Fundamental Branches in Investigation of Linear Functional Differential Equations

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Abstract – **There are given a survey of author results in the theory of linear functional-differential equations that are based on the properties of Fredholm operator established by academician S. M. Nikol'skii in 1936-1943.**

Keywords – **reconstructive modeling, factorization of linear operator, minimal rank perturbation, minimal sets of cyclic vectors, equation with boundary inequalities**

I. INTRODUCTION

Theory of functional-differential equations describes and investigates properties of such dynamical processes, the motion of which is depended on their previous history and a planning future of the states of these processes. In view of a complicated behavior of such processes the main effort of researches was directed to create a well-proportioned and fruitful theory of linear processes. Now we can certify that the sum of bounded continuously invertible and finite dimensional operators forms the ground stone of this theory. This conclusion is based on the fundamental constructive additive decomposition of Fredholm operator established by academician S. M. Nikol'skii in [1]. This work allowed us to substantiate the following branches in investigation of linear theory of functional differential equations [2-7]:

1) Main stages of reconstructive modeling;
2) Abstract scheme of constructing of functional spaces;
3) Additive-multiplicative factorization of functional differential operators;
4) Factorization of Green operators in the problem of approximation by finite dimensional operators;
5) Perturbations of minimal rank for functional-differential operators;
6) Fredholm theorems for functional differential equations with boundary inequalities;
7) Minimal family of cyclic vectors of Green operators.
8) Ryaben'kii-Filippov theorem in the theory of boundary value problems for functional differential equations.

II. MAIN STAGES OF RECONSTRUCTIVE MODELING

Deterministic linear system of type input-output is modeled by functional relation $x = Gf$ between an external action f on this system and a response of the system x to this effect. Here G denotes the linear mapping between Banach spaces that we want to restore. Natural desire is to obtain this system by means of reconstruction of already known deterministic system with injective transformation Λ. The latter is called the base system. Reconstructive modeling includes the following basic phases:

1. *Choice of base deterministic system.* At this stage, it is specified the nature of the values of external action f to base system and reaction $x = \Lambda f$ of the system on this impact. In addition, there is selected the Banach space B so that its norm adequately reflects the magnitude of the deviation of two different values of the forcing from each other. Next, it is pointed out a sufficiently wide linear variety Z over the field of real or complex numbers, containing the values of characteristic x both the base and modeled systems. Linear mapping $\Lambda : B \to Z$ is necessary to describe the hypothesis about specific essence of starting model process.

2. *A finite dimensional extension of the set of all values of the reaction of base system.* At this stage we take the hypothesis that the set GB of all values of modeled system lies in a linear variety D which is wider than ΛB , but can be represented as a direct sum $D = \Lambda B \oplus E$, where the second term is finite. The role of finite-dimensional space E $(n = \dim E)$ consists of desire to reflect the possible, but limited environmental impact on the base system. In the space D we introduce the norm $\|x\|_D = \|\delta x\|_B + \sum_{j=1}^{n} |r_j(x)|$ under which it becomes a Banach space. There are used here a linear operator $\delta : D \to B$ and linear functionals $r_j(x), j = 1,\ldots,n$, which are uniquely determined from the additive-multiplicative representation of identity operator: $x = \Lambda\delta x + \sum_{j=1}^{n} u_j r_j(x),\ \ x \in D$, where system of elements $u_1,\ldots,u_n$ forms a fixed basis in finite-dimensional space E . In this representation all indicated operators will be bounded relative to introduced norm.

3. *Checking structural equivalence of modeled and base systems.* Under structural equivalence of operators $G : B \to D$ and $\Lambda : B \to D$ respectively modeled and

base systems we understand a factorization of the form $G=(\Lambda+K)P^{-1}$, where $K: B \to D, P: B \to B$ are bounded linear operators, moreover K is finite dimensional and P is continuously invertible. In the mentioned factorization finite dimensional operator reflects the influence of the environment, and continuously invertible operator expresses the requirement of prior linear homeomorphic transformation of space B in the formation of external influence on the base system. Structurally equivalent operators belong to the same operator ideals and have the same order of approximation numbers. A criterion for structural equivalence follows directly from Nikol'skii Theorem on Fredholm operator. *The operator $G: B \to D$ and an injective operator $\Lambda: B \to D$ are structurally equivalent if and only if the functionals $r_j(Gf), f \in B, j=1,\dots,n$ are bounded and the product $\delta G: B \to B$ is a Fredholm operator.*

III. ADDITIVE-MULTIPLICATIVE FACTORIZATION OF FUNCTIONAL-DIFFERENTIAL OPERATORS

Let $L: D \to B$ be a linear bounded operator. By virtue of above factorization of identity operator in constructed space D we have the following representation $Lx = Q\delta x + \sum_{j=1}^{n} q_j r_j(x),\ x \in D$. Here $Q = L\Lambda: B \to B$ is a linear bounded operator and the elements $q_j = Lu_j, j=1,\dots,n$ belong to Banach space B. The following is a direct consequence of Nikol'skii Theorem on Fredholm operator. *The product of L and operator G, which is structurally equivalent to injection Λ generating the space D, is a Fredholm operator if and only if "the principal part» Q of operator L is a Fredholm operator as well.* This important mathematical result provides a wide class of functional-differential operators with good properties and having important practical significance. On the set of values of structurally equivalent operator $G: B \to D$ it is vanished a finite number of linearly independent functionals from dual space D^*: $l_1(x),\dots,l_m(x)$. And also its number m is not less than n. If the product LG is identity operator in the main space B then $m=n$, and the map $G: B \to D$ is the Green operator of boundary task $Lx=f,\ l_1(x)=0,\dots,l_n(x)=0$. Representation of defective functionals $l_i \in D^*$ of structurally equivalent operator $G: B \to D$ in the next suitable form $l_i(x) = \varphi_i(\delta x) + \sum_{j=1}^{n} \alpha_{ij} r_j(x),\ \varphi_i \in B^*, \alpha_{ij} = l_i(u_j)$ allows us effectively construct such an operation of type $L_0\, x = \delta x + \sum_{j=1}^{n} p_j r_j(x),\ x \in D, p_j \in B, j=1,\dots,n$ that the problem $L_0\, x = f,\ l_1(x)=0,\dots,l_n(x)=0$ is uniquely solvable in the space D and the Green operator of this task is of form $W = \Lambda + K_0$, where finite-dimensional operator $K_0: B \to D$ is written explicitly. Factorization $G=(\Lambda+K)P^{-1}$ of operator $G: B \to D$ can now be obtained through the above operators L_0, W and not yet used affected functionals $l_{n+1},\dots,l_m$ of this operator. Let ν be the rank of finite dimensional operator K. Then for approximation numbers $s_j(G) = \inf\|G-H\|, rank\, H < j$ of operator G we have for $j > \nu$ the following bilateral estimation $s_{j+\nu}(\Lambda)/\|P\| \le s_j(G) \le \gamma(P) s_{j-\nu}(\Lambda)/\|P\|$, where $\gamma(P) = \|P\|\cdot\|P^{-1}\|$ is the condition number of linear homeomorphism P.

In the most interesting in practice case, when $D \subset B$ and the injection $\Lambda: B \to B$ generating the space D is compact, every operator $L: D \to B$ with Fredholm main part $Q = L\Lambda$ will be unbounded but closed operator in the main space B. Therefore it is significant the following statement about density of kernels intersection for defective functionals from D^* in the space B. *Let $m \ge n$. Kernels intersection $F = \bigcap_{i=1}^{m} \ker l_i$ of linear bounded functionals $l_i \in D^*$ is dense in B if and only if when it is found a finite-dimensional operator $K: B \to D$ such that the range of values $(\Lambda+K)B$ of operator $(\Lambda+K): B \to B$ coincides with F, and the null-space of conjugate operator $(\Lambda+K)^*: B^* \to B^*$ consists of one zero.*

IV. MINIMAL RANK PERTURBATIONS OF FUNCTIONAL-DIFFERENTIAL OPERATORS

Let injection $\Lambda: B \to B$ generating the space $D = \Lambda B \oplus E \subset B$ be compact. We choose a linear bounded operator $L: D \to B$ with Fredholm main part $Q = L\Lambda$. Suppose that the intersection $F = \bigcap_{i=1}^{n} \ker l_i$ of kernels $n = \dim E$ functionals $l_i \in D^*$ is dense in the global space B and the linear boundary value problem $Lx=f,\ l_1(x)=0,\dots,l_n(x)=0$ is uniquely solvable in the subspace D. Then the restriction $L_{|F}$ of operator L on the subspace F will be a dense defined closed operator in the space B. Moreover, this restriction will have compact resolvent and, as a consequence, a discrete spectrum. In applied problems the discrete spectrum of restriction $L_{|F}$ describes the frequencies of natural vibrations of different purposes systems. So we are interested in the next task of discrete spectrum control: $rank\, H \to \min,\ P(L_{|F} - H) \subset \Omega$ .Here the notation $P(L_{|F} - H)$ denotes the resolvent set of perturbation operator $L_{|F} - H$ and Ω is a given nonempty bounded subset of the complex plane. We have the following duality theorem. *The minimal value of rank of finite dimensional perturbation H in the problem of discrete spectrum control is equal to the maximal geometric multiplicity of eigenvalues of operator $L_{|F}$, which lie in the set Ω.* In the proof of Theorem 1 on the properties equivalence of Fredholm operator S. M. Nicol'skii used a special construction of a finite dimensional operator, which corrects a Fredholm operator to a continuously reversible operator. Similar structure has been applied by V. A. Trenogin in studying the problem of solutions bifurcation of operator equations, and more earlier similar perturbations were introduced by E. Schmidt. It is found that optimal finite dimensional operator in the problem of discrete spectrum control also belongs to the class of Schmidt-Nikolsky-Trenogin perturbations. Clarify this assertion. Let I denotes the identity operator in any Banach space. Let further $n(\mu)$ be dimension of the subspaces $\ker(L_{|F} - \mu I)$ and $\ker(L^*_{|F} - \bar{\mu} I)$. Finite dimensional operator H will be called perturbation of Schmidt-Nikols'kii-Trenogin type, which correspondes to the eigenvalue μ, if it can be represented in the next form $Hx = \sum_{i=1}^{n(\mu)} x_i < x, y_i >$, where $< x, y_i >= y_i(x)$ $\{y_i\}_1^{n(\mu)}$ is a system functionals B^* biorthogonal to any basis of eigensubspace $\ker(L_{|F} - \mu I)$, and $\{x_i\}_1^{n(\mu)}$ - a system of elements of B biorthogonal to any basis of conjugate subspace $\ker(L^*_{|F} - \bar{\mu} I)$. The importance of such a construction is disclosed in the following assertion. *The perturbation H of minimal rank in the problem of discrete spectrum control is a perturbation of Schmidt-Nikols'kii-Trenogin type corresponding to the eigenvalue μ with maximum geometric multiplicity.*

V. FREDHOLM THEOREMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH BOUNDARY INEQUALITIES

Now, let B and D be real spaces, and linear bounded operator $L: D \to B$ has a closed range. Here are mentioned two criteria of solvability in the space D for equation $Lx = f$ with boundary inequalities $l_i(x) \ge \beta_i, i = 1,...,m$, where $l_i \in D^*$. In terms of canonical representations of linear operator $Lx = Q\delta x + \sum_{j=1}^{n} q_j r_j(x)$, $Q = L\Lambda, q_j = Lu_j$ and bounded functionals $l_i(x) = (\delta x, \varphi_i) + \sum_{j=1}^{n} \alpha_{ij} r_j(x)$, $\varphi_i = l_i\Lambda \in B^*, (\delta x, \varphi_i) = \varphi_i(\delta x), \alpha_{ij} = l_i(u_j)$ takes place the following analogue of Fredholm theorem. Let $f \in B$, and there are given real numbers $\beta_i, i = 1,...,m$ such that the system of inequalities $l_i(x) \ge \beta_i, i = 1,...,m$ is consistent. *In order to exist a solution $x \in D$ of equation $Lx = f$ with boundary inequalities $l_i(x) \ge \beta_i, i = 1,...,m$ it is necessary and sufficient that for any $y \in B^*$, satisfying for some nonnegative numbers $\lambda_1,...,\lambda_m$ to a system of equations*

$$\begin{cases} Q^* y = \sum_{i=1}^{m} \lambda_i \varphi_i, \\ (q_i, y) = \sum_{i=1}^{m} \lambda_i \alpha_{ij}, j = 1,...,n, \end{cases}$$

it is valid the inequality $(f, y) \ge \sum_{i=1}^{m} \lambda_i \beta_i$. In applications it occurs the case when B is isomorphic and isometric to the space X^* dual to a Banach space X. The nature of elements of the second conjugate space can be very complicated (as in the following typical case of $X = L^n{}_1[a,b], B = L^n{}_\infty[a,b]$) and this causes quite definite difficulties. Assertion about simultaneous Fredholm property of operator and its adjoint in the Theorem 1 of work [1] allows us select and identify a class of operators L and functional l_i of D^*, for which the claim solvability of equations with boundary inequalities can be formulated in terms of dual space X. Let a bilinear form $< \cdot,\cdot >$ sets duality between X and B , Q be conjugate to an operator $\Gamma : X \to X$ and $l_i(x) \equiv < \varphi_i, \delta x > + \sum_{j=1}^{n} \alpha_{ij} r_j(x), \varphi_i \in X, i = 1,...,m$.

The problem $Lx = f$ *,* $l_i(x) \geq \beta_i, i = 1,...,m$ *solvable in* D *if and only if for any* $\varphi \in X$ *, satisfying for some numbers* $\lambda_1 \geq 0,..., \lambda_m \geq 0$ *to linear system of equations*

$$\begin{cases} \Gamma\varphi = \sum_{i=1}^{m} \lambda_i \varphi_i, \\ < \varphi, q_i > = \sum_{i=1}^{m} \lambda_i \alpha_{ij}, j = 1,...,n, \end{cases}$$

the inequality $< \varphi, f > \geq \sum_{i=1}^{m} \lambda_i \beta_i$ *.is valid.*

VI. MINIMAL FAMILIES OF CYCLIC VECTORS OF GREEN OPERATORS

Let B be a complex separable Hilbert space, an injection $\Lambda : B \to B$ generating the subspace $D = \Lambda B \oplus E \subset B$ belongs to the Shatten class $\Xi_q^s(B)$ ($q > q_0 \geq 1$) and has a real spectrum, and its resolvent operator admits the following upper estimate $\|(\Lambda - \lambda I)^{-1}\| \leq \gamma |\mathrm{Im}\,\lambda|^{-1} (\mathrm{Im}\,\lambda \neq 0)$. Suppose, further, that $L : D \to B$ is a bounded linear operator, the main part of which will be represented in the next form $Q = L\Lambda = I - F_1 + F_2$. Here, F_1, F_2 are bounded linear operators in the main space B , and also the second operator is completely continuous and the norm of the first admits the bound $\|F_1\| < (\gamma \cos ec\left(\frac{\pi}{2q_0}\right) + 2)^{-1}$. Suppose that the kernels intersection $F = \bigcap_{i=1}^{n} \ker l_i$ of $n = \dim E$ bounded linear functionals $l_i \in D^*$ is dense in B and the linear boundary value problem $Lx = f$, $l_1(x) = 0,..., l_n(x) = 0$ is uniquely solvable in the subspace D. *Then the Green operator of this problem* $G : B \to D$ *has a complete system of root vectors in the main space* B.

The family of linearly independent elements $f_1,..., f_m$ of the space B is called a family of cyclic vectors of the Green operator G if the linear span of vectors $\{G^k f_1,..., G^k f_m\}_{k=0}^{\infty}$ is dense in B . The number of elements μ of minimal family of cyclic vectors of the Green operator G is equal to

$$\mu = \max_{\lambda \neq 0} \dim \ker(G - \lambda I) =$$
$$\max_{\lambda \neq 0} \min_{S} \|GS - \lambda S + G / \lambda\|^2 =$$
$$\max_{\lambda \neq 0} \|GT_\lambda - \lambda T_\lambda + G / \lambda\|.$$

Here $\min$ is taken over all Hilbert-Schmidt operators S acting in the space B . The norm of Hilbert-Schmidt operator S is defined over operator trace tr by equality $\|S\|^2 = tr(S^* S)$. Operator T_λ denotes any solution in Hilbert-Schmidt class of the following normalized operator equation $(U_\lambda^* U_\lambda)T = U_\lambda^* V_\lambda$, where we set $U_\lambda = \lambda I - G, V_\lambda = G / \lambda$. One of the solution of the above mentioned equation can be found by iterations method $W_0 = 0,\quad W_{k+1} = (I - \tau U_\lambda^* U_\lambda) W_k + \tau U_\lambda^* V_\lambda$, where positive parameter τ is sufficiently small.

VII. RYABEN'KII-FILIPPOV THEOREM IN THE THEORY OF BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS

An abstract form of Ryaben'kii-Filippov theorem on relationship between approximation, stability and convergence concepts for linear equation (see, e.g., K. I. Babenko, Fundamentals of numerical analysis, M.-Izhevsk, SRC "RCD", 2002) is formulated in terms of finite-dimensional approximations of infinite-dimensional normed spaces and operators. We use it in analysis of numerical solution methods of boundary value problems for functional differential equations by means of minimal finite dimensional approximation of injection Λ generating functional space D . Let B be a separable Hilbert space with scalar product $(\cdot,\cdot)$, Λ be a compact linear operator in this space. Consider the extremal problem of $rank\,K \to \min,\ \|\Lambda - K\| \leq \varepsilon$, which consists in minimizing rank of a finite dimensional operator $K : B \to B$ approximating injection Λ with accuracy ε by norm of space of bounded operators. The minimal value of functional in this problem is equal to the minimal value of nonnegative index j , for which j -th approximation number $s_j(\Lambda) = \inf_{rank\,K < j} \|\Lambda - K\|$ does not exceed the error of approximation ε . If the integer $j^* = j(\varepsilon)$ solves the last problem, then the solution K of initial extremal problem is uniquely given by segment of length j^* of the following Schmidt expansion $\Lambda = \sum_{i=1}^{\infty} s_i(\Lambda) \psi_i(\cdot, \varphi_i)$ for operator Λ . Here $\{\psi_i\}_{i=1}^{\infty}$ and $\{\varphi_i\}_{i=1}^{\infty}$ are two special systems of orthonormal vectors in Hilbert space B . Note that the replacement of compact operator Λ by finite dimensional approximation $\sum_{i=1}^{j(\varepsilon)} s_i(\Lambda)\psi_i(\cdot,\varphi_i)$ allow us to obtain a discrete approximation of minimal dimension for functional space D with specified error of model ε. Further, for a fixed index j values s_j,ψ_j,φ_j form the triple (s,ψ,φ) which can be found from the spectral problem $s^2\psi=\Lambda\varphi,\ \varphi=\Lambda^*\psi$, where Λ^* is the adjoint operator. Necessary for numerical solution of obtained above spectral problem mathematical and algorithmic constructions are derived from the next assertion on equivalence of this problem to the infinite system of equations. *Let* $\{e_i\}_{i=1}^{\infty}$ *be an arbitrary orthonormal basis of Hilbert space* B *and* R *be the real line. Then the original spectral problem is equivalent to the problem of finding the triple* $(s,\psi,\varphi)\in R\times B\times B$ *with nonzero components of the following system of equations*

$$s^2(\psi,e_i)=(\Lambda^*\psi,\Lambda^*e_i),$$
$$s^2(\varphi,e_i)=(\Lambda\varphi,\Lambda e_i),\ i=1,2,...$$

Regarding the Fourier coefficients $\alpha_i=(\psi,e_i)$ and $\beta_i=(\varphi,e_i)$ one can get two systems with symmetric positive definite matrices of infinite dimension. After truncation of these matrices up to finite length l we apply a special algorithm for computation of the spectrum of truncate matrices and corresponding eigenvectors $(\alpha_1,...,\alpha_l)$ and $(\beta_1,...,\beta_l)$. The effectiveness of proposed approach was tested on the example of the following operator of m - multiple integration $(\Lambda x)(t)=\int_a^t \frac{(t-s)^{(m-1)}}{(m-1)!}x(s)ds$ considered in the space $L_2[a,b]$ of quadratically summable on the interval $[a,b]$ scalar functions.

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