Some problems of the balance model of economy

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Abstract – **We discuss some problems of the balance model of economy such that the testing of productivity, product and pricing control, acceleration of computations.**

Keywords – **productive models, product and pricing control, multipurpose algorithms, universal operation on matrix structures**

I. INTRODUCTION

Interdependence of industries is reflected in the balance equations between amount of produced, consumed and output products. In the balanced economy the prices for produced and consumed products and the profits in the economy sectors also satisfy to balance relations. The issue of pricing in today's environment requires close attention from the state for several reasons. Spontaneous market of demand and supply leads to formation of correct prices of goods and services only asymptotically and this complex process constantly feels negative impact by various anticompetitive agreements and criminal organizations. To compensate for this negative impact on the economy market in addition to an "invisible hand" of Adam Smith it is required a "visible hand" of the state that stabilizes the process of pricing. We believe that in economy the product flows are primary and leading but the financial flows are secondary and slave. In addition, we consider the balance method of describing a structure of economy as the basic tool for predicting of evolution both the production and financial sectors.

The balance method of material and finance resources is used for accounting, control and planning of their reproduction and consumption in the course of any economic activity [1]. Historically, this method was based on the simple economic model, consisting of two equations: $x - Ax = y,\ p - pA = q.$ First (primary) equation describes the balance between the volumes of produced and output productions and second (dual) equation represents balance between the values of price and profit of productions. Here A is $n \times n$ -matrix of cost rate with non-negative coefficients, n denotes the number of types of the products to be taken into account. Components of n -dimensional column vector x contain the volumes of produced products and thus can only be nonnegative. Column vector y is called export-import vector. Positive components of this vector correspond to the final (output) product and comply with export, while negative components characterize volumes of import. Product Ax describes the costs in manufacturing sector. Components of n -dimensional row vector p include prices of manufactured products and can only be nonnegative. Row vector q is called value vector of the unit productions. Positive components of this vector correspond to profit and characterize profitable products, while negative components point out on losses from the sale of unit productions. The product pA describes the cost of manufactured goods in economy.

Describing economy on the base of balance method includes number of tasks. First task is the solving primary and dual equations. Second task, which we attribute to category of the most important, is to evaluate relative error of the result under variations of initial parameters of the model: matrix $A,$ vectors y and $q.$ As is known, in this task the leading role belongs to condition number $cond\ B = \|B\| \cdot \|B^{-1}\|$ of matrix $B = E - A,$ where E is identity matrix of order n ,

$$\|B\| = \max_{1 \le i \le n} \sum_{j=1}^{n} |\,\delta_{ij} - a_{ij}\,|,$$

δ_{ij} denotes Kronecker symbol. Another task is calculation of $\det B$ and inverse matrix B^{-1} of full costs. Finally, equally important tasks are testing on productivity the balance model and investigation of nonproductive case. In the section IV we describe some multipurpose algorithm that solves all these tasks in the course of its execution.

II. PRODUCTIVE MODELS

Direct task for balance model is to calculate export-import vector y for a given plan $x \ge 0.$ If all components y are positive then x is called an *effective* plan. The plan x is said to be *ineffective*, if all components of export-import vector y are not positive. In inverse problem for a given y we must specify the set of all plans $x \ge 0$ that satisfy to the primary balance equation. If this set is not empty then we have a *consistent* model. Inconsistent models may occur due to inconsistency of primary or dual balance equations or absence of nonnegative solutions of these equations. Consistent model is called *correct* if the set of plans consists of one element. Incorrect model is characterized by infinite set of plans. Correct model with effective plan is called *productive*. Correct model with ineffective plan is called nonproductive. Consistency test of balance model is formulated as follows.

Solvability criterion. E*quation* $x = Ax + y$ *has nonnegative solution* $x \geq 0$ *if and only if for every* n*-dimensional row vector* z, *satisfying to the system of inequalities* $z \geq zA$, *the inequality* $zy \geq 0$ *takes place.*

In practice, it suffices find finite system of extreme directions (generators) of the cone $\{z \in R^n : z \geq zA\}$ and limit verifying the inequality $zy \geq 0$ only on these generators. Our approach to definition of finite system of generators is based on the using of generalized inverse matrix, which is calculated by applying *universal operation on matrix structures* (see below). Let G be any generalized inverse matrix for matrix $B = E - A$ $(B = BGB)$. Then we have

Theorem 1. *Balance equation* $x = Ax + y$ *has non-negative solution* $x \geq 0$ *if and only if* $y = BGy$ *and for every* n*-dimensional row vector* $h \geq 0$, *satisfying to equation* $h = hGB$, *the inequality* $hGy \geq 0$ *holds.*

Conclusion implication in this assertion should be checked only on elements of finite fundamental system of nonnegative solutions of homogeneous equation $h = hGB$. Efficient algorithm of searching such system was developed by N. V. Chernikova [2, pp. 255]. Note that equality $y = BGy$ is necessary and sufficient condition for consistency of the linear system $x = Ax + y$.

Let $r(A)$ denotes spectral radius of matrix A. Condition $r(A) < 1$ selects important class of productive balance models, which able to produce any volumes of final production and values of profit. As is known, this condition identifies the class of economies that can grow through internal resources, whereas in the case of $r(A) \geq 1$ economy can exist entirely due to the import. For productive matrix A from $y = x - Ax \leq 0$ it follows that $x \leq 0$, so that ineffective plans can not be in the productive economy. Different criteria and algorithms for verification of productive balance model are noteworthy and in our interest. This direction of research is important for many reasons. Innovative processes in society, fast development of free market relations and integration into global economic space will inevitably lead to qualitative and quantitative changes in the economy. Qualitatively, this is reflected in the fact that new industries appear and some of old disappear. Further, structure of economic relations can be greatly changed. Quantitatively, this translates into changing dimension, structure and values of the elements of matrix A. Therefore, it is necessary effective tools for testing productive property of dynamically changing economy and fast methods of calculating plans. Checking productive property of balance model can be carried out in various ways, which, however, differ greatly in volume of calculations. Of interest are time-consuming works of programs that check matrices with large sizes on presence of productive property. Experiments in package Mathematica for moderate values of dimension n show that criterion

$$Max[Inverse[IdentityMatrix[n] - A]] \geq 0$$

gains compared with

$$Max[Abs[Eigenvalues[A]]] < 1.$$

Other tests of productive property can be found in [1]. In section IV we present some criterion that relies on Hawkins-Simon test. Now we give heuristic algorithm for testing which is based on the equality

$$r(A) = \min_x[\max[A.x / x] = \max_x[\min[A.x / x].$$

Here $\min$ and $\max$ are taken over all vectors x of dimension n with positive components.

Algorithm 1. *Let* $\psi = 0$ *be zero vector of dimension* n. *In endless loop we generate random probability vector* v *of dimension* n. *Next, calculate residual* $\varphi = v - Av$. *If* $\alpha = \min_{1 \leq i \leq n} \varphi_i > 0$ *then matrix* A *must be productive, and hence we exit loop. If* $\beta = \max_{1 \leq i \leq n} \varphi_i < 0$ *then* A *is not productive. Under condition* $\alpha \leq 0$ *and* $\beta \geq 0$ *we compute* $\psi := \psi + \varphi$. *In the case of* $\bar{\alpha} = \min_{1 \leq i \leq n} \psi_i > 0$ *matrix* A *must be productive, but if* $\bar{\beta} = \max_{1 \leq i \leq n} \psi_i < 0$ *then* A *is not productive. In the case of* $\bar{\alpha} \leq 0$ *and* $\bar{\beta} \geq 0$ *we go to the top of infinite loop.*

Analysis shows that it is possible theoretically an infinite loop of execution but computational experiments on test examples have shown high efficiency of this heuristic algorithm.

Among balance models are highlights irreducible models (graph of matrix A is strongly connected). Economically, this means that the product of any industry directly or indirectly uses products of other industries. For irreducible models you can specify weaker validation criteria of productivity. Balance model with decomposable matrix indicates that in corresponding economy there is a space in establishment of economic ties between industries. So there is an interesting task to specify a minimum set of economic relations, establishment of which will provide irreducibility property of new economy.

For productive economic models the iterative process

$$x(0)=y,\ \ x(i+1)=Ax(i)+y,\ i=1,2,\ldots$$

converges to the unique solution $x(\infty)$ of equation $x=Ax+y$ for any y. This iterative process became known among economists as reorder method. Computational experiments with matrices of large size show that convergence $x(i)$ to solution $x(\infty)$ is usually very slow. This is not the only shortcoming of reorder method. It cannot be applied to nonproductive economics, meaning of which is increasing in era of integration into global community. What is our refinement of reorder method? We propose to use an implicit reorder method in the next form: $x(0)=y$,

$$(E-H)x(i+1)=(A-H)x(i)+y,\ i=1,2,\ldots,$$

which requires the inversion of matrix $E-H$. One can show that the minimal rank of matrix H, which provides the geometric rate of convergence $\delta^i\ (\delta<1)$ in a suitable norm, is equal to the maximum of geometric multiplicity of the eigenvalues that lie outside the circle of radius δ.

The next way of the constructing of well-posed equation and acceleration of computations is based on the following fact.

Theorem 2. *There exists* $n\times n$*-matrix* Q *such that* $P=E-Q(E-A)$ *has preassigned norm* $\delta<1$. *Iterative process* $x(i+1)=Px(i)+Qy,\ i=0,1,\ldots$ *converges to solution* $x(\infty)$ *of equation* $x=Ax+y$ *with the rate*

$$\|x(i)-x(\infty)\|\le\frac{\delta^i}{1-\delta}\|Px_0+Qy-x_0\|.$$

Here $x(0)$ denotes any starting vector. In addition we have the upper bound for condition number

$$cond\,(E-P)\le\frac{1+\delta}{1-\delta}.$$

To find a matrix Q mentioned in this theorem we used a parallel algorithm of random searching on a system of graphic accelerators supporting CUDA technology.

III. PRODUCT AND PRICING CONTROL

Consider evolution of diversified economy in discrete periods $k=0,1,2,\ldots$ with following economic indicators corresponding to the k-th period:

$x[k]$ - n-dimensional nonnegative column vector of volumes of produced production;

$y[k]$ - n-dimensional column vector of volumes of export-import production (positive components are responsible for export and negative – for import);

$u[k]$ - n-dimensional nonnegative column vector of control, it provides a subsidy of production resources from the state reserve. The target production $x[k+1]$ in the next period is calculated from the equation

$$x[k]+u[k]=Ax[k+1]+y[k].$$

In the left side of this equation we have the sum of total reserves for each product at the beginning of $(k+1)$-th period. This sum covers production costs of this period and export-import of the previous period.

We consider two fundamentally different ways of forming control vector. By the first method $u[k]=Hx[k]$ (H - matrix of control) the grant from the state reserve is calculated by the rule of delay feedback. By the second method $u[k]=Hx[k+1]$. Evolution of economy in the first case is described by equation

$$(E+H)x[k]=Ax[k+1]+y[k], \tag{1}$$

In the second case we have

$$x[k]=(A-H)x[k+1]+y[k]. \tag{2}$$

Let $\rho>0$ be a given rate of change of export-import vector: $y[k+1]=\rho\, y[k], k=0,1,2,\ \ldots$, where $y[0]$ is specified. It is natural require similar property from the productions: $x[k+1]=\rho\, x[k], k=0,1,2,\ \ldots,$ where $x(0)$ should be calculated. Under these assumptions, from (1) and (2) we have respectively

$$(E+H)x[0]=\rho Ax[0]+y[0], \tag{1'}$$

$$x[0]=\rho(A-H)x[0]+y[0]. \tag{2'}$$

For solvability of these equations for any choice of starting vector of exports-imports $y[0]$ it is necessary and sufficient that matrices $E+H-\rho A$ and $E-\rho(A-H)$ are nonsingular for given $\rho>0$. We note again that export-import vector $y[0]$ may have coordinates of any sign. However, solutions $x[0]$ of equations (1') and (2') must have nonnegative components as a starting vectors of production volumes. For both rules of economic governance with help of resources of the state reserve the lower value of rank of the control matrix H is given in the next assertion.

Theorem 3. *For minimal rank of matrix* H, *that provides any rate* ρ *of change of export and import from the given interval* $[\mu,\nu]$, *we have*

$$\min rank\,H=\max_{\mu\le\rho\le\nu}\dim\ker\left(\frac{1}{\rho}E-A\right). \tag{3}$$

Similarly, we investigate evolution of the prices $p[k]$ of manufactured products in the following pricing model with minimal state intervention

$$p[k]+v[k]=p[k+1]A+q[k],$$

were control row vector is taken as $v[k]=p[k]H$ or $v[k]=p[k+1]H$. Here $q[k]$ denotes n-dimensional row vector of profits.

Additionally, one can consider the case when different components of final product may have different rates of change: $y[k+1] = diag(\rho_1, \rho_2, ..., \rho_n) y[k]$.

IV. MULTIPURPOSE ALGORITHMS

High-performance computing is usually associated with high speed of computer processors, memory and channels of information transmissions. However, this is not the only possible way of accelerating the calculations. We believe that the huge reserves in increasing the performance of computing one can find in the correct choice of multipurpose algorithms for specific types of problems, when at the time of computing process we are able to solve the entire class of tasks related to this problem. Confirmation of this may be found in [3-5].

Now we show that all the tasks mentioned in the first section can be solved during the multipurpose algorithm presented below. The initial information table of this algorithm has the form.

		$\hat{p}$	
	1	$-q$	
$\hat{x}$	y	B	p^T
		x^T	

The upper index T denotes transpose operation,

$$\hat{x} = (x_{n+1}, x_{n+2}, ..., x_{2n})^T, \ \hat{p} = (p_{n+1}, p_{n+2}, ..., p_{2n}),$$
$$y = (y_1, y_2, ..., y_n)^T, \ -q = (-q_1, -q_2, ..., -q_n),$$
$$x^T = (x_1, x_2, ..., x_n), \ p^T = (p_1, p_2, ..., p_n)^T.$$

The colored part of this table is numerical and is changed in the loop "for $k = 1,2,...,n$ do". Let matrix D with elements $d_{ij}, i, j = 0,1,...,n$ denote the current value of this part. It is surrounded by one-dimensional column and row vectors of x-variables and p-variables. Resolved elements of the loop are taken as $\gamma_k = d_{kk} \neq 0, \ k = 1,2,...,n.$ Element γ_k define the following rule of calculation the next table from the current table. Variables x_k and x_{k+n} , respectively p_k and p_{k+n}, change places but the other variables remain in their places. The elements of numerical matrix D converted by the rule of universal matrix operation:

$$d_{00} := d_{00} * \gamma_k ,$$

$$d_{ij} := \frac{1}{\gamma_k} \begin{cases} 1, \ if \ \ i = k, j = k; \\ d_{ij}, \ if \ i = k; \\ -d_{ij}, \ if \ j = k; \\ d_{ij} * \gamma_k - d_{ik} * d_{kj}, \ if \ \ i \neq k, j \neq k. \end{cases}$$

According to Hawkins-Simon criterion [1] the matrix A is productive if and only when the sequence of main minors $b_1, b_2, ,..., b_n$ of the matrix B is greater than zero. It is easy to see that the sequence $1, b_1, b_2, ,..., b_n$ after division of the current element to the previous element gives the sequence $\gamma_1, \gamma_2, ,..., \gamma_n$ of resolved elements of our algorithm. This means that the positivity of all resolved elements is necessary and sufficient condition for productivity of the matrix A. Further, it is obvious that the product of all resolved elements is equal to determinant of the matrix B and will be located in the zero row and column of the final (last) table D. Note also that in the last table on the place of matrix B we have the matrix of full costs B^{-1}, which will allow us to calculate the condition number of matrix B. Finally, from the last table D on the position of components of the vectors y and $-q$ we find respectively the volumes and prices x and p of producible products.

If during multipurpose algorithm the current diagonal resolved element γ_k will not be positive then matrix A is not productive, and one must follow to strategy of choice of the resolved elements as in [3].

This algorithm was tested on hybrid computing systems, hardware of which are composed of multi-core CPU and multiprocessors of company NVIDIA, forming a parallel computing architecture CUDA. Multipurpose algorithm requires intensive interaction with CPU only at the stage of transfer the matrix A, vectors y, q and issuance of the final result x, p.

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