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Coupled Motion of a Rigid Body and Point Vortices on a Two-Dimensional Spherical Surface

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Abstract—The paper is concerned with a class of problems which involves the dynamical interaction of a rigid body with point vortices on the surface of a two-dimensional sphere. The general approach to the 2D hydrodynamics is further developed. The problem of motion of a dynamically symmetric circular body interacting with a single vortex is shown to be integrable. Mass vortices on S^2 are introduced and the related issues (such as equations of motion, integrability, partial solutions, etc.) are discussed. This paper is a natural progression of the author's previous research on interaction of rigid bodies and point vortices in a plane.

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Contents

In this paper we consider the problem of motion of a circular body interacting dynamically with point vortices on the surface of a two-dimensional sphere. In its spirit this problem traces back to the classical texts of Beltrami, Bogomolov, Gromeka, Lamb, and Zermelo, which will be discussed in detail in the main text. Since this journal issue is dedicated to the 60th birthday of our teacher

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Valery Vasil'evich Kozlov, we would like to give him credit for his constant encouragement and emphasize his role in directing our interest to this field.

First of all let us take a quick look at V. V. Kozlov's achievements in the vortex hydrodynamics. On the one hand we should mention his works concerned with the general vortex theory [49] in which Valery Vasil'evich (V. V.) attempts to look through the prism of the general concepts of this theory at various problems from optic, thermodynamics, quantum mechanics and other fields. In particular, V. V. has elaborated a new approach to integration of Hamiltonian system (an extension of the Hamilton–Jacobi method). On the other hand many of his recent publications address the development of the "weak limit" theory and its application to the study of asymptotic behavior and statistical properties of large clusters of point vortices.

In addition to these two lines of investigation, V.V. Kozlov has a long-standing interest in the classical problem of motion of a rigid body in an ideal fluid. Among Kozlov's papers dealing with this topic we mention the following: study of integrability of the equations of motion [51], the falling motion of a heavy plate in an ideal fluid [45], motion of a heavy 2D rigid body with circulation around it [46], motion of a body in a resisting medium [47]. Starting with the "most trivial" model of inviscid ideal fluid and then introducing new effects (such as circulation, dissipation, and vorticity), V. V. obtained more and more realistic models for analytical treatment of motion of bodies in real medium. This naturally led him to the problem of dynamical interaction between point vortices and rigid bodies.

It is known from the classical hydrodynamics that vortices shed from sharp edges of a body and then form rather curious vortical structures (e.g., Karman's wakes). Nevertheless, in spite of the ample literature on the subject (dealing for the most part with numerical analysis of various empirical models), until quiet recently there have been no rigorous analytical expressions for the hydrodynamic reaction that point vortices exert on the body. It is also well known that the formation of new vortices (the very process of shedding) cannot be explained within the realm of the ideal-fluid model. At the same time, this model provides a fairly good description of interaction between the body and the already shed vortices. So we hope that author's strict analytical results in this direction are both of theoretical interest and are useful for verification of more complicated models.

The use of point vortices in obtaining a more realistic model of interaction between a body and the ambient medium was illustrated by V.V. by the following popular example. Consider a moving car with a small flag on the hood. The flag is made of a rough, almost inflexible material and can be thought of as a stiff rectangular piece of cardboard which can rotate about the vertical axis. When the car moves at a low speed, the flag is in equilibrium: its plane is aligned with car's velocity. As soon as the speed exceeds a certain critical value, this equilibrium loses stability: the flag starts to oscillate about the vertical. This phenomenon, the birth of a limit cycle from an unstable equilibrium, is referred to as the Hopf bifurcation. Kozlov tried to obtain this limit cycle analytically by introducing circulation and dissipative effects (using Rayleigh's dissipation function), but all in vain: the behavior of the flag changed (new equilibriums emerged), but oscillations never occurred. In this connection, V. V. suggested that, despite everything, the existence of the oscillation regime can be proved analytically if the shedding of vortices from the flag's rear edge is taken into consideration. Thus, the oscillation is due to the interaction between the flag and the point vortices in the incident flow.

From the preceding it is now clear why V. V. became interested in the problem of falling motion of a rigid body (e.g. a plate) in an ideal fluid in the presence of point vortices. This is precisely the problem V. V. posed about 14 years ago to one of the authors (S. M. Ramodanov). In this general statement, this problem remains unsolved. (In this connection, a mention should be made of a recent intriguing paper [48], which addresses this problem: the shedding of vortices in a prescribed manner from body's edges is postulated, the effect on the falling body is calculated numerically and in particular it is shown that due to the vortices the broadside-on fall can become unstable.) Adopting some simplifications (according to which there is just one vortex, the body is a circular cylinder and there is no gravity), S. M. Ramodanov derived the equations of motion for the "body+vortices" system and partly investigated its behavior. Upon inspection of numerically plotted orbits, Ramodanov came to the conclusion that the system of a circular cylinder interacting with a single point vortex is integrable, yet he failed to prove that. In its turn Kozlov asked for assistance from A. V. Borisov and I. S. Mamaev. It should be noted that exploration of integrability in vortex dynamics is rather specific and requires a sound knowledge of the theory of Poisson structures, Lie algebras, and modern methods of reduction of dynamical system. With this technique the determination of an additional first integral (of course, for integrable systems) is almost straightforward. Here is how A. V. Borisov, I. S. Mamaev and S. M. Ramodanov, owing to V. V. Kozlov, became a team and soon proved the predicted result. Since then we have worked together and solved a number of interesting problems being inspired by V. V.: motion of a body with circulation [20], interaction of several rigid bodies in an ideal fluid [22, 50], etc. In the limit of infinitely small bodies new concepts of *massive vortices* [22, 38] and *dynamical advection* [6] were introduced. The analysis of motion of two spheres in an ideal fluid [50] stimulated further investigation on non-linear reduction (the Poisson bracket of phase variables is not their linear combination).

This paper addresses a class of problems which involves the dynamical interaction of a rigid body with point vortices on the surface of a two-dimensional sphere. Our analysis is strongly based on XIX and XX century fundamental works on hydrodynamics in curved spaces and, at the same time, this paper is a natural progression of the author's previous research on interaction of rigid bodies and point vortices in a plane, initiated by V. V. Kozlov.

INTRODUCTION

The investigation of flows of an ideal incompressible fluid with the velocities of the fluid particles parallel to a certain plane (so-called plane-parallel flows) is a classical, well-elaborated area of fluid dynamics. In contrast, the problem of fluid motion over an arbitrary two-dimensional surface appears to be poorly studied. The earliest investigations of the motion of a curved fluid layer trace back to the second half of the 19th century; i.e., to electrodynamical studies by Beltrami, Boltzmann, Kirchhoff, Umov, etc. (parallels between electrodynamics and fluid mechanics were discussed in detail, e.g., even by Poincaré [36]); however, these studies are restricted to the case where a single-valued potential exists, i.e., the case of irrotational flows.

The first systematic study of *vortical* fluid motion on two-dimensional surfaces was done by the Russian mechanician I. S. Gromeka [7]. He postulated the very concept of point vortices (invoking hydrodynamic considerations), derived the equations of their motion (restricting himself, however, to the cases of a sphere and a circular cylinder), and analyzed, in particular, the problem of motion of a point vortex in bounded regions on a sphere [7]. Some his results were independently obtained by modern researchers (see, e.g., Kidambi and Newton [30] and Crowdy [27]).

The famous German scientist E. Zermelo (widely known for his fundamental achievements in the set theory and works on statistical mechanics) undertook the most complete and systematic investigation of two-dimensional fluid mechanics. Studies in vortical fluid mechanics were the subject of his dissertation, which consists of two parts [43, 44] (only the first of which has been published). A detailed discussion of Zermelo's dissertation in a scientific–historical context was given by Borisov *et al.* [3].

Gromeka's studies were principally aimed at obtaining the equations of motion of point vortices over a spherical surface. As for Zermelo, his investigation is a fundamental construction of two-dimensional fluid mechanics; he considers the flow of an ideal fluid over an arbitrary surface and proves analogs of the basic theorems of the classical fluid mechanics on a plane, (the Bernoulli integral, the Helmholtz theorem, an the conservation of energy). Only upon a comprehensive development of general theory [43, § 1], Zermelo concentrates on the problem of motion of point vortices [43, § 2], [44]. His study [44] (which is scientifically highly valuable but exists only as a hand-written manuscript; see [3]) presents a detailed investigation of the problem of motion of three point vortices on a sphere; this problem is reduced to quadratures, and a general view of motion is formed using the theory of elliptic functions (which was evolved in detail at Zermelo's time but has been largely forgotten by now).

Zermelo also obtained for the first time a Hamiltonian form of the equations of motion of point vortices. Similar equations were derived in the 1970s by V.A. Bogomolov [1]. In the modern literature, Bogomolov's studies [1, 2] are cited as historically the first systematic investigations of the dynamics of point vortices on a sphere; however (see [3]), as we can see, these problems were rigorously analyzed about a century earlier by Zermelo and Gromeka. We also note that many other results presented in Zermelo's dissertation were independently obtained and developed in modern studies (e.g., [16, 24, 31]). An alternative hydrodynamic model of the motion of point vortices on a sphere is proposed in [15].

An attempt of finding the equations of motion of point vortices over a surface of revolution was made by Hally [29], while Boatto and Koiller consider in their preprint [14] the general case of motion on surfaces. The results obtained in [14, 29] are largely tentative.

Here, we briefly describe the ideas of and methods used by Gromeka and Zermelo [7, 43, 44], whose approaches to the problem of motion of ideal fluid over a surface differ in their ideological backgrounds. We will also develop here these approaches considering two model problems:

(1) the motion of a circular rigid body (spherical segment) over a spherical surface with the presence of circulation and
(2) the motion of a circular rigid body interacting with point vortices.

For a plane, various forms of the equations of motion that describe a planar interaction between a rigid circular cylinder and vortices in an ideal fluid were recently (nearly simultaneously) obtained [38–40]. A Hamiltonian form of these equations of motion with a nontrivial Poissonian structure was found and the integrability of the equations of motion of a circular cylinder interacting with a point vortex was noted [21, 23]. A different Hamiltonian structure of the equations of motion was also found [40]. The relationship between these two Poissonian structures is investigated in a recent review article [42].

1. FLUID DYNAMICS ON TWO-DIMENSIONAL SURFACES

1.1. The Dynamics of Ideal Fluid on Two-Dimensional Surfaces

Equations of motion in an Eulerian form. Let S be a two-dimensional surface with a metric $ds^2 = Ed\xi^2 + Gd\eta^2$. The motion of a point with a mass m over S under the action of force with a potential mV is governed by the regular Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\xi}\right) = \frac{\partial L}{\partial \xi}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\eta}\right) = \frac{\partial L}{\partial \eta}, \tag{1.1}$$

where $L = \frac{m}{2}(E\dot\xi^2 + G\dot\eta^2) - mV$.

Remark 1. As a rule, we assume here that S is a two-dimensional surface immersed in three-dimensional Eucledian space, although this constraint is not necessary with our approach.

It is known (and also noted by Zermelo in his study) that a transition from the equations of motion of discrete masses (1.1) to the equations of motion of ideal fluid requires

– replacing the mass m with the surface density $\rho(\xi, \eta)$ and
– adding the pressure gradient $p(\xi, \eta)$ to the right-hand side of the equations.

Thus, we finally obtain

$$\begin{aligned}
&\frac{d}{dt}\left(u\sqrt{E}\right) + u^2\sqrt{E}\frac{\partial(1/\sqrt{E})}{\partial\xi} + v^2\sqrt{G}\frac{\partial(1/\sqrt{G})}{\partial\xi} = -\frac{1}{\rho}\frac{\partial p}{\partial\xi} - \frac{\partial V}{\partial\xi},\\
&\frac{d}{dt}\left(v\sqrt{G}\right) + u^2\sqrt{E}\frac{\partial(1/\sqrt{E})}{\partial\eta} + v^2\sqrt{G}\frac{\partial(1/\sqrt{G})}{\partial\eta} = -\frac{1}{\rho}\frac{\partial p}{\partial\eta} - \frac{\partial V}{\partial\eta},
\end{aligned} \tag{1.2}$$

where $u = \dot\xi\sqrt{E}$ and $v = \dot\eta\sqrt{G}$ are the components of the particle velocity in the directions of the coordinate lines ξ and η, and the total derivative with respect to time has the form $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{\sqrt{E}}\frac{\partial}{\partial\xi} + \frac{v}{\sqrt{G}}\frac{\partial}{\partial\eta}$, V being the volumetric potential of the external forces.

To obtain a closed system, we have to complement equations (1.2) with the continuity equation

$$\sqrt{EG}\frac{\partial\varrho}{\partial t} + \frac{\partial(u\varrho\sqrt{G})}{\partial\xi} + \frac{\partial(v\varrho\sqrt{E})}{\partial\eta} = 0. \tag{1.3}$$

From here on, we will everywhere assume that the *barotropy* condition is satisfied, viz., the pressure p and density ρ are related by the conditions that $\frac{dp}{\rho}$ is again the differential of a certain function P, i.e.,

$$\frac{1}{\rho}\frac{\partial p}{\partial \xi} = \frac{\partial P}{\partial \xi}, \quad \frac{1}{\rho}\frac{\partial p}{\partial \eta} = \frac{\partial P}{\partial \eta}.$$

Therefore, in the case of borotropy, the differential of the function $\Phi = P + V$ appears in the right-hand side of equation (1.2).

Remark 2. The barotropy condition is necessarily satisfied for incompressible fluid, $\rho = \text{const}$.

Equations of motion in the Gromeka–Lamb form. The Helmholtz theorem. We rewrite equations (1.2) in a somewhat different form to obtain an analog of the Cauchy–Lagrange integral, from which, in turn, we will be able to express the pressure of the fluid. We introduce the *vorticity* of the fluid motion as

$$\Omega = \frac{1}{\sqrt{EG}}\Big(\frac{\partial}{\partial \xi}(v\sqrt{G}) - \frac{\partial}{\partial \eta}(u\sqrt{E})\Big) \tag{1.4}$$

and use (1.2) to obtain equations analogous to those known in classical fluid mechanics as the *Gromeka–Lamb* equations,

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Omega v - \frac{1}{\sqrt{E}}\frac{\partial}{\partial \xi}\Big(\Phi + \frac{u^2+v^2}{2}\Big),\\ \frac{\partial v}{\partial t} &= -\Omega u - \frac{1}{\sqrt{G}}\frac{\partial}{\partial \eta}\Big(\Phi + \frac{u^2+v^2}{2}\Big).\end{aligned} \tag{1.5}$$

Equations (1.5) and the continuity equation (1.3) can be used to easily deduce a statement referred to by Zermelo as the *Helmholtz theorem*.

Theorem 1. *For a barotropic flow of an ideal, incompressible fluid in a potential field of mass forces, the circulation* $\varGamma_C = \int\limits_C u\sqrt{E}\,d\xi + v\sqrt{G}\,d\eta$ *along a closed curve* C *consisting of the same fluid particles remains invariable in time.*

This theorem (for a planar case) is more widely known in the literature [34] as the *Thomson theorem*. The generally known Helmholtz theorem on the vorticity transfer by the flow follows from the former as a simple consequence.

Incompressible fluid. The stream function. We now dwell on the case of a *incompressible, homogeneous* fluid ($\rho = \text{const}$). Assume that there are no mass forces $V = 0$ and, therefore, $\Phi = \frac{p}{\rho}$. By analogy with the planar case, we introduce a stream function $\psi(\xi, \eta)$. We fix a point O on S; then we can set ψ at point A equal to the fluid flux through the curve connecting O and A:

$$\psi(A) = \int\limits_{OA} \boldsymbol{v}_n\,ds,$$

where $\boldsymbol{v}_n$ is the velocity component normal to the curve OA. In view of the incompressibility of the fluid, this flux has the same value for all homotopic curves connecting O and A. As on the plane, the velocity of fluid particles is equal to the skewed gradient of the stream function:

$$(u, v) = \mathbf{J}\nabla\psi,$$

where $\mathbf{J}$ is the operator of rotation by 90 degrees; alternatively, in a coordinate notation,

$$u = \frac{1}{\sqrt{G}}\frac{\partial \psi}{\partial \eta}, \quad v = -\frac{1}{\sqrt{E}}\frac{\partial \psi}{\partial \xi}. \tag{1.6}$$

Therefore, the vorticity (1.3) and stream function ψ are related by the equation

$$\Omega = -\Delta\psi, \tag{1.7}$$

where

$$\Delta = \frac{1}{\sqrt{EG}}\Big(\frac{\partial}{\partial\xi}\Big(\sqrt{\frac{G}{E}}\frac{\partial}{\partial\xi}\Big) + \frac{\partial}{\partial\eta}\Big(\sqrt{\frac{E}{G}}\frac{\partial}{\partial\eta}\Big)\Big) \tag{1.8}$$

is the Laplace–Beltrami operator on S. We can use (1.3), (1.5), and (1.7) to show [43] that the vorticity Ω satisfies the partial differential equation

$$\frac{\partial\Omega}{\partial t} = \frac{1}{\sqrt{EG}}\left(\frac{\partial\psi}{\partial\eta}\frac{\partial\Omega}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial\Omega}{\partial\eta}\right), \tag{1.9}$$

an analog of the Helmholtz equation for the case of a plane. We also note that, in view of (1.7), equation (1.9) is a *third-order partial differential equation that must be satisfied by the stream function of ideal incompressible fluid,* $\psi(\xi,\eta,t)$.

Potential flows. If there is no vorticity ($\Omega = 0$), the flow has a velocity potential φ,

$$u = \frac{1}{\sqrt{E}}\frac{\partial\varphi}{\partial\xi}, \quad v = \frac{1}{\sqrt{G}}\frac{\partial\varphi}{\partial\eta}, \tag{1.10}$$

and the stream function ψ and the potential φ are harmonic conjugate functions on S, i.e.,

$$\begin{gathered}\Delta\varphi = \Delta\psi = 0,\\ \frac{\partial\varphi}{\partial\xi} = \sqrt{\frac{E}{G}}\frac{\partial\psi}{\partial\eta}, \quad \frac{\partial\varphi}{\partial\eta} = -\sqrt{\frac{G}{E}}\frac{\partial\psi}{\partial\xi}.\end{gathered} \tag{1.11}$$

The Cauchy–Lagrange integral. Under the assumption of constant vorticity, an analog of the Cauchy–Lagrange integral can easily be obtained from (1.5) (to the point, it was not noted by Zermelo). We find (at $\rho = 1$) from (1.5) and (1.7) that

$$\begin{aligned}\frac{\partial}{\partial\xi}\Big(p + \frac{u^2+v^2}{2}\Big) &= \Omega\frac{\partial\psi}{\partial\xi} + \sqrt{\frac{E}{G}}\frac{\partial}{\partial\eta}\Big(\frac{\partial\psi}{\partial t}\Big),\\ \frac{\partial}{\partial\eta}\Big(p + \frac{u^2+v^2}{2}\Big) &= \Omega\frac{\partial\psi}{\partial\eta} - \sqrt{\frac{G}{E}}\frac{\partial}{\partial\xi}\Big(\frac{\partial\psi}{\partial t}\Big).\end{aligned} \tag{1.12}$$

Since we have $\Omega = \mathrm{const}$ for the vorticity, it can be found from (1.7) that $\Delta\frac{\partial\psi}{\partial t} = 0$, i.e., the following statement is valid:

Proposition 1. *Let* $\psi(\xi,\eta,t)$ *be the stream function of barotropic flows on an arbitrary two-dimensional surface of an incompressible fluid with* $\Omega = \mathrm{const}$*; then the function* $\frac{\partial\psi}{\partial t}$ *is harmonic.*

Consider a function $\varphi(\xi,\eta,t)$ such that $\frac{\partial\varphi(\xi,\eta,t)}{\partial t}$ is a function harmonic on S and conjugate to $\frac{\partial\psi(\xi,\eta,t)}{\partial t}$ (the fact that the function $\frac{\partial\psi}{\partial t}$ is harmonic in the case of constant vorticity was noted repeatedly [34]). In view of the last two relationships (1.11), the right-hand sides of equations (1.12) are derivatives of the same function; therefore, the following theorem is valid:

Theorem 2 (The Cauchy–Lagrange integral). *For barotropic flows of an ideal, incompressible fluid with a constant vorticity* $\Omega = \mathrm{const}$*, the relationship*

$$p + \frac{u^2+v^2}{2} = \Omega\psi - \frac{\partial\varphi}{\partial t} + f(t). \tag{1.13}$$

holds.

Note that the arbitrary function of time, $f(t)$, can vanish if an appropriate gauge transformation $\varphi \to \varphi + \int f(t)\,dt$, which does not affect the velocity field, is applied.

1.2. Point Vortices on Two-Dimensional Surfaces

Let us survey once again certain currently known results concerning surfacial motion. As already noted, the studies [7] and [43, 44] present ideologically different approaches to defining a point vortex on the sphere. The motion of point vortices over a sphere is discussed in detail in [16, 24, 25, 35]. A model of vortical motion over the sphere differing from the classical one was recently introduced in [15]. As we noted, the equations of motion of point vortices over surfaces of revolution were obtained by Hally [29] under the assumption that the total intensity of the vortices is zero and the solution of the Poisson equation (1.7) is known. A generalization of these particular results to the case of an arbitrary compact two-dimensional surface, interesting from the standpoint of differential geometry, is given in the recent study [14]. Note also [32], where the case of Lobachevsky's plane H^2 is considered along with the case of the sphere S^2. In [14], the hypothesis that the vortical dipole moves over a two-dimensional surface in a geodesic [32] is substantiated. However, little is currently known even on the motion of a single point vortex (self-advance) on an arbitrary two-dimensional surface (possibly, a solution for a triaxial ellipsoid can be obtained in elliptic quadratures). We consider various possible situations in greater detail.

Point vortices on compact surfaces. A vortical flow on a compact surface S has the following particular property distinguishing it from those on a plane:

the total vorticity over the whole surface is zero.

Indeed, let D be a region in S. According to the Stokes theorem, the circulation of velocity along the boundary ∂D is equal to the double integral of the vorticity Ω over D; on the other hand, it is equal to the double integral of Ω over M/D, with the sign reversed; thus, $\int\limits_S \Omega\, dS = 0$.

Therefore, vorticity cannot be concentrated at one point of a compact surface. A point vortex can exist on S only with the presence of

1. either another vortex with the same but sign-reversed intensity [1, 7],
2. or a background vorticity [14, 43, 44].

Equation of motion of point vortices on a plane. Thus, prior to explicitly writing the equations of motion for a given surface S, it is necessary to solve the Poisson equation

$$\Delta\psi = \Omega, \tag{1.14}$$

where Ω is a function constant on S (except for the points where the vortices are located) and Δ is the Beltrami–Laplace operator on S.

Remark 3. More precisely, Ω is the sum of Dirac δ functions and a certain constant.

The motion of a point vortex $\boldsymbol{s}_0 = (\xi_0, \eta_0)$ can be described in terms of a "desingularized" stream function known in Riemann geometry as the Robin function [26],

$$\dot{\xi}_0 = \frac{1}{G}\frac{\partial\psi_R(\boldsymbol{s}_0)}{\partial\eta_0}, \qquad \dot{\eta}_0 = -\frac{1}{E}\frac{\partial\psi_R(\boldsymbol{s}_0)}{\partial\xi_0}, \qquad \psi_R(\boldsymbol{s}_0) = \lim_{s\to s_0}\left(\psi(\boldsymbol{s}, s_0) - \frac{\ln d(\boldsymbol{s}, \boldsymbol{s}_0)}{2\pi}\right), \tag{1.15}$$

where the stream function ψ is a solution of the Poisson equation (1.14) with a singularity at the point $\boldsymbol{s}_0$ and $d(\boldsymbol{s}, \boldsymbol{s}_0)$ is the geodesic distance from $\boldsymbol{s}$ to $\boldsymbol{s}_0$. It is noted in [14] that, in contrast to the case of a plane or sphere, a point vortex on a varying-curvature surface S can move under the action of the flow produced by this vortex itself.

Note in this context that the explicit solution of equation (1.14) is known only for very few surfaces. In particular, the equations of motion of vortices on a triaxial ellipsoid have not yet been found in an explicit form. Let us also mention the study [26] in this connection, which considers the motion of vortices over the surface of an ellipsoid of revolution obtained by slightly perturbing a sphere, $x^2 + y^2 + x^2/(1+\varepsilon) = R^2$. Approximate, to terms of order $O(\varepsilon^2)$, equations of motion of vortices are derived in [26]; the cases of two and (numerically) three vortices are also investigated.

The Zermelo–Bogomolov point vortex on the sphere. On the unit sphere $x^2 + y^2 + z^2 = 1$, we introduce spherical coordinates $x = \sin\theta\cos\lambda, y = \sin\theta\sin\lambda, z = \cos\theta$ and seek a solution of equation (1.14), which now becomes

$$\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\lambda}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\lambda}\right)\right) = -\Omega, \tag{1.16}$$

in the form $\psi = \psi(\theta)$ setting $\Omega = \mathrm{const}$. Then, we immediately find

$$\psi = -2\Omega\ln\sin\frac{\theta}{2}. \tag{1.17}$$

The circulation of the fluid along the parallel specified by the angle θ is $-2\pi\Omega(1 + \cos\theta)$.

Definition ([43]). *We will say that a point vortex of intensity $2\pi\Omega$ is located at the point $\theta = 0$ if the flow of the fluid is described by the stream function* (1.17)*. Note that the condition of zero total vorticity is satisfied in this case.*

Remark. If this "naive" definition of the point vortex is adopted, it is not easy to prove that the vortices move together with the flow. The proof suggested by Zermelo [44] can hardly be called transparent. Unfortunately, for unknown reasons, even some classical textbooks assume that this statement is obvious and does not require any proof. It is more correct to define point vortices using a limiting process, i.e., first considering the dynamics of vortical spots of a small radius r and a constant vorticity Ω and then requiring that $r \to 0$ and $\Omega \to \infty$ with the product Ωr^2 remaining constant (see [7, 11, 14]).

For the system of several vortices with spherical coordinates (θ_i, λ_i) and intensities $\Gamma_i = 2\pi\varepsilon_i$, the stream function is the sum of functions of the form (1.17). We assume that each vortex moves with a velocity induced by the other point vortices, thus representing the equations of motion in the form

$$\varepsilon_i\sin\theta_i\frac{d\theta_i}{dt} = -\frac{\partial H}{\partial\lambda_i}, \quad \varepsilon_i\sin\theta_i\frac{d\lambda_i}{dt} = \frac{\partial H}{\partial\theta_i}, \quad i = 1, \ldots, n. \tag{1.18}$$

Here, the Hamiltonian function H is

$$H = \sum_{i<j}\varepsilon_i\varepsilon_j\ln\sin\frac{r_{ij}}{2}, \tag{1.19}$$

where $r_{ij} = 2R^2(1 - \cos\theta_i\cos\theta_j - \sin\theta_i\sin\theta_j\cos(\lambda_i - \lambda_j))$ is the chord distance between the ith and jth vortices.

As already noted, Zermelo's equations are identical in their form with Bogomolov's equations, which read as [1]:

$$\dot\theta_k = \{H, \theta_k\}, \qquad \dot\lambda_k = \{H, \lambda_k\}, \qquad \{\lambda_k, \cos\theta_k\} = \frac{\delta_{ik}}{R^2\Gamma_i};$$

here, H differs from (1.19) only by an unimportant factor.

Antipodal point vortices on the sphere. Except the solution (1.17), equation (1.16) also admits a solution of the form

$$\psi = \alpha\cdot\ln\left(\mathrm{tg}\,\frac{\theta}{2}\right), \quad \alpha = \mathrm{const},$$

which corresponds to the presence of vortices of opposite intensities at the poles, with $\Omega = 0$ at all other points of the sphere.

It turns out that such a vortex pair can be considered a whole hydrodynamic object, so-called *antipodal vortex.*

The motion of antipodal point vortices is investigated in the recent study [15]; there, it is particularly shown that, if two vortices of opposite intensities are initially located at the ends of the same diameter, they will preserve this arrangement in the course of motion. In addition, a classification of motions is given in [15] for the *integrable* problem of three antipodal vortices.

Studies by I. S. Gromeka. In contrast to Zermelo, I. S. Gromeka comes to considering vortical motions on two-dimensional surfaces from studying flows of ideal fluid in three-dimensional space, which are parallel to a certain family of surfaces.

We choose an orthogonal system of curvilinear coordinates ξ_1, ξ_2, ξ_3, in which the considered family of surfaces is specified by the equation $\xi_3 = \text{const}$ and whose metric has the form

$$ds^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2 + h_3^2 d\xi_3^2,$$

where h_i are functions of the coordinates (the Lamé coefficients). We also assume, following I. S. Gromeka, that

$$h_3 = \text{const}. \tag{1.20}$$

Remark. It is well known [28] that any two-dimensional surface can be embedded in a triorthogonal family; i.e., orthogonal curvilinear coordinates can be specified in the neighborhood of any surface. At the same time, not any one-parameter family of surfaces can be completed to forming a triorthogonal family (the corresponding families are termed the Lamé families).

Let u_1, u_2, u_3 be the components of the fluid-particle velocity in the chosen coordinate system; then the condition of the flow being parallel to the family $\xi_3 = \text{const}$ can be represented as

$$u_3 = 0, \quad \frac{\partial}{\partial \xi_3}(h_1 u_1) = \frac{\partial}{\partial \xi_3}(h_2 u_2) = 0. \tag{1.21}$$

Under the above assumption (1.20) on the family of surfaces, it can be easily shown that, if the flow is initially parallel to the surfaces $\xi_3 = \text{const}$, it will remain parallel to $\xi_3 = \text{const}$ at any later time.

By substituting (1.21) into the equations specifying the vorticity $\boldsymbol{\omega} = \operatorname{rot} \boldsymbol{v}$ of the fluid,

$$\omega_i = \sum_{j,k} \varepsilon_{ijk} \frac{1}{h_j h_k} \frac{\partial h_k u_k}{\partial \xi_j},$$

(where ε_{ijk} is the antisymmetric Levi-Civita tensor), we find

$$\omega_1 = \omega_2 = 0, \quad \omega_3 = \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 u_2)}{\partial \xi_1} - \frac{\partial(h_1 u_1)}{\partial \xi_2}\right) = \Omega(\xi_1, \xi_2, \xi_3). \tag{1.22}$$

We use the continuity equation for incompressible fluid,

$$\sum_i \frac{\partial}{\partial \xi_i}\left(\frac{h_1 h_2 h_3}{h_i} u_i\right) = 0,$$

and determine the stream function ψ for the flows parallel to the surfaces $\xi_3 = \text{const}$ according to the formulas

$$u_1 = -\frac{1}{h_2 h_3}\frac{\partial \psi}{\partial \xi_2}, \quad u_2 = \frac{1}{h_1 h_3}\frac{\partial \psi}{\partial \xi_1}.$$

Thus, to find the flows of an incompressible fluid parallel to the surface $\xi_3 = \text{const}$ with a given vorticity (1.22), it is necessary to solve the equation

$$\frac{1}{h_1 h_2}\left(\frac{\partial}{\partial \xi_1}\left(\frac{h_2}{h_1}\frac{\partial \psi}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{h_1}{h_2}\frac{\partial \psi}{\partial \xi_2}\right)\right) = \Delta\psi = -h_3 \Omega(\xi_1, \xi_2, \xi_3),$$

where Δ is the Laplace–Beltrami operator on S; the variable ξ_3 here appears as a parameter.

The simplest surfaces satisfying the above-formulated conditions are

- parallel planes;
- coaxial circular cylinders;
- concentric spheres.

The curl vector of the velocity (1.22) is normal to these surfaces.

Let us dwell on the case of the sphere. The Lamé coefficients for the regular spherical coordinates (r, θ, φ) in $\mathbb{R}^3$ are $h_r = 1$, $h_\theta = r$, $h_\varphi = r \sin\theta$, i.e., the conditions (1.20) is satisfied.

Remark. I. S. Gromeka himself uses either a stereographic projection or the coordinates of a Mercator projection, for which $h_1 = h_2$. However, this has obviously no effect on the result.

Later, I. S. Gromeka assumes that

$\Omega = 0$ *everywhere but at the points where the vortic4es are located.*

This assumption certainly eliminates the background vorticity. In view of this, I. S. Gromeka concentrates on the problem of *the motion of n point vortices of arbitrary intensity in a subregion of the sphere bounded by motionless, impermeable walls.*

In this case, the total vorticity over the whole sphere remains zero due to mirror images of the vortices with the opposite vorticity outside the subregion considered. Thus, the limiting assumption that the total vorticity Ω is zero, introduced by Gromeka himself, prevented him from obtaining the equations of motion of the vortices on the entire sphere (1.18) by shrinking the boundary of the region to a point.

2. THE MOTION OF A CIRCULAR RIGID BODY WITH A CIRCULATION ON S^2

It is well known [33] that, in regular Eucledian space E^3 and on the plane $\mathbb{R}^2$, the motion of a rigid body in an infinite volume of ideal incompressible fluid that flows irrotationally and rests at infinity can be described by a finite-dimensional Hamiltonian system of equations known as the Kirchhoff equations. Chaplygin [12] has shown that, in the case of a plane-parallel motion of a rigid body with the presence of a constant circulation around the body, terms linear in velocities appear in the right-hand sides of the Kirchhoff equations. (Chaplygin has also proved the integrability of this system.) The study [20] contains a proof of the nonintegrability of the Chaplygin system in the case of a gravitational field present and a brief survey of the known results. Let us note equations similar to the Chaplygin equations for the particular case where the moving body is bounded by a circular contour and circulation is present on the surface of a two-dimensional sphere.

Equation of motion. Let the sphere be specified in a motionless Cartesian coordinate system by the equation

$$x^2 + y^2 + z^2 = R^2$$

and let the body be a circular spot (spherical segment) of radius R_1 (see Fig. 1); we assume that, outside the body, the sphere is covered with a homogeneous, incompressible fluid (and its density is equal to unity). Let the circulation of the fluid around the body be $\Gamma^* = \text{const}$; according to the Helmholtz theorem (see Section 1), it remains conserved in the course of motion. Let $\boldsymbol{r}_0$ be a vector connecting the center of the sphere with the center of the body.

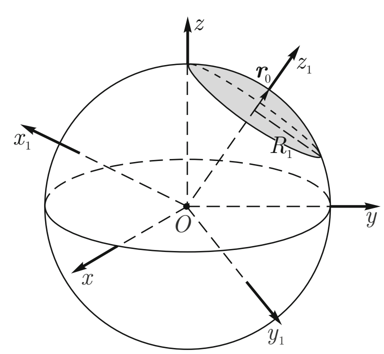

Fig. 1.

We will obtain the equations of motion of the body in view of the fact that only the normal reaction of the constraint (on the side of the sphere) and the hydrodynamic pressure act on the body. We fix the moving coordinate system $Ox_1y_1z_1$ to the body (see Fig. 1) assuming that the Oz_1

axis runs through its center. It can be seen from the figure that the motion of the two-dimensional body over the sphere is equivalent to the motion of a normal (three-dimensional) body with a fixed point coinciding with the center of the sphere, O. We denote the moments of inertia of the equivalent three-dimensional rigid body as A, B, C and write the theorem on angular-momentum changes in the projections onto the moving axes (the Euler equations):

$$A\dot{\omega}_1 + (C - B)\omega_2\omega_3 = \mathcal{M}_1, \quad B\dot{\omega}_2 + (A - C)\omega_3\omega_1 = \mathcal{M}_2, \quad C\dot{\omega}_3 + (B - A)\omega_1\omega_2 = \mathcal{M}_3. \tag{2.1}$$

Here, ω_i and $\mathcal{M}_i$ are the projections of the (three-dimensional) body's angular velocity and of the external-forces moment onto the moving axes $Ox_1y_1z_1$. We assume the mass distribution in the body to be arbitrary, so that, generally, $A \neq B$.

Since the direction of the normal reaction acting on the body runs through the center of the sphere, its moment is zero; the moment of the pressure forces exerted by the fluid, $\boldsymbol{\mathcal{M}}^L$, is calculated in Appendix (Section 5); it is equal to

$$\begin{gathered} \boldsymbol{\mathcal{M}}^L = -a\boldsymbol{r}_0 \times \boldsymbol{a}_0 - \Gamma^* \frac{R + d}{2}\boldsymbol{v}_0, \\ a = \pi R_1^2, \quad d = \sqrt{R^2 - R_1^2}, \end{gathered} \tag{2.2}$$

where $\boldsymbol{v}_0$ and $\boldsymbol{a}_0$ are the velocity and acceleration of the body's center, respectively. The first term is due to the effect of associated masses (as in the planar case, it is proportional to the acceleration of the body), and the second term is an analog of Zhukowski's lifting force.

We write the velocity $\boldsymbol{v}_0$ and acceleration $\boldsymbol{a}_0$ in terms of the angular velocity of the body as

$$\boldsymbol{v}_0 = \boldsymbol{\omega} \times \boldsymbol{r}_0, \quad \boldsymbol{a}_0 = \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_0 + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_0) \tag{2.3}$$

to obtain the equations of motion in the form

$$\mathbf{I}\dot{\boldsymbol{\omega}} + a\boldsymbol{r}_0 \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}_0) = \big(\mathbf{I}\boldsymbol{\omega} + a\boldsymbol{r}_0 \times (\boldsymbol{\omega} \times \boldsymbol{r}_0)\big) \times \boldsymbol{\omega} - \Gamma^* \frac{R + d}{2} \boldsymbol{\omega} \times r_0, \tag{2.4}$$

where $\mathbf{I} = \operatorname{diag}(A, B, C)$.

In view of the fact that $\boldsymbol{r}_0 = (0, 0, R)$ in the chosen moving coordinate system, we represent equation (2.4) in the form

$$\tilde{\mathbf{I}}\dot{\boldsymbol{\omega}} = (\tilde{\mathbf{I}}\boldsymbol{\omega} + \boldsymbol{k}) \times \boldsymbol{\omega}, \tag{2.5}$$

where $\tilde{\mathbf{I}} = \operatorname{diag}(A + aR^2, B + aR^2, C)$, $\boldsymbol{k} = \left(0, 0, \Gamma^* \frac{R(R+d)}{2}\right)$.

A comparison of (2.5) with the equations of the equilibrated gyrostat without an external field [4] leads us to the conclusion that the following theorem is valid:

Theorem. *If an axisymmetric (two-dimensional) rigid body (with an arbitrary mass distribution) on a sphere is submerged in an ideal incompressible fluid with a nonzero circulation, the dynamics of this body is equivalent to the dynamics of a top in the Zhukowski–Volterra case provided that the gyrostatic moment is directed along the principal axis.*

Remark. As is known, the Zhukowski–Volterra system describes free motion of a rigid body inside which there is a balanced, rotating rotor with a constant gyrostatic-moment vector $\boldsymbol{k}$. Zhukowski [8] noted an analogy between this problem and the dynamics of a rigid body with cavities that are not singly connected (of a torus type) and are filled with a potentially moving nonideal fluid. Our theorem reveals another possible analogy, with a Zhukowski–Volterra system, which is also hydrodynamic.

The following natural generalization of this result appears to be valid;

Conjecture. *The equations describing the motion of an arbitrary body on a sphere with a nonzero circulation are equivalent to the Zhukowski–Volterra system with an arbitrary gyrostatic-moment vector.*

The first integrals. The system (2.5), as is known [4], admits two first integrals — energy and squared momentum:

$$H = \frac{1}{2}(\boldsymbol{\omega}, \tilde{\mathbf{I}}\boldsymbol{\omega}), \qquad F = (\tilde{\mathbf{I}}\boldsymbol{\omega} + \boldsymbol{k}, \tilde{\mathbf{I}}\boldsymbol{\omega} + \boldsymbol{k}).$$

Therefore, this system is integrable; it is qualitatively analyzed in the book [4], where a detailed bibliography concerning various aspects of the dynamics of the Zhukowski–Volterra system is also presented. Let us consider one very simple case in greater detail.

The case of dynamical symmetry. In this case, $A = B$; in particular, if the mass is uniformly distributed in the body,

$$\mu = (C - A)(\boldsymbol{\omega}, \boldsymbol{r}_0)/R^2,$$
$$A = m\frac{4R^3 - d\big(3R^2 + d^2\big)}{6(R - d)}, \quad C = m(d + 2R)(R - d).$$

The equations of motion (2.5) in this case also admit the linear integral

$$\omega_3 = \text{const}. \tag{2.6}$$

To find the trajectory of the motion, we utilize the fact that the momentum vector $\tilde{\mathbf{I}}\boldsymbol{\omega} + \boldsymbol{k}$ is constant in the motionless axes $Oxyz$; we use the integral (2.6) to represent it in the form

$$\begin{gathered}\tilde{\mathbf{I}}\boldsymbol{\omega} + \boldsymbol{k} = \lambda\boldsymbol{\omega} + \mu\boldsymbol{r}_0, \\ \lambda = A + aR^2, \quad \mu = \bar{\Gamma}d + \frac{C - A - aR^2}{R}\omega_3.\end{gathered} \tag{2.7}$$

Since λ and μ are scalar constants, this form remains unchanged in the motionless axes. We take the cross product of (2.7) by $\boldsymbol{r}_0$ and, taking into account the relationship $\dot{\boldsymbol{r}}_0 = \boldsymbol{\omega} \times \boldsymbol{r}_0$, find

$$\lambda\dot{\boldsymbol{r}}_0 = \boldsymbol{K} \times \boldsymbol{r}_0,$$

where $\boldsymbol{K}$ is a constant vector. Therefore, the body moves at a constant velocity v_0 about the vector $\boldsymbol{K}$ in a circle of radius

$$\frac{\lambda R v_0}{\sqrt{\mu^2 R^4 + \lambda^2 v_0^2}} \leqslant R.$$

Thus, we note that *the presence of a circulation prevents the body from moving in a geodesic*: the body moves in a great circle only in the limit of $v_0 \to \infty$.

3. THE COUPLED MOTION OF A RIGID BODY WITH A CIRCULATION AND POINT VORTICES ON S^2

Equations of motion. Assume that, under the conditions of the above-considered problem, not only the body but also a vortex of intensity Γ moves over the surface of the sphere; as previously, the contour of the body is circular and the mass distribution is arbitrary. The position of the vortex is specified by the position vector issuing from the center of the sphere, $\boldsymbol{r}_1$. In this case, an additional term appears in the expression (2.2) for the moment of the pressure forces of the fluid (see Appendix for the derivation); it is proportional to the absolute velocity of the vortex itself, $\boldsymbol{v}_1$, and to the velocity of a certain point, $\tilde{\boldsymbol{v}}_1$, so that

$$\begin{gathered}\boldsymbol{\mathcal{M}}^L = -a\boldsymbol{r}_0 \times \boldsymbol{a}_0 - \Gamma^*\frac{R + d}{2}\boldsymbol{v}_0 + \Gamma R(\tilde{\boldsymbol{v}}_1 - \boldsymbol{v}_1), \\ \tilde{\boldsymbol{r}}_1 = C_0\boldsymbol{r}_0 + C_1\boldsymbol{r}_1, \quad C_0 = \frac{Rd - (\boldsymbol{r}_0, \boldsymbol{r}_1)}{R^2 - (\boldsymbol{r}_0, \boldsymbol{r}_1)}, \quad C_1 = \frac{R(R - d)}{R^2 - (\boldsymbol{r}_0, \boldsymbol{r}_1)}.\end{gathered} \tag{3.1}$$

The geometrical meaning of this vector is clarified in Fig. 2, where $\boldsymbol{r}_1^i$ is the position vector of the point inversely symmetric to the vortex, $\boldsymbol{r}_1$; thus, in a stereographic projection, their images are inversely symmetric with respect to the projection of the body's boundary.

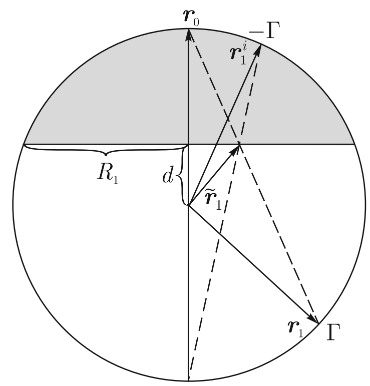

Fig. 2.

In the limit of $R \to \infty$, the vector $\tilde{\boldsymbol{r}}_1$ changes into $\boldsymbol{r}_1^i$ and expression (3.1) becomes completely analogous to the expression for the force acting on the circular cylinder in the planar case [38].

In the motionless coordinate system, the velocity of the vortex is determined by the desingularized stream function (the Kirchhoff–Routh function) according to relationships (1.15). Therefore, in the moving system $Ox_1y_1z_1$ fixed to the body, the evolution of the vortex position is described by the equation

$$\dot{\boldsymbol{r}}_1 = \boldsymbol{r}_1 \times \boldsymbol{\omega} - \frac{1}{R}\boldsymbol{r}_1 \times \frac{\partial \psi_R}{\partial \boldsymbol{r}_1}, \tag{3.2}$$

where (see Appendix)

$$\begin{gathered}\psi_R = \frac{d-R}{R^2 - (\boldsymbol{r}_0, \boldsymbol{r}_1)}(\boldsymbol{r}_1 \times \boldsymbol{v}_0, \boldsymbol{r}_0) + \frac{\Gamma}{4\pi}\ln\big((\boldsymbol{r}_0, \boldsymbol{r}_1) - Rd\big) \\ -\frac{\bar{\Gamma}}{4\pi}\ln\big((\boldsymbol{r}_0, \boldsymbol{r}_1) - R^2\big),\end{gathered} \tag{3.3}$$

and $\frac{\partial \psi_R}{\partial \boldsymbol{r}_1}$ is the vector whose components are the derivatives of the function ψ_R with respect to the corresponding coordinates of the vector $\boldsymbol{r}_1$. We also note that the circulation Γ^* around the body can be found if we place a point vortex in its center, with an intensity $\bar{\Gamma}$ specified by the formula

$$\bar{\Gamma} = \Gamma^* \frac{2R}{d+R}. \tag{3.4}$$

Equations (2.1) (in which it is necessary to set $\mathcal{M}_i = \mathcal{M}_i^L$, where $\mathcal{M}_i^L$ are defined according to (2.7)) in combination with (3.1) form the complete system of equations describing the motion of the body–vortex system in the moving axes $Ox_1y_1z_1$. It can easily be shown that these equations can be represented in a nearly Lagrangian (but not Lagrangian, as is typical of the vortical-dynamics problems) form.

Proposition. *Equations of motion of a rigid body bounded by a circular contour and of a vortex on the surface of a two-dimensional sphere have the form*

$$\begin{gathered}\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right) = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega}, \qquad \Gamma\dot{\boldsymbol{r}}_1 = -\frac{1}{R}\boldsymbol{r}_1 \times \frac{\partial L}{\partial \boldsymbol{r}_1}, \\ L = \frac{1}{2}(\boldsymbol{\omega}, \tilde{\mathbf{I}}\boldsymbol{\omega}) + (\boldsymbol{k}, \boldsymbol{\omega}) + \Gamma\tilde{\psi}_R, \\ \tilde{\psi}_R = \psi_R - R(\boldsymbol{\omega}, \boldsymbol{r}_1) + d(\boldsymbol{\omega}, \boldsymbol{r}_0) = -R(\boldsymbol{\omega}, \tilde{\boldsymbol{r}}_1 - \boldsymbol{r}_1) + \frac{\Gamma}{4\pi}\ln\big(Rd - (\boldsymbol{r}_0, \boldsymbol{r}_1)\big) - \frac{\bar{\Gamma}}{4\pi}\ln\big(R^2 - (\boldsymbol{r}_0, \boldsymbol{r}_1)\big),\end{gathered} \tag{3.5}$$

where, as before, $\tilde{\mathbf{I}} = \operatorname{diag}(A + aR^2, B + aR^2, C)$, $\boldsymbol{k} = (0, 0, \Gamma^* \frac{R(R+d)}{2})$.

Proof. According to (2.5) and (3.1), the equations of motion of a rigid body in the moving axes are

$$\tilde{\mathbf{I}}\dot{\boldsymbol{\omega}} = (\tilde{\mathbf{I}}\boldsymbol{\omega} + \boldsymbol{k}) \times \boldsymbol{\omega} + \Gamma R(\tilde{\boldsymbol{v}}_1 - \boldsymbol{v}_1), \tag{3.6}$$

where $\tilde{\boldsymbol{v}}_1$ and $\boldsymbol{v}_1$ are the absolute velocities of the vortex and the point $\tilde{\boldsymbol{r}}_1$. On the other hand,

$$\left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right)^{\boldsymbol{\cdot}} = \tilde{\mathbf{I}}\dot{\boldsymbol{\omega}} - \Gamma R(\dot{\tilde{\boldsymbol{r}}}_1 - \dot{\boldsymbol{r}}_1), \quad \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega} = (\tilde{\mathbf{I}}\boldsymbol{\omega} + \boldsymbol{k}) \times \boldsymbol{\omega} - \Gamma R(\tilde{\boldsymbol{r}}_1 - \boldsymbol{r}_1) \times \boldsymbol{\omega};$$

we note that the velocities in the moving axes are linked to the absolute velocities via the relationships $\boldsymbol{v}_1 = \dot{\boldsymbol{r}}_1 + \boldsymbol{\omega} \times \boldsymbol{r}_1$, $\tilde{\boldsymbol{v}}_1 = \dot{\tilde{\boldsymbol{r}}}_1 + \boldsymbol{\omega} \times \tilde{\boldsymbol{r}}_1$ and arrive at equations (3.6). Similarly, equation (3.2) for the vortex can be obtained.

Furthermore, the first term in the stream function (3.2), which describes the motion of the fluid induced only by the displacement of the body, can be represented as

$$\psi_R^{(0)} = \frac{d - R}{R^2 - (\boldsymbol{r}_0, \boldsymbol{r}_1)}(\boldsymbol{r}_1 \times \boldsymbol{v}_0, \boldsymbol{r}_0) = R(\boldsymbol{\omega}, \tilde{\boldsymbol{r}}_1) - d(\boldsymbol{\omega}, \boldsymbol{r}_0)$$

and, as in the planar case [22], proves to be in a remarkable relationship with the image of the vortex, $\tilde{\boldsymbol{r}}_1$, in terms of which the moment of the forces (3.1) exerted by the fluid on the body can be expressed. □

The Hamiltonian form of the equations of motion and first integrals. We carry out the Legendre transformation of system (3.5) with respect to the angular velocities $\boldsymbol{\omega}$:

$$\begin{gathered}\boldsymbol{M} = \frac{\partial L}{\partial \boldsymbol{\omega}} = \tilde{\mathbf{I}}\boldsymbol{\omega} + \tilde{\boldsymbol{k}}, \quad \tilde{\boldsymbol{k}} = \boldsymbol{k} - \Gamma R(\tilde{\boldsymbol{r}}_1 - \boldsymbol{r}_1), \\ H = (\boldsymbol{M}, \boldsymbol{\omega}) - L = \frac{1}{2}(\boldsymbol{M} - \tilde{\boldsymbol{k}}, \tilde{\mathbf{I}}^{-1}(\boldsymbol{M} - \tilde{\boldsymbol{k}})) - \frac{\Gamma^2}{4\pi}\ln(Rd - (\boldsymbol{r}_0, \boldsymbol{r}_1)) + \frac{\Gamma\bar{\Gamma}}{4\pi}\ln(R^2 - (\boldsymbol{r}_0, \boldsymbol{r}_1)).\end{gathered} \tag{3.7}$$

A direct verification demonstrates the validity of the following

Theorem. *The equations of motion of a body bounded by a circular contour and of a point vortex on the sphere in the moving axes $Ox_1y_1z_1$ can be represented in the Hamiltonian form*

$$\dot{M}_i = \{M_i, H\}, \quad \dot{x}_1 = \{x_1, H\}, \quad \dot{y}_1 = \{y_1, H\}, \quad \dot{z}_1 = \{z_1, H\},$$

where the Hamiltonian is given by relationship (3.7), *and the Poissonian structure has the form*

$$\{M_i, M_j\} = -\varepsilon_{ijk}M_k, \quad \{x_1, y_1\} = \frac{1}{\Gamma R}z_1, \quad \{y_1, z_1\} = \frac{1}{\Gamma R}x_1, \quad \{z_1, x_1\} = \frac{1}{\Gamma R}y_1. \tag{3.8}$$

The Lie–Poisson bracket (3.8) corresponds to the Lie algebra $so(3) \oplus so(3)$, is degenerate, and has two Casimir functions (which are obviously first integrals of the equations of motion)

$$\Phi_1 = M_1^2 + M_2^2 + M_3^2, \quad \Phi_2 = x_1^2 + y_1^2 + z_1^2 = R^2. \tag{3.9}$$

We restrict the system to the symplectic sheet specified by relationships (3.9) to obtain a Hamiltonian system with two degrees of freedom. According to the Liouville theorem, its integrability requires another, additional (except the Hamiltonian) first integral.

Dynamically symmetric case. Assume that $A = B$; then the system is integrable due to the presence of the additional, Lagrangian-type integral [4]

$$\omega_3 = \frac{\partial H}{\partial M_3} = \frac{1}{C}\left(M_3 - \Gamma^* \frac{R(R + d)}{2} + \Gamma R(\tilde{z}_1 - z_1)\right) = \text{const}.$$

Corollary. *The problem of the motion of a circular, dynamically symmetric body and one point vortex on a spherical surface is Liouville-integrable.*

A similar result for the plane was obtained in [5] and analyzed in detail in [22, 23, 38, 39]. Note that, as numerical experiments show, the integrability of the system fails at $A \neq B$, and chaotic motions originate.

Remark. The statement on the integrability can also be obtained in a simpler way. Rewriting the equation of motion of the rigid body (3.6) in the motionless coordinate system and integrating it yields

$$\lambda \boldsymbol{\omega} = \Gamma R(\tilde{\boldsymbol{r}}_1 - \boldsymbol{r}_1) - \mu \boldsymbol{r}_0 + \boldsymbol{K}, \tag{3.10}$$

where $\boldsymbol{K}$ is a vector fixed in absolute space with $\boldsymbol{K}^2 = \Phi_1$ (see (3.9)) and the coefficients λ and μ are specified in (2.7). We take the cross product of equation (3.10) by $\boldsymbol{r}_0$, thus obtaining

$$\lambda \dot{\boldsymbol{r}}_0 = \Gamma R \boldsymbol{r}_0 \times (\boldsymbol{r}_1 - \tilde{\boldsymbol{r}}_1) - \boldsymbol{r}_0 \times \boldsymbol{K}. \tag{3.11}$$

The equation of motion of the vortex (3.2) in the motionless coordinate system has the form

$$\dot{\boldsymbol{r}}_1 = -\frac{1}{R}\boldsymbol{r}_1 \times \frac{\partial \psi_R}{\partial \boldsymbol{r}_1}. \tag{3.12}$$

The system of six equations (3.11) and (3.12) for $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ has four first integrals

(a) the Hamiltonian function (3.7) in which $\boldsymbol{\omega}$ should be expressed from (3.10);

(b) $\boldsymbol{r}_0^2 = R^2$;

(c) $\boldsymbol{r}_1^2 = R^2$;

(d) $(\Gamma R(\tilde{\boldsymbol{r}}_1 - \boldsymbol{r}_1) - \mu \boldsymbol{r}_0 + \boldsymbol{K}, \boldsymbol{r}_0) = \mathrm{const}$ (the projection of the angular velocity onto the symmetry axis of the body is constant).

The system also preserves a standard measure, being thus integrable, according to the theory of the last Jacobi multiplier.

4. MASS VORTICES

Equations of motion. Assume that the body is dynamically symmetric and let the radius of the body approach zero, leaving its mass unchanged; we then obtain an object that, as in the planar case [22], we will term a *mass vortex*. The same reasoning as in [22] shows that the dynamics of two mass vortices in the motionless axes $Oxyz$ is described by the equations

$$\begin{aligned} m_1 R^2 \boldsymbol{r}_1 \times \ddot{\boldsymbol{r}}_1 &= -\Gamma_1 R \dot{\boldsymbol{r}}_1 + \frac{\Gamma_1\Gamma_2}{4\pi}\frac{\boldsymbol{r}_2 \times \boldsymbol{r}_1}{R^2 - (\boldsymbol{r}_1, \boldsymbol{r}_2)}, \\ m_2 R^2 \boldsymbol{r}_2 \times \ddot{\boldsymbol{r}}_2 &= -\Gamma_2 R \dot{\boldsymbol{r}}_2 + \frac{\Gamma_1\Gamma_2}{4\pi}\frac{\boldsymbol{r}_1 \times \boldsymbol{r}_2}{R^2 - (\boldsymbol{r}_1, \boldsymbol{r}_2)}. \end{aligned} \tag{4.1}$$

The problem of the motion of two mass vortices is, in the general case, not integrable even on the plane [9, 14]. Therefore, system (4.1) also does not appear to be integrable. Nevertheless, it would be interesting to qualitatively analyze equations (4.1).

Hamiltonian form and first integrals. We define the new variables

$$\boldsymbol{M}_\alpha = m_\alpha \boldsymbol{r}_\alpha \times \dot{\boldsymbol{r}}_\alpha + \frac{\Gamma_\alpha}{R}\boldsymbol{r}_\alpha, \quad \alpha = 1, 2, \tag{4.2}$$

and rewrite equation (4.1) as follows:

$$\begin{gathered} \dot{\boldsymbol{M}}_\alpha = -\boldsymbol{r}_\alpha \times \frac{\partial U}{\partial \boldsymbol{r}_\alpha}, \quad m_\alpha \dot{\boldsymbol{r}}_\alpha = -\boldsymbol{r}_\alpha \times \boldsymbol{M}_\alpha, \\ U = -\frac{\Gamma_1\Gamma_2}{4\pi R^2}\ln\Big(R^2 - (\boldsymbol{r}_1, \boldsymbol{r}_2)\Big). \end{gathered} \tag{4.3}$$

It can easily be verified that these equations are Hamiltonian with the Poisson bracket (which is a Lie–Poisson bracket specified by the algebra $e(3) \oplus e(3)$)

$$\{M_{\alpha i}, M_{\alpha j}\} = -\varepsilon_{ijk} M_{\alpha k}, \quad \{M_{\alpha i}, x_{\alpha j}\} = -\varepsilon_{ijk} x_{\alpha k}, \quad \alpha = 1, 2, \quad i, j, k = 1, 2, 3, \tag{4.4}$$

(here, the zero brackets are omitted), and with the Hamiltonian

$$H = \frac{1}{2}\Big(\frac{1}{m_1}(\boldsymbol{M}_1, \boldsymbol{M}_1) + \frac{1}{m_2}(\boldsymbol{M}_2, \boldsymbol{M}_2)\Big) + U. \tag{4.5}$$

For the Poisson bracket (4.4), the constants of the Casimir functions should be fixed as follows:

$$K_\alpha^{(1)} = (\boldsymbol{r}_\alpha, \boldsymbol{r}_\alpha) = R^2, \quad K_\alpha^{(2)} = (\boldsymbol{M}_\alpha, \boldsymbol{r}_\alpha) = R^{-1}\Gamma_\alpha.$$

It is remarkable that the equation of motion (4.3), the Poissonian structure (4.4), and the Hamiltonian (4.5) coincide with those in the case of a conventional two-body problem on the sphere [17]. The difference is in that, for interacting bodies, all the constants $K_\alpha^{(2)} = 0$, while they are proportional to the circulation in the case of mass vortices.

System (4.3) has a vector integral of the total momentum

$$\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2 = \text{const}.$$

We also note that, in [17], results of numerical experiments are presented, which testify to the nonintegrability of the two-body problem on the sphere with a Newtonian and a Hookian potential.

The problem of n mass vortices. These equations can be generalized without difficulties to the case of an arbitrary number of mass vortices, the Hamiltonian assuming the form

$$H = \frac{1}{2}\sum_{\alpha=1}^{n} \frac{1}{m_\alpha}(\boldsymbol{M}_\alpha, \boldsymbol{M}_\alpha) - \sum_{\alpha<\beta}^{n} \frac{\Gamma_\alpha\Gamma_\beta}{4\pi R^2}\ln\big(R^2 - (\boldsymbol{r}_\alpha, \boldsymbol{r}_\beta)\big),$$

where $\boldsymbol{r}_\alpha$ is the position vector of the corresponding vortex and $\boldsymbol{M}_\alpha$ is the momentum defimed according to formula (4.2).

The Poisson structure is determined by the algebra $\overset{n}{\underset{\alpha=1}{\oplus}} e(3)$ and has the form

$$\{M_{\alpha i}, M_{\alpha j}\} = -\varepsilon_{ijk}M_{\alpha k}, \quad \{M_{\alpha i}, x_{\alpha j}\} = -\varepsilon_{ijk}x_{\alpha k}, \quad ijk = 1, 2, 3, \quad \alpha = 1 \dots n,$$

the corresponding Kirchhoff functions being fixed as follows:

$$(\boldsymbol{r}_\alpha, \boldsymbol{r}_\alpha) = R^2, \quad (\boldsymbol{M}_\alpha, \boldsymbol{r}_\alpha) = R^{-1}\Gamma_\alpha, \quad \alpha = 1 \dots n.$$

The integral of the total momentum is

$$\boldsymbol{M} = \sum_{\alpha=1}^{n} M_\alpha.$$

Remark 4. The equations of the mass vortices on the plane were independently obtained in [9, 22, 37], where physical and hydrodynamic problems were also noted in which this model can be used. The mass vortices on the sphere can be considered a real alternative to various known models of vortical motions in the Earth's atmosphere and oceans (cyclones, tornadoes, oceanic vortical flows, etc.). This model is physically more realistic due to the consideration of the vortical-column mass, whose presence is natural, since powerful vortical features, such as tornadoes, suck in large foreign bodies in the course of their motion. Anyway, the development of applications and the experimental verification of a theoretical model are decisive for its applicability. As some other vortical models, we also note the motion of vorticity sources and vortical spots [18, 19].

APPENDIX

Stream function. The fluid flow on the sphere in a region D outside the body bounded by the contour C (Fig. 3) is completely determined by the stream function $\psi(\xi, \eta, t)$, which, as shown above, must satisfy

– the system of differential equations

$$\frac{\partial\Omega}{\partial t} = \frac{1}{\sqrt{EG}}\left(\frac{\partial\psi}{\partial\eta}\frac{\partial\Omega}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial\Omega}{\partial\eta}\right), \quad \Delta\psi = -\Omega; \tag{A.1}$$

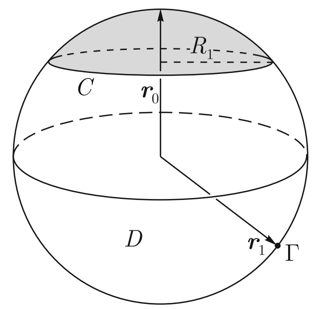

Fig. 3.

– the unpermeability condition at the boundary of the body, C:

$$(\boldsymbol{v},\boldsymbol{n})\big|_C = \frac{\partial\psi}{\partial\boldsymbol{\tau}}\Big|_C, \tag{A.2}$$

where $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are the vectors normal and tangent to the boundary C and $\boldsymbol{v}$ is the velocity vector of the corresponding point of the body's boundary;

– the condition that, in the absence of a vortex ($\Gamma = 0$), the circulation around the contour encircling the body is

$$\int\limits_C (\boldsymbol{v},\boldsymbol{\tau})\,dl = \Gamma^*. \tag{A.3}$$

According to the superposition principle, the sought-for stream function can be represented in the form

$$\psi = v_{ox}\psi_x^{(0)} + v_{oy}\psi_y^{(0)} + v_{oz}\psi_z^{(0)} + \psi_c + \psi_v,$$

where $\boldsymbol{v}_0 = (v_{ox}, v_{oy}, v_{oz})$ is the velocity of the body's center and the terms ψ_c and ψ_v are proportional to Γ^* and Γ, respectively.

The fluid flow induced only by the motion of the body with the velocity $\boldsymbol{v}_0$ can be described by the stream function

$$\psi^{(0)}(\boldsymbol{r}) = v_{ox}\psi_x^{(0)} + v_{oy}\psi_y^{(0)} + v_{oz}\psi_z^{(0)} = \frac{d-R}{R^2 - (\boldsymbol{r},\boldsymbol{r}_0)}(\boldsymbol{r}\times\boldsymbol{v}_0,\boldsymbol{r}_0). \tag{A.4}$$

The validity of the unpermeability condition (A.2) can be directly checked. The function $\psi^{(0)}$ is harmonic (outside the body, $\Delta\psi^{(0)} = 0$) and, therefore, satisfies the condition (1.9). The corresponding flow has zero circulation along the body's boundary C.

Remark 5. Formula (A.4) can be empirically obtained as follows. It is known that the velocity field produced by a circular cylinder moving over the plane is the flow induced by a dipole. Assume that the same result holds for the sphere. First consider a flow on the plane such that the velocities of all particles are the same. As is known, such a flow is induced by an infinitely distant dipole. If we map this flow onto the sphere via the stereographic projection, simple rearrangements will lead us to formula (A.4).

As in the planar case, the flow induced by the point vortex ψ_v can be obtained by adding an inversely symmetric (in terms of the stereographic projection) vortex of the opposite vorticity, so that

$$\psi_v(\boldsymbol{r}) = -\frac{\Gamma}{4\pi}\ln\Big(R^2 - (\boldsymbol{r}_1,\boldsymbol{r})\Big) + \frac{\Gamma}{4\pi}\ln\Big(R^2 - (\boldsymbol{r}_1^i,\boldsymbol{r})\Big),$$

$$\boldsymbol{r}_1^i = \Big(R^2 + d^2 - \frac{2d}{R}(\boldsymbol{r}_0,\boldsymbol{r}_1)\Big)^{-1}\Big((R^2-d^2)\boldsymbol{r}_1 + 2(Rd - (\boldsymbol{r}_0,\boldsymbol{r}_1))\boldsymbol{r}_0\Big),$$

where $\boldsymbol{r}_1^i$ is the position vector of the point inversely symmetric to the vortex. The function ψ_v is harmonic outside the body everywhere (except for the point $\boldsymbol{r}_1$) and, therefore, satisfies equation (A.1).

Let us discuss the function ψ_c, which describes a purely circulative flow around the rigid body. In contrast to the case of the plane, the circulations along homotopic contours encircling the body on the sphere are generally different. As shown in Section 1.2, if a vortex of intensity Γ is placed at the point $\theta = 0$ on the sphere, the circulation along the parallel $\theta = \theta_0$ will be $\Gamma(1 + \cos\theta_0)/2$. Therefore, to satisfy the condition (A.3), a point vortex should be placed at the center of the body, and its intensity $\bar{\Gamma}$ and the circulation Γ^* should be linked by the relationship (3.4), so that

$$\psi_c = -\frac{\Gamma^*}{4\pi}\frac{2R}{R+d}\ln\Big(R^2 - (\boldsymbol{r}_0, \boldsymbol{r})\Big).$$

The function ψ_c thus defined satisfies the equation $\Delta\psi = \frac{\bar{\Gamma}}{4\pi R^2}$ in the region D and, therefore, it satisfied equation (A.1).

Thus, the stream function is completely specified. We note in view of the following that, in the region D, it satisfies the equation

$$\Delta\psi = \frac{\bar{\Gamma}}{4\pi R^2} - \Gamma\delta(\boldsymbol{r} - \boldsymbol{r}_1), \tag{A.5}$$

where Δ is the Laplace–Beltrami operator (1.8), $\boldsymbol{r}_1$ is the position vector of the vortex, and $\delta(\boldsymbol{r} - \boldsymbol{r}_1)$ is the Dirac delta.

The Kirchhoff–Routh function. The absolute velocity of the vortex, $\boldsymbol{v}_1$, is given by relationship (1.15), which can be written in Cartesian coordinates as

$$\boldsymbol{v}_1 = -\frac{1}{R}\boldsymbol{r}_1 \times \frac{\partial}{\partial \boldsymbol{r}}\Big[\psi(\boldsymbol{r}) + \frac{\Gamma}{4\pi}\ln\big(R^2 - (\boldsymbol{r}_1, \boldsymbol{r})\big)\Big]\Big|_{\boldsymbol{r}=\boldsymbol{r}_1}.$$

Upon simple rearrangements, we find that

$$\boldsymbol{v}_1 = -\frac{1}{R}\boldsymbol{r}_1 \times \frac{\partial\psi_R(r_1)}{\partial \boldsymbol{r}_1},$$

$$\psi_R(\boldsymbol{r}_1) = \frac{d-R}{R^2 - (\boldsymbol{r}_0, \boldsymbol{r}_1)}(\boldsymbol{r}_1 \times \boldsymbol{v}_0, \boldsymbol{r}_0) + \frac{\Gamma}{4\pi}\ln(Rd - (\boldsymbol{r}_0, \boldsymbol{r}_1))$$
$$-\frac{\Gamma^*}{4\pi}\frac{2R}{R+d}\ln\big(R^2 - (\boldsymbol{r}_0, \boldsymbol{r}_1)\big).$$

The planar analog of the function $\psi_R(\boldsymbol{r}_1)$ is traditionally referred to in vortical dynamics as the Kirchhoff–Routh function (see, e.g., [10]).

The moment of the fluid-pressure forces. To calculate the moment of the fluid-pressure forces $\boldsymbol{\mathcal{M}}^L$ exerted on the boundary of the body C, we use the theorem of the kinetic-momentum changes. The variation in the kinetic momentum of the fluid, $\boldsymbol{K}^L$, that fills the region D (the exterior of the contour C) is

$$\dot{\boldsymbol{K}}^L = -\boldsymbol{\mathcal{M}}^L, \tag{A.6}$$

where $\boldsymbol{\mathcal{M}}^L$ is the sought-for moment of the fluid-pressure forces exerted on the contour C. We set the fluid density equal to unity and represent the kinetic momentum of the fluid, $\boldsymbol{K}^L$, in the region D as

$$\boldsymbol{K}^L = \iint\limits_D \boldsymbol{r} \times \boldsymbol{v} ds = R\iint\limits_D \nabla\psi ds,$$

where all the vectors are considered to be expressed in local orthogonal coordinates ξ, η on the sphere according to the formulas

$$\boldsymbol{v} = \frac{1}{\sqrt{G}}\frac{\partial\psi}{\partial\eta}\boldsymbol{e}_\xi - \frac{1}{\sqrt{E}}\frac{\partial\psi}{\partial\xi}\boldsymbol{e}_\eta, \quad \nabla\psi = \frac{1}{\sqrt{E}}\frac{\partial\psi}{\partial\xi}\boldsymbol{e}_\xi + \frac{1}{\sqrt{G}}\frac{\partial\psi}{\partial\eta}\boldsymbol{e}_\eta, \quad \boldsymbol{r} = R\boldsymbol{e}_\xi \times \boldsymbol{e}_\eta.$$

In the Cartesian coordinates $Oxyz$, the components of the vector $\boldsymbol{K}^L$ can be represented in the form

$$K_x^L = R\iint\limits_D (\nabla x, \nabla\psi)ds, \quad K_y^L = R\iint\limits_D (\nabla y, \nabla\psi)ds, \quad K_z^L = R\iint\limits_D (\nabla z, \nabla\psi)ds.$$

Note that these integrals diverge at the point $\boldsymbol{r}_1$*; therefore, they should be understood in terms of the theory of generalized functions (as principal values).* We use the Stokes formula and employ a vector notation to obtain

$$\boldsymbol{K}_C = R\int\limits_C \boldsymbol{r}\sqrt{\frac{E}{G}}\psi'_\eta d\xi - \boldsymbol{r}\sqrt{\frac{G}{E}}\psi'_\xi d\eta. \tag{A.7}$$

Thus, in view of (A.6), *the moment of the fluid-pressure forces acting on the boundary of the body is specified by the relationship*

$$\boldsymbol{\mathcal{M}}^L = -\dot{\boldsymbol{K}}_C + \left(R\iint\limits_D \boldsymbol{r}\Delta\psi ds\right)^{\cdot}. \tag{A.8}$$

Remark. The moment $\boldsymbol{\mathcal{M}}^L$ could be found by directly integrating the moment of the pressure forces along the boundary of the body,

$$\boldsymbol{\mathcal{M}}^L = \int\limits_C \boldsymbol{r}\times p\boldsymbol{n}dl = R\int\limits_C p\boldsymbol{\tau}dl = R\int\limits_C pd\boldsymbol{r},$$

where p is the fluid pressure expressed from the generalized Cauchy–Lagrange integral (1.13) (we use here the relationship $\boldsymbol{r}\times\boldsymbol{n} = R\boldsymbol{\tau}$, which is valid for the vectors $\boldsymbol{n}$ and $\boldsymbol{\tau}$ normal and tangent to the contour on the sphere). However, the expression for $\boldsymbol{\mathcal{M}}^L$ thus obtained proves to be highly cumbersome and to have no clear physical meaning.

We calculate the double integral in the right-hand side of (A.8) using equation (A.5) and the relationship $\iint\limits_D \boldsymbol{r}ds = -\pi(R^2-d^2)\boldsymbol{r}_0$; finally, we obtain

$$R\iint\limits_D \boldsymbol{r}\Delta\psi ds = -\Gamma R\boldsymbol{r}_1 + \frac{d^2-R^2}{4R}\bar{\Gamma}\boldsymbol{r}_0. \tag{A.9}$$

Now we calculate the curvilinear integral $\boldsymbol{K}_C$ given by formula (A.7). In this formula, we sequentially substitute the function ψ with $\psi^{(0)}$, ψ_v, and ψ_c to directly obtain

$$\boldsymbol{K}_C\big|_{\psi=\psi^{(0)}} = a\boldsymbol{r}_0\times\boldsymbol{r}_0, \quad \text{where } a = \pi R_1^2,$$
$$\boldsymbol{K}_C\big|_{\psi=\psi_v} = -R\Gamma\tilde{\boldsymbol{r}}_1,$$
$$\boldsymbol{K}_C\big|_{\psi=\psi_c} = \frac{\bar{\Gamma}d(R+d)}{2R}\boldsymbol{r}_0.$$

The coefficient a is the added mass of the circular body of radius R_1. The vector $\tilde{\boldsymbol{r}}_1$ (see Fig. 2) is defined in (3.1) (the heads of the vectors $\tilde{\boldsymbol{r}}_1$, $\boldsymbol{r}_1$, and $\boldsymbol{r}_0$ are in the same straight line).

We substitute these expressions together with (A.9) into (A.8) to obtain the following formula for the moment:

$$\boldsymbol{M}^L = -a\boldsymbol{r}_0\times\ddot{\boldsymbol{r}}_0 + \Gamma R(\dot{\tilde{\boldsymbol{r}}}_1 - \dot{\boldsymbol{r}}_1) - \Gamma^*\frac{R+d}{2}\boldsymbol{v}_0.$$

UNRESOLVED PROBLEMS

To conclude, we formulate certain issues that would be worth further investigating based on the results of this study.

- Studying various cases of the motion of a rigid body and point vortices on the sphere. In particular, analyzing the problem of the motion of an arbitrary two-dimensional body and several vortices. In the general case, the equations of motion of such a system will likely be not integrable. Nevertheless, it would be interesting to obtain a general equation in a Hamiltonian form and analyze various particular solutions, e.g., stationary configurations [21, 41]. The most interesting problem is the description of the motion of N rigid bodies on the sphere with allowances for circulation around each body. A particular case of this problem is the interaction of several rigid bodies and point vortices. Regrettably, a general formalism of such a problem has not yet been developed even for the plane. Thus, the hydrodynamics of several rigid bodies and point vortices still remains a challenging unresolved problem.

- A more systematic study of various particular cases of the dynamics of mass vortices. The equations of motion of mass vortices on the sphere were presented here; they have already been partially analyzed for planar cases [9, 22, 41]. This form of equations makes it possible to consider a number of interesting problems. For example, a very simple application of these equations to the description of dynamical advection, i.e., the motion of an insoluble, fine-grained admixture in the fluid flow, is considered in [6]. In our opinion, the equations obtained in [6] appear to have a more applied character compared to the classical model of chaotic advection.

  In addition, some issues in the dynamics of mass vortices, whose analogs are constantly under consideration in celestial mechanics, remain unexplored, viz., the existence of stationary and static configurations, their classification and stability. Note that static configurations of mass vortices, even with the same intensity, are also possible on the sphere. Various problems related to the classification of symmetric static configurations and search for nonsymmetric ones arise in this context. These problems have not yet completely explored even for the dynamics of regular vortices. Results concerning the static and stationary configurations of classical vortices are systematically discussed in the survey [13].

- Investigation of vortical motions on surfaces of other types. In view of possible applications, of considerable interest is the investigation of vortical dynamics not only on the sphere but also on other compact surfaces that model, in one approximation or another, the Earth's surface. From a general theoretical standpoint, it would be interesting to consider the dynamics of rigid bodies and vortices on a Lobachevsky plane; this problem is simpler than that of motion on compact surfaces, since it does not involve difficulties due to the ambiguity of the definition of the point vortex.

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