely controllable, then it is completely controllable on any interval of length $n\omega$. (However, one cannot claim [2, formula (2.26)] that it is completely controllable on any interval of length equal to the period [7].) Set $\vartheta = n\omega$. Then system (1) is ϑ-periodic and ϑ-UCC. We set $T = 2\vartheta$ and construct a control $U_1(t)$ just as in Theorem 2. For arbitrary $k \in \mathbb{Z}$ and $t \in [0,T]$, we have

$$X(t+kT, kT) = X(t,0), \qquad W(t+kT,kT) = W(t,0).$$

Therefore, the control $U_1(t)$ is T-periodic. Let the function $Y_k(t)$, $t \in \Delta_k$, be given by relation (19). Then $Y_{k+1}(t+T) = Y_k(t)$ for arbitrary $k \in \mathbb{Z}$ and $t \in \Delta_k$.

Indeed, let $t \in \Delta_k$, then $t+T \in \Delta_{k+1}$. We have

$$\begin{aligned} Y_{k+1}(t+T) &= X(t+T,(k+1)T)\left(I + \int\limits_{(k+1)T}^{t+T} X((k+1)T,s)B(s)U_1(s)\,ds\right) = |s = \tau + T| \\ &= X(t+T,kT+T)\left(I + \int\limits_{kT}^{t} X(kT+T,\tau+T)B(\tau+T)U_1(\tau+T)\,d\tau\right) \\ &= X(t,kT)\left(I + \int\limits_{kT}^{t} X(kT,\tau)B(\tau)U_1(\tau)\,d\tau\right) = Y_k(t). \end{aligned}$$

Consequently, the control $U(t)$ defined by the relation $U(t) = U_1(t)Y_k^{-1}(t)$ for $t \in [kT,(k+1)T)$ is also T-periodic. Therefore, system (13) is T-periodic, and relation (14) holds. The transformation $x = L(t)z$, where $L(t) = X_U(t,0)Z(0,t)$, is a Lyapunov transformation. It reduces system (13) to system (15). The relation $X_U(T,0) = Z(T,0)$ holds by virtue of (14), and since systems (13) and (15) are T-periodic, we have $X_U(t+T,T) = X_U(t,0)$ and $Z(t+T,T) = Z(t,0)$ for all $t \in \mathbb{R}$. Then

$$\begin{aligned} L(t+T) &= X_U(t+T,0)Z(0,t+T) = X_U(t+T,T)X_U(T,0)Z(0,T)Z(T,t+T) \\ &= X_U(t+T,T)Z(T,t+T) = X_U(t,0)Z(0,t) = L(t) \end{aligned}$$

for any $t \in \mathbb{R}$. Therefore, the Lyapunov transformation is T-periodic. The proof of the theorem is complete.

Any T-periodic system can be reduced by a real T_1-periodic Lyapunov transformation ($T_1 = 2T$) to a system with a constant matrix. In this case, under the assumptions of Theorem 2, one can globally shift the spectrum of Lyapunov exponents of system (13). Therefore, Theorem 2 implies the following assertion.

Corollary 2. *Let $A(\cdot)$ and $B(\cdot)$ be ω-periodic functions, and let system* (1) *be completely controllable. Then there exists a $T_1 > 0$ (equal to $4n\omega$) such that, for any $\varkappa > 0$, one can indicate a T_1-periodic admissible control $U(\cdot)$ and a T_1-periodic Lyapunov transformation $x = L_1(t)w$ reducing the T_1-periodic system* (13) *to the system $\dot{w} = Qw$ with a constant matrix such that $\operatorname{Re} q_i \le -\varkappa < 0$ for all eigenvalues q_i of the matrix Q.*

Remark 4. If the coefficients of system (1) are ω-periodic bounded piecewise continuous functions, then we have the following assertion [9] stronger than Theorem 2: the condition of complete controllability of system (1) is a necessary and sufficient condition for the global Lyapunov reducibility of system (13), that is, reducibility to an arbitrary system $\dot{z} = Q(t)z$ with bounded piecewise continuous matrix rather than only a system of the form (15).

Now consider the system

$$\dot{\widetilde{x}} = (A(t) - V(t)C^*(t))\widetilde{x}. \tag{22}$$

By $\widetilde{X}_V(t,s)$ we denote the Cauchy matrix of system (22) with an admissible control $V(\cdot)$, and by $\widetilde{X}(t,s)$ we denote the Cauchy matrix of the system $\dot{\widetilde{x}} = A(t)\widetilde{x}$ [it coincides with $X(t,s)$]. By virtue of the duality principle, one can state an assertion similar to Theorem 1.

Corollary 3. *Let system* (11) *be UCO. Then there exists a $T > 0$ such that, for each $\mu \in \mathbb{R}$, one can indicate an admissible control $V : \mathbb{R} \to M_{nk}$ providing the relation*

$$\widetilde{X}_V((k+1)T, kT) = e^{\mu T}\widetilde{X}((k+1)T, kT) \tag{23}$$

for the Cauchy matrix $\widetilde{X}_V(t,s)$ of system (22) *for all $k \in \mathbb{Z}$.*

Proof. Let system (11) be ϑ-UCO. Then system (12) is ϑ-UCC. We set $T = 2\vartheta$ and fix an arbitrary $\mu \in \mathbb{R}$. By $P(t,s)$ we denote the Cauchy matrix of the system $\dot{x} = -A^*(t)x$, and by $P_{V^*}(t,s)$ we denote the Cauchy matrix of the system

$$\dot{x} = (-A^*(t) + C(t)V^*(t))x. \tag{24}$$

Then $P(t,s) = \widetilde{X}^*(s,t)$ and $P_{V^*}(t,s) = \widetilde{X}_V^*(s,t)$ for all $t, s \in \mathbb{R}$. By Theorem 2, there exists an admissible control $V^* : \mathbb{R} \to M_{kn}$ ensuring that the Cauchy matrix $P_{V^*}(t,s)$ of system (24) satisfies the relation $P_{V^*}((k+1)T, kT) = e^{-\mu T}P((k+1)T, kT)$ for all $k \in \mathbb{Z}$; consequently, $\widetilde{X}_V^*(kT,(k+1)T) = e^{-\mu T}\widetilde{X}^*(kT,(k+1)T)$. By transposing the last relation and by taking the inverse matrices on the left- and right-hand sides, we obtain relation (23). The proof of the corollary is complete.

Corollary 4. *Let $A(\cdot)$ and $C(\cdot)$ be ω-periodic functions, and let system* (11) *be completely observable. Then there exists a T (equal to $2n\omega$) such that, for each $\mu \in \mathbb{R}$, there exists a T-periodic admissible control $V(\cdot)$ providing the relation* (23) *for all $k \in \mathbb{Z}$ for the Cauchy matrix $\widetilde{X}_V(t,s)$ of the T-periodic system* (22). *In this case, there exists a T-periodic Lyapunov transformation $\widetilde{x} = \widetilde{L}(t)\widetilde{z}$ reducing the T-periodic system* (22) *to the T-periodic system $\dot{\widetilde{z}} = (A(t) + \mu I)\widetilde{z}$.*

The proof of Corollary 4 is similar to that of Theorem 2.

Let us proceed to the study of system (7). By $\mathbf{X}_{U,V}(t,s)$ we denote the Cauchy matrix of system (7), where $U(\cdot)$ and $V(\cdot)$ are admissible controls. Let us introduce the system

$$\begin{pmatrix} \dot{z} \\ \dot{\widetilde{z}} \end{pmatrix} = \begin{pmatrix} A(t) + \lambda I & S(t) \\ 0 & A(t) + \mu I \end{pmatrix} \begin{pmatrix} z \\ \widetilde{z} \end{pmatrix}. \tag{25}$$

Here $\lambda, \mu \in \mathbb{R}$ are some numbers and $S(t)$ is some matrix function with integrally bounded norm. By $\mathbf{Z}(t,s)$ we denote the Cauchy matrix of system (25).

Theorem 3. *Let system* (1) *be UCC, and let system* (11) *be UCO. Then there exists a $T > 0$ such that, for arbitrary $\lambda, \mu \in \mathbb{R}$, one can indicate controls $U(t)$ and $V(t)$ and a piecewise continuous bounded matrix $S(t)$, $t \in \mathbb{R}$, such that the Cauchy matrix $\mathbf{X}_{U,V}(t,s)$ of system* (7) *with these controls satisfies the relation*

$$\mathbf{X}_{U,V}((k+1)T, kT) = \mathbf{Z}((k+1)T, kT) \tag{26}$$

for all $k \in \mathbb{Z}$.

Proof. Let system (1) be ϑ-UCC, and let system (11) be ϑ-UCO. We set $T = 2\vartheta$ and fix numbers $\lambda, \mu \in \mathbb{R}$. By using Theorems 2 and 4, we construct controls $U(t)$ and $V(t)$, $t \in \mathbb{R}$, such that relations (14) and (23) hold for all $k \in \mathbb{Z}$. We construct the Cauchy matrix of system (7) with these $U(t)$ and $V(t)$:

$$\mathbf{X}_{U,V}(t,s) = \begin{pmatrix} X_U(t,s) & \int\limits_s^t X_U(t,\tau)(-B(\tau)U(\tau))\widetilde{X}_V(\tau,s)\,d\tau \\ 0 & \widetilde{X}_V(t,s) \end{pmatrix}. \tag{27}$$

We write out the Cauchy matrix of system (25):

$$\mathbf{Z}(t,s) = \begin{pmatrix} e^{\lambda(t-s)}X(t,s) & \int\limits_s^t e^{\lambda(t-\tau)}X(t,\tau)S(\tau)\widetilde{X}(\tau,s)e^{\mu(\tau-s)}\,d\tau \\ 0 & e^{\mu(t-s)}\widetilde{X}(t,s) \end{pmatrix}. \tag{28}$$

Obviously, the diagonal blocks of the matrices (27) and (28) satisfy relation (26) by virtue of (14) and (23). We write out relation (26) for the right upper blocks of the matrices (27) and (28). By taking into account (14) and by performing simple manipulations, we obtain

$$\int\limits_{kT}^{(k+1)T} X_U(kT,\tau)(-B(\tau)U(\tau))\widetilde{X}_V(\tau,kT)\,d\tau = \int\limits_{kT}^{(k+1)T} e^{\lambda(kT-\tau)}X(kT,\tau)S(\tau)\widetilde{X}(\tau,kT)e^{\mu(\tau-kT)}\,d\tau.$$

We denote the left-hand side of the last relation by H_k,

$$K_k(\tau) := e^{\lambda(kT-\tau)}X(kT,\tau), \qquad N_k(\tau) := \widetilde{X}(\tau,kT)e^{\mu(\tau-kT)}, \qquad \tau \in \Delta_k.$$

The last relation can be represented in the form

$$H_k = \int\limits_{kT}^{(k+1)T} K_k(\tau)S(\tau)N_k(\tau)\,d\tau. \tag{29}$$

The matrices $X_U(kT,\tau)$, $\widetilde{X}_V(\tau,kT)$, $K_k(\tau)$, and $N_k(\tau)$, $\tau \in \Delta_k$, are bounded in norm uniformly for all $k \in \mathbb{Z}$; the matrix $B(\tau)U(\tau)$ is integrally bounded on $\mathbb{R}$. Therefore, there exist numbers $\varrho_1, \varrho_2 > 0$ (independent of k) such that the inequalities $|H_k| \le \varrho_1$, $|K_k(t)| \le \varrho_1$, $|N_k(t)| \le \varrho_1$, $\det K_k(t) \ge \varrho_2 > 0$, and $\det N_k(t) \ge \varrho_2 > 0$ hold for all $k \in \mathbb{Z}$ and for arbitrary $t \in \Delta_k$. We define a matrix $S_k(t)$ on the interval $t \in [kT, (k+1)T)$ by the relation

$$S_k(t) = T^{-1}K_k^{-1}(t)H_kN_k^{-1}(t),$$

and let $S(t) = S_k(t)$ for $t \in [kT, (k+1)T)$. Then the matrix $S(t)$ is bounded on $\mathbb{R}$, is piecewise continuous, and satisfies relation (29). The proof of the theorem is complete.

Corollary 5. *Let system* (1) *be UCC, and let system* (11) *be UCO. Then for arbitrary* $\lambda, \mu \in \mathbb{R}$, *there exist admissible controls* $U(\cdot)$ *and* $V(\cdot)$ *such that system* (7) [*and system* (6)] *with these controls can be reduced to system* (25) *with some piecewise continuous bounded matrix* $S(t)$.

Proof. We construct controls $U(t)$ and $V(t)$ and a matrix $S(t)$, $t \in \mathbb{R}$, in accordance with Theorem 3. Then the Cauchy matrices of systems (7) and (25) coincide on the set $\{kT,\ k \in \mathbb{Z}\}$. This implies their asymptotic equivalence.

Corollary 6. *Let* $A(\cdot)$, $B(\cdot)$, *and* $C(\cdot)$ *be* ω-*periodic functions, let system* (1) *be completely controllable, and let system* (11) *be completely observable. Then for arbitrary* $\lambda, \mu \in \mathbb{R}$, *there exist* $T = 2n\omega$-*periodic admissible controls* $U(t)$ *and* $V(t)$ *and a* T-*periodic piecewise continuous bounded matrix* $S(t)$, $t \in \mathbb{R}$, *such that the Cauchy matrix* $\mathbf{X}_{U,V}(t,s)$ *of system* (7) *with these controls satisfies relation* (26) *for all* $k \in \mathbb{Z}$. *In addition, there exists a* T-*periodic Lyapunov transformation reducing the* T-*periodic system* (7) *to the* T-*periodic system* (25).

Proof. System (1) is ϑ-UCO, and system (11) is ϑ-UCO for $\vartheta = n\omega$. Set $T = 2\vartheta$. For arbitrary $\lambda, \mu \in \mathbb{R}$, we construct $U(t)$, $V(t)$, and $S(t)$, $t \in \mathbb{R}$, in accordance with Theorem 3. The functions $U(t)$ and $V(t)$ are T-periodic by Theorem 2 and Corollary 4. Therefore, systems (13) and (22) are T-periodic. Consequently, $X_U(t,0) = X_U(t+kT, kT)$ and $\widetilde{X}_V(t,0) = \widetilde{X}_V(t+kT, kT)$ for arbitrary $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. Hence it follows that $H_k = H_{k+1}$ for an arbitrary $k \in \mathbb{Z}$. One can readily verify the relations $K_k(t) = K_{k+1}(t+T)$ and $N_k(t) = N_{k+1}(t+T)$, $t \in \Delta_k$. Hence it follows that $S(t)$ is a periodic function. Therefore, system (25) is T-periodic [as well as system (7)], and the relations $\mathbf{X}_{U,V}(t+T,T) = \mathbf{X}_{U,V}(t,0)$ and $\mathbf{Z}(t+T,T) = \mathbf{Z}(t,0)$ hold. Then from (26) one can readily find (for example, just as in Theorem 2) that the Lyapunov matrix $\mathbf{L}(t) = \mathbf{X}_{U,V}(t,0)\mathbf{Z}(0,t)$ is T-periodic.

Remark 5. Consider the system

$$\begin{pmatrix} \dot{z} \\ \dot{\widetilde{z}} \end{pmatrix} = \begin{pmatrix} A(t)+\lambda I & 0 \\ 0 & A(t)+\mu I \end{pmatrix} \begin{pmatrix} z \\ \widetilde{z} \end{pmatrix}. \tag{30}$$

The central exponents of system (25) and system (30) coincide. Indeed, let $x = T(t)\xi$ be a Perron transformation [1, p. 263] reducing the system $\dot{x} = A(t)x$ to the triangular system $\dot{\xi} = R(t)\xi$ with matrix $R = T^{-1}AT - T^{-1}\dot{T}$. Then the transformation

$$\begin{pmatrix} z \\ \widetilde{z} \end{pmatrix} = \begin{pmatrix} T(t) & 0 \\ 0 & T(t) \end{pmatrix} \begin{pmatrix} \xi \\ \widetilde{\xi} \end{pmatrix}$$

is a Perron transformation reducing system (25) to the triangular system

$$\begin{pmatrix} \dot{\xi} \\ \dot{\widetilde{\xi}} \end{pmatrix} = \begin{pmatrix} R(t)+\lambda I & T^{-1}(t)S(t)T(t) \\ 0 & R(t)+\mu I \end{pmatrix} \begin{pmatrix} \xi \\ \widetilde{\xi} \end{pmatrix} \tag{31}$$

with integrally bounded coefficient matrix and system (30) to the triangular system

$$\begin{pmatrix} \dot{\xi} \\ \dot{\widetilde{\xi}} \end{pmatrix} = \begin{pmatrix} R(t)+\lambda I & 0 \\ 0 & R(t)+\mu I \end{pmatrix} \begin{pmatrix} \xi \\ \widetilde{\xi} \end{pmatrix}. \tag{32}$$

A Perron transformation is a Lyapunov transformation; consequently, it preserves the exponents. Therefore, the central exponents of systems (25) and (31) and systems (30) and (32) coincide. By [1, p. 120], the central exponents of a triangular system and its diagonal approximation system coincide. Consequently, these exponents coincide for systems (31) and (32), since the diagonals of these systems coincide. Therefore, these exponents are the same for systems (25) and (30). A similar assertion is true for singular exponents.

These considerations, together with Theorem 3, imply corollaries on the global controllability of central and singular exponents. Their behavior is specified by diagonal blocks, and by using an appropriate shift of λ (and μ), one can move them arbitrarily. Let us formulate some corollaries.

Corollary 7. *Let system* (1) *be UCC, and let system* (11) *be UCO. Then system* (7) *has the property of global controllability of the upper central exponent; i.e., for any* $\alpha \in \mathbb{R}$, *there exist admissible controls* U *and* V *such that the upper central exponent* $\Omega_{U,V}$ *of system* (7) *satisfies the relation* $\Omega_{U,V} = \alpha$.

Proof. Let $\Omega(A)$ be the upper central exponent of the system $\dot{x} = A(t)x$. Then the upper central exponent of the system

$$\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} A(t) & 0 \\ 0 & A(t) \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$$

is also equal to $\Omega(A)$. For any α, we take $\lambda = \mu = \alpha - \Omega(A)$ and construct controls U and V for these λ and μ in accordance with Theorem 3. Then $\Omega_{U,V}$ for system (7) coincides with the upper central exponent of the system

$$\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} A(t) + \lambda I & 0 \\ 0 & A(t) + \lambda I \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix},$$

which is equal to α.

Similar assertions can be stated for the lower central singular exponent ω, the upper singular exponent Ω^0, and the lower singular exponent ω_0. Next, system (7) is said to be uniformly stabilizable [7] if for any $\alpha > 0$ there exist admissible controls U and V such that the upper singular exponent $\Omega^0_{U,V}$ of system (7) satisfies the inequality $\Omega^0_{U,V} < -\alpha$. The following assertion is obvious.

Corollary 8. *Let system* (1) *be UCC, and let system* (11) *be UCO. Then system* (7) *is uniformly stabilizable.*

In conclusion, we state assertions that follow from Popova's result (mentioned in Remark 3) on the global scalarization and from the above-obtained results. Suppose that the function $A(\cdot)$ is bounded and piecewise continuous and that $B(\cdot)$ and $C(\cdot)$ are bounded and piecewise uniformly continuous.

Corollary 9. *Let system* (1) *be UCC, and let system* (11) *be UCO. Then for arbitrary bounded piecewise continuous functions* $p, q : \mathbb{R} \to \mathbb{R}$, *there exist piecewise continuous bounded controls* $U(\cdot)$ *and* $V(\cdot)$ *and a matrix* $S(\cdot)$ *such that system* (7) *with these controls is asymptotically equivalent to the system*

$$\begin{pmatrix} \dot{z} \\ \dot{\tilde{z}} \end{pmatrix} = \begin{pmatrix} p(t)I & S(t) \\ 0 & q(t)I \end{pmatrix} \begin{pmatrix} z \\ \tilde{z} \end{pmatrix}. \tag{33}$$

Proof. Take arbitrary bounded piecewise continuous functions $p, q : \mathbb{R} \to \mathbb{R}$, and for arbitrary $t, s \in \mathbb{R}$ set

$$\varphi(t,s) = \exp \int_s^t p(\tau)\, d\tau, \qquad \psi(t,s) = \exp \int_s^t q(\tau)\, d\tau.$$

By Theorem 1 in [8], we construct a control $U(\cdot)$ such that

$$X_U(k\vartheta, (k-1)\vartheta) = F_{k+1} Q_k F_k^{-1} \varphi(k\vartheta, (k-1)\vartheta), \qquad k \in \mathbb{Z}.$$

(For the definition of F_k and Q_k, see [8].) By multiplying these matrices, for each $k \in \mathbb{Z}$, we obtain the relation $X_U(k\vartheta, 0) = F_{k+1} T_k F_1^{-1} \varphi(k\vartheta, 0)$. Here $T_k := Q_k Q_{k-1} \cdots Q_1$ (in Theorem 1 in [8], this matrix is denoted by $\widetilde{Q}_k$). Next, we construct the matrix $L(t) := X_U(t,0)\varphi(0,t)$. It was proved in Theorem 1 in [8] that $L(t)$ is a Lyapunov matrix.

Further, since system (11) is UCO, it follows that system (12) is UCC. By using Theorem 1 in [8], we construct a control $V^*(\cdot)$ such that the Cauchy matrix $P_{V^*}(t,s)$ of system (24) satisfies the relations

$$P_{V^*}(k\vartheta,(k-1)\vartheta) = \widetilde{F}_{k+1}\widetilde{Q}_k\widetilde{F}_k^{-1}\psi^{-1}(k\vartheta,(k-1)\vartheta), \quad k \in \mathbb{Z}.$$

(The definition of the matrices $\widetilde{F}_k$ and $\widetilde{Q}_k$ follows from the proof of Theorem 1 in [8].) By multiplying these matrices, for all $k \in \mathbb{Z}$, we obtain the relations $P_{V^*}(k\vartheta, 0) = \widetilde{F}_{k+1}\widetilde{T}_k\widetilde{F}_1^{-1}\psi^{-1}(k\vartheta,0)$; here $\widetilde{T}_k := \widetilde{Q}_k\widetilde{Q}_{k-1}\cdots\widetilde{Q}_1$. We transpose the last relation and take the inverse matrices for the left- and right-hand sides. Further, since $P_{V^*}(t,s) = \widetilde{X}_V^*(s,t)$ for all $t, s \in \mathbb{R}$, we have

$$\widetilde{X}_V(k\vartheta,0) = \widetilde{F}_1^*(\widetilde{T}_k^{-1})^*(\widetilde{F}_{k+1}^{-1})^*\psi(k\vartheta,0).$$

Set $\widetilde{L}(t) := \widetilde{X}_V(t,0)\psi(0,t)$. In this case, $\widetilde{L}(t)$ is a Lyapunov matrix (which can be proved in a similar way). Let $\mathbf{L}(t) = \operatorname{diag}\{L(t), \widetilde{L}(t)\}$. Then the Lyapunov transformation $\mathbf{x} = \mathbf{L}(t)\mathbf{z}$, where $\mathbf{x} = \operatorname{col}(x, \widetilde{x})$ and $\mathbf{z} = \operatorname{col}(z, \widetilde{z})$, reduces system (7) to system (33). The matrix $S(t)$ can be found from a relation of the form (29), where $T = \vartheta$, $K_k(\tau) = \varphi(k\vartheta,\tau)I$, and $N_k(\tau) = \psi(\tau, k\vartheta)I$. It is piecewise continuous and bounded, since $\mathbf{L}(t)$ is a Lyapunov transformation and $U(\cdot)$ and $V(\cdot)$ are piecewise continuous and bounded. The proof of the corollary is complete.

It follows from Corollary 9 that Corollaries 1–4 in [8] can be restated for system (7). One can also readily generalize Theorem 3 on the global controllability of the complete spectrum of Lyapunov exponents in [8] to the case of system (7).

Corollary 10. *Let $A(\cdot)$ be bounded and piecewise continuous, and let $B(\cdot)$ and $C(\cdot)$ be bounded and piecewise uniformly continuous. Let system* (1) *be UCC, and let system* (11) *be UCO. Then for arbitrary numbers $\lambda_1 \le \cdots \le \lambda_{2n}$, there exist piecewise continuous bounded controls $U(\cdot)$ and $V(\cdot)$ such that the complete spectrum of Lyapunov exponents of system* (7) *with these controls is the set $\lambda_1, \ldots, \lambda_{2n}$.*

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