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SHORT COMMUNICATIONS

Control of Spectrum in Bilinear Systems

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Abstract—For a bilinear stationary control system, we obtain necessary and sufficient conditions for the solvability of the spectrum control problem for the case in which the coefficients have a special form.

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Let $\mathbb{R}^n$ be a Euclidean space of dimension n; let $e_1, \ldots, e_n$ be a canonical basis in $\mathbb{R}^n$ [i.e., $e_1 = \operatorname{col}(1, 0, \ldots, 0)$, $\ldots$, $e_n = \operatorname{col}(0, \ldots, 0, 1)$]; let $M_{m,n}$ be the space of real $m \times n$ matrices; let $M_n := M_{n,n}$; let $I = [e_1, \ldots, e_n] \in M_n$ be the unit matrix; let $*$ be the operation of transposition of a vector or a matrix [if $x \in \mathbb{R}^n$ is a column vector, then $x^* \in \mathbb{R}^{n*}$ is a row vector]; let $J_0 := I$; let J_1 be the first unit superdiagonal [i.e., $J_1 := \sum_{i=1}^{n-1} e_i e_{i+1}^* \in M_n$]; let $J_k := J_1^k$, $k \in \mathbb{N}$ (i.e., $J_k = 0 \in M_n$ for $k \geq n$); let $\chi(A; \lambda)$ be the characteristic polynomial of a matrix A; and let $\operatorname{Sp} A$ be the trace of a matrix A.

Consider the linear control system

$$\dot{x} = A(t)x + B(t)u, \qquad y = C^*(t)x, \qquad (t, x, u, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k. \tag{1}$$

Let the control in system (1) be constructed by the linear incomplete feedback principle in the form $u = Uy$. Then system (1) becomes the homogeneous system

$$\dot{x} = (A(t) + B(t)UC^*(t))x. \tag{2}$$

Along with system (2), consider the bilinear control system

$$\dot{x} = (A(t) + u_1 A_1(t) + u_2 A_2(t) + \cdots + u_r A_r(t))x, \qquad u_l \in \mathbb{R}. \tag{3}$$

Any system of the form (2) can be represented in the form (3) if we set $r := mk$, take the coefficients of the matrix U for u_l, $l = 1, \ldots, r$, and take the matrices $b_i(t)c_j^*(t) \in M_n$ for the matrices $A_l(t)$, $l = 1, \ldots, r$, where the $b_i(t)$, $i = 1, \ldots, m$, are the columns of the matrix $B(t)$ and the $c_j(t)$, $j = 1, \ldots, k$, are the columns of the matrix $C(t)$,

$$\dot{x} = \left(A(t) + \sum_{i=1}^{m} \sum_{j=1}^{k} u_{ij}(b_i(t)c_j^*(t)) \right) x.$$

Systems of the form (2) were studied in [1–8], and systems of the form (3), in [7, 9–11]. The consistency property was introduced in [1] for system (2) and in [9] for system (3). This property was analyzed in [1, 4, 5, 7, 9]. The consistency property was used in the proof of the local controllability of Lyapunov exponents of system (2) in [2–6], the stability of Lyapunov exponents of system (2) in [3], the local attainability and local Lyapunov reducibility of system (3) in [9–11], and the local controllability of Lyapunov exponents in [10] and Izobov exponents of system (3) in [11].

In the present paper, we consider stationary systems of the form (2) and (3),

$$\dot{x} = (A + BUC^*)x, \qquad x \in \mathbb{R}^n, \tag{4}$$

$$\dot{x} = (A + u_1 A_1 + \cdots + u_r A_r)x, \qquad x \in \mathbb{R}^n. \tag{5}$$

In the case of stationary systems, the asymptotic behavior of a system is characterized by the spectrum of the system matrix. We consider the problem on the global control of the spectrum.

Definition 1. We say that the *spectrum control problem for the matrix* $A + u_1A_1 + \cdots + u_rA_r$ *of system* (5) *is solvable* if, for any polynomial $p(\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \cdots + \gamma_n$ of degree n with real coefficients γ_i, there exists a constant control $u = \operatorname{col}(u_1, \ldots, u_r) \in \mathbb{R}^r$ that ensures the relation

$$\chi(A + u_1A_1 + \cdots + u_rA_r; \lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \cdots + \gamma_n \tag{6}$$

for the characteristic polynomial χ of the coefficient matrix of the system.

This problem is also referred to as the eigenvalue assignment problem [12, p. 159]. For $C = I$, the spectrum control problem for system (4) is solvable if and only if system (1) is completely controllable [13, 14]. A survey of known sufficient and necessary solvability conditions in this problem for system (4) can be found in [12, pp. 179–181]. New solvability conditions for this problem for system (4) were obtained in [15]; in contrast to the conditions given in [12], they are both necessary and sufficient. In the present paper, the results obtained for system (4) in [15] are generalized to system (5).

Let the coefficients of system (5) have the following form: the matrix A has the Hessenberg form, and the first $p-1$ rows and the last $n-p$ columns in the matrices A_l, $l = 1, \ldots, r$, are zero; i.e.,

$$\begin{aligned} &A = \{a_{ij}\}_{i,j=1}^n, \quad a_{i,i+1} \neq 0, \quad i = 1, \ldots, n-1; \quad a_{ij} = 0, \quad j > i+1; \\ &A_l = \{a_{ij}^l\}_{i,j=1}^n, \quad a_{ij}^l = 0, \quad i = 1, \ldots, p-1, \quad j = 1, \ldots, n; \\ &a_{ij}^l = 0, \quad i = 1, \ldots, n, \quad j = p+1, \ldots, n; \quad l = 1, \ldots, r; \quad p \in \{1, \ldots, n\}. \end{aligned} \tag{7}$$

By $\lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_n$ we denote the characteristic polynomial of the matrix A.

For the matrix $A = \{a_{ij}\}_{i,j=1}^n$, we construct the matrix S just as in [15]. Namely, we first construct the matrix

$$S_1 = \{s_{ij}^1\}_{i,j=1}^n, \quad s_{11}^1 := 1, \quad s_{1j}^1 := 0, \quad j = 2, \ldots, n; \quad s_{ij}^1 := a_{i-1,j}, \quad i = 2, \ldots, n, \quad j = 1, \ldots, n.$$

Further, for each $l = 2, \ldots, n$ and for the matrix $S_{l-1} = \{s_{ij}^{l-1}\}_{i,j=1}^n$, we construct the matrix $S_l = \{s_{ij}^l\}_{i,j=1}^n$ with entries $s_{11}^l := 1$, $s_{1j}^l := s_{j1}^l := 0$, $j = 2, \ldots, n$, and $s_{ij}^l := s_{i-1,j-1}^{l-1}$, $i, j = 2, \ldots, n$. Set $S = S_nS_{n-1} \cdots S_1$. All matrices S_l and S are lower triangular and nonsingular. Next, we construct the matrices $H_i := SA_iS^{-1}$, $i = 1, \ldots, r$, and $H := SAS^{-1}$. Then the matrices H_i, $i = 1, \ldots, r$, have the same form as A_i, $i = 1, \ldots, r$; i.e., the first $p-1$ rows and the last $n-p$ columns of the matrices H_i, $i = 1, \ldots, r$, are zero. The matrix H has the following form [15, Lemma 3]: $H = J_1 + e_n\xi$, where $\xi = (-\alpha_n, \ldots, -\alpha_1) \in \mathbb{R}^{n*}$. We have

$$\chi(A + u_1A_1 + \cdots + u_rA_r; \lambda) = \chi(H + u_1H_1 + \cdots + u_rH_r; \lambda). \tag{8}$$

Let us construct the matrix $G := \sum_{i=1}^n \alpha_{i-1}J_{i-1}^*$, where $\alpha_0 := 1$.

Let $h_j^i \in \mathbb{R}^n$ be the jth column of the matrix H_i, $i = 1, \ldots, r$. For the matrices $H_i = [h_1^i, h_2^i, \ldots, h_n^i]$, $i = 1, \ldots, r$, we construct the $n \times r$ matrices $P_1 = [h_1^1, \ldots, h_1^r]$, $\ldots$, $P_n = [h_n^1, \ldots, h_n^r]$. Then we construct the $n \times r$ matrix $Q = J_0GP_1 + J_1GP_2 + \cdots + J_{n-1}GP_n$.

Theorem 1. *Let system* (5) *with the matrices* (7) *be given, and let*

$$\lambda^n + \gamma_1\lambda^{n-1} + \cdots + \gamma_n := \chi(A + u_1A_1 + \cdots + u_rA_r; \lambda).$$

Then coefficients γ_i *of the characteristic polynomial of the matrix* $A + u_1A_1 + \cdots + u_rA_r$ *can be expressed via the coefficients of system* (5), (7) *as follows*:

$$\gamma = \alpha - Qu, \tag{9}$$

where $\gamma = \operatorname{col}(\gamma_1, \ldots, \gamma_n)$ *and* $\alpha = \operatorname{col}(\alpha_1, \ldots, \alpha_n)$.

Theorem 2. *The spectrum control problem for system* (5) *with the matrices* (7) *is solvable if and only if the rows of the matrix* Q *are linearly independent*; *in addition*, *the control* u *bringing* $\chi(A + u_1A_1 + \cdots + u_rA_r; \lambda)$ *to a given polynomial* $p(\lambda)$ *with coefficients* γ_i *can be found from system* (9).

Theorem 2 obviously follows from Theorem 1. Indeed, if the rows of the matrix Q are linearly independent, then for any γ the linear system (9) is solvable for u, and the solution u of this system ensures relation (6). If the rows of the matrix Q are linearly dependent, then system (9) is not necessarily solvable for any γ; consequently, the spectrum control problem is unsolvable. It remains to prove Theorem 1.

Remark 1. It follows from Theorem 2 that the condition $r \geq n$ is a necessary solvability condition in the spectrum control problem for system (5).

Remark 2. Of the matrices P_i, $i = 1, \dots, n$, the last $n - p$ matrices are zero; therefore, for Q one can take the matrix $Q = J_0GP_1 + J_1GP_2 + \cdots + J_{p-1}GP_p$.

Proof of Theorem 1. To prove the desired assertion, we need Lemma 4 in [15].

Let system (5) be a given system with the matrices (7), and let relation (6) hold. For the matrices A and A_i, $i = 1, \dots, r$, we construct matrices H and H_i, $i = 1, \dots, r$. By virtue of (8), we obtain $\chi(H + u_1H_1 + \cdots + u_rH_r; \lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \cdots + \gamma_n$. Therefore, by Lemma 4 in [15], we obtain the relations

$$\begin{aligned}\gamma_i &= \alpha_i - \mathrm{Sp}(u_1H_1 + \cdots + u_rH_r)J_{i-1}G \\ &= \alpha_i - (u_1 \,\mathrm{Sp}(H_1J_{i-1}G) + \cdots + u_r \,\mathrm{Sp}(H_rJ_{i-1}G)), \qquad i = 1, \dots, n. \end{aligned} \tag{10}$$

Relations (10) can be represented in the vector form $\gamma = \alpha - Tu$. Here $T \in M_{n,r}$ and

$$T = \| \mathrm{Sp}(H_jJ_{i-1}G) \|, \qquad i = 1, \dots, n, \qquad j = 1, \dots, r.$$

It remains to show that $T = Q$. We introduce the mapping $\mathrm{vec} : M_n \to \mathbb{R}^{n^2}$, which "expands" the matrix $F \in M_n$, $F = \{f_{ij}\}$, $i, j = 1, \dots, n$, by rows in the column vector

$$\mathrm{vec}\, F := \mathrm{col}(f_{11}, \dots, f_{1n}, \dots, f_{n1}, \dots, f_{nn}) \in \mathbb{R}^{n^2}.$$

Obviously, $\mathrm{Sp}(F_1^*F_2) = (\mathrm{vec}\, F_1)^*(\mathrm{vec}\, F_2) = (\mathrm{vec}\, F_2)^*(\mathrm{vec}\, F_1)$. Consider the first row of the matrix T. Then for all $l = 1, \dots, r$, we have $\mathrm{Sp}(H_lJ_0G) = (\mathrm{vec}(J_0G))^*(\mathrm{vec}(H_l^*))$. For the second row of the matrix T, we have the relation

$$\mathrm{Sp}(H_lJ_1G) = (\mathrm{vec}(J_1G))^*(\mathrm{vec}(H_l^*)), \qquad l = 1, \dots, r,$$

and so on; for the ith row of the matrix T ($i = 1, \dots, n$), we have

$$\mathrm{Sp}(H_lJ_{i-1}G) = (\mathrm{vec}(J_{i-1}G))^*(\mathrm{vec}(H_l^*)), \qquad l = 1, \dots, r.$$

Let us construct the matrices

$$L = \left\| \begin{matrix} (\mathrm{vec}(J_0G))^* \\ (\mathrm{vec}(J_1G))^* \\ \cdots \\ (\mathrm{vec}(J_{n-1}G))^* \end{matrix} \right\| \in M_{n,n^2}, \qquad N = \left\| \mathrm{vec}(H_1^*), \dots, \mathrm{vec}(H_r^*) \right\| \in M_{n^2,r}.$$

Then $T = LN$.

Consider the matrices J_iG. We have

$$J_0G = \begin{Vmatrix} 1 & 0 & 0 & \dots & 0 \\ \alpha_1 & 1 & 0 & \dots & 0 \\ \alpha_2 & \alpha_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \dots & 1 \end{Vmatrix},$$

$$J_1G = \begin{Vmatrix} \alpha_1 & 1 & \dots & 0 \\ \alpha_2 & \alpha_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \alpha_{n-1} & \alpha_{n-2} & \dots & 1 \\ 0 & 0 & \dots & 0 \end{Vmatrix}, \qquad J_{n-1}G = \begin{Vmatrix} \alpha_{n-1} & \dots & 1 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{Vmatrix}.$$

We expand these matrices by rows in column vectors, transpose them into rows, and form the matrix L. Then one can note that the resulting matrix has the block form

$$L = \begin{Vmatrix} [J_0G], [J_1G], \dots, [J_{n-1}G] \end{Vmatrix} \in M_{n,n^2}.$$

Now consider the matrix N. We have

$$\operatorname{vec}(H_1^*) = \begin{Vmatrix} h_1^1 \\ h_2^1 \\ \vdots \\ h_n^1 \end{Vmatrix} \in \mathbb{R}^{n^2}, \quad \dots, \quad \operatorname{vec}(H_r^*) = \begin{Vmatrix} h_1^r \\ h_2^r \\ \vdots \\ h_n^r \end{Vmatrix} \in \mathbb{R}^{n^2} \quad \Longrightarrow \quad N = \begin{Vmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Vmatrix} \in M_{n^2,r}.$$

Consequently, $T = LN = J_0GP_1 + J_1GP_2 + \dots + J_{n-1}GP_n = Q$. The proof of Theorem 1 is complete.

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REFERENCES

1. Popova, S.N. and Tonkov, E.L., Control Over the Lyapunov Exponents of Consistent Systems. I, *Differential Equations*, 1994, vol. 30, no. 10, pp. 1556–1564.
2. Popova, S.N. and Tonkov, E.L., Control Over the Lyapunov Exponents of Consistent Systems. II, *Differential Equations*, 1994, vol. 30, no. 11, pp. 1800–1807.
3. Popova, S.N. and Tonkov, E.L., Control Over the Lyapunov Exponents of Consistent Systems. III, *Differential Equations*, 1995, vol. 31, no. 2, pp. 209–218.
4. Popova, S.N. and Tonkov, E.L., Uniform Consistency of Linear Systems, *Differential Equations*, 1995, vol. 31, no. 4, pp. 672–674.
5. Popova, S.N. and Tonkov, E.L., Consistent Systems and Control of Lyapunov Exponents, *Differential Equations*, 1997, vol. 33, no. 2, pp. 226–235.
6. Makarov, E.K. and Popova, S.N., Local Controllability of Characteristic Lyapunov Exponents of Systems with Simple Exponents, *Differ. Uravn.*, 1997, vol. 33, no. 4, pp. 495–499.
7. Zaitsev, V.A., Consistency and Control of Lyapunov Exponents, *Izv. Inst. Mat. Inform. Udmurt. Gos. Univ.*, Izhevsk, 1999, vol. 2 (17), pp. 3–40.
8. Popova, S.N., Local Attainability for Linear Control Systems, *Differential Equations*, 2003, vol. 39, no. 1, pp. 51–58.

9. Zaitsev, V.A. and Tonkov, Ye.L., Attainability, Consistency and the Rotation Method by V.M. Millionshchikov, *Russian Mathematics*, 1999, vol. 43, no. 2, pp. 42–52.
10. Tonkov, E.L., Uniform Attainability and Lyapunov Reducibility of Bilinear Control System, *Proc. Steklov Inst. Math.*, 2000, Suppl. 1, pp. S228–S253.
11. Tonkov, E.L., Lyapunov Reducibility of a Linear System, Stabilization, and Control of Izobov Exponents, *Tr. Inst. Mat. Nats. Akad. Nauk Belarusi*, Minsk, 2000, vol. 4, pp. 146–155.
12. Leonov, G.A. and Shumafov, M.M., *Metody stabilizatsii lineinykh upravlyaemykh sistem* (Stabilization Methods for Linear Control Systems), St. Petersburg, 2005.
13. Popov, V.M., *Giperustoichivost' avtomaticheskikh sistem* (Hyperstability of Automatic Systems), Moscow: Nauka, 1970.
14. Wonham, W.M., On Pole Assignment in Multi-Input Controllable Linear Systems, *IEEE Trans. Automat. Control*, 1967, vol. AC-12, no. 6, pp. 660–665.
15. Zaitsev, V.A., Spectrum Control in Linear Systems with Incomplete Feedback, *Differential Equations*, 2009, vol. 45, no. 9, pp. 1348–1357.