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SHORT COMMUNICATIONS

# Modal Control of a Linear Differential Equation with Incomplete Feedback

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Consider the linear stationary controlled system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbf{R}^n, \qquad u \in \mathbf{R}^m. \tag{1}$$

Suppose that all coordinates of the state vector $x$ can be measured. Taking a feedback control $u = Ux$, we obtain the closed system

$$\dot{x} = (A + BU)x. \tag{2}$$

The problem of placing the eigenvalues $\lambda_i(A + BU)$, $i = 1, \ldots, n$, of system (2) at arbitrary given points is called the *modal control problem*. The term "modal control" is due to the fact that the eigenvalues correspond to components of free motion of the system, sometimes referred to as *modes* [1, p. 5]. A similar problem for a nonstationary system (2) is referred to as the control problem for Lyapunov characteristic exponents. One says that the Lyapunov exponents of system (2) are *globally controllable* if, by using some admissible control, one can bring them to arbitrary given points in $\mathbf{R}$; the exponents are said to be *locally controllable* if, by using small-norm controls, one can bring them to arbitrary given points in a neighborhood of the exponents of the nonperturbed system (2) with $U = 0$. We consider only stationary systems. The requirement of complete controllability of system (1) is known to be a necessary and sufficient condition for the global controllability of Lyapunov exponents. For example, consider the single-input $n$th-order linear differential equation

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = u, \qquad x \in \mathbf{R}, \qquad u \in \mathbf{R}. \tag{3}$$

The controlled system (3) is equivalent to the matrix system (1), where $A$ is a Frobenius matrix and $B = e_n$ is the $n$th column of the identity matrix $I$. This matrix system is completely controllable, and the control $U = (a_n - \gamma_n, \ldots, a_1 - \gamma_1)$ brings the characteristic polynomial of system (3) to an arbitrary given form $p(\lambda) = \sum_{i=0}^{n} \gamma_i \lambda^{n-i}$ ($\gamma_0 = 1$). Accordingly, the control

$$u = (a_1 - \gamma_1)\, x^{(n-1)} + \cdots + (a_n - \gamma_n)\, x \tag{4}$$

brings Eq. (3) to an equation with given coefficients $\gamma_i$ and provides the desired asymptotics of solutions of system (3), (4).

Consider an $n$th-order object whose input is a linear combination of $m$ signals and their derivatives of order $\leq n - p$. We assume that $k$ distinct linear combinations of the object state $z$ and its derivatives of order $\leq p - 1$ can be measured:

$$\begin{aligned} &z^{(n)} + a_1 z^{(n-1)} + \cdots + a_n z \\ &\qquad = b_{p1} v_1^{(n-p)} + b_{p+1,1} v_1^{(n-p-1)} + \cdots + b_{n1} v_1 + \cdots + b_{pm} v_m^{(n-p)} + \cdots + b_{nm} v_m, \\ &\qquad\quad z \in \mathbf{R}, \qquad 1 \leq p \leq n, \end{aligned} \tag{5}$$

$$y_i = c_{1i} z + \cdots + c_{pi} z^{(p-1)}, \qquad i = 1, \ldots, k, \tag{6}$$

where $v = \operatorname{col}(v_1, \ldots, v_m) \in \mathbf{R}^m$ is the control vector and $y = \operatorname{col}(y_1, \ldots, y_k) \in \mathbf{R}^k$ is the output vector. The modal control problem is the problem of constructing an incomplete feedback control $v = Uy$ that brings system (5), (6) to a closed system

$$z^{(n)} + \gamma_1 z^{(n-1)} + \cdots + \gamma_n z = 0 \tag{7}$$

with given coefficients. Such a control was constructed in [1, p. 36] for a completely controllable, completely observable object (5), (6) for the case in which $m+k-1 \geq n$, i.e., the total number of input and output signals exceeds the dimension of the object. (This case is rather trivial.) In the present paper, we obtain a necessary and sufficient condition for the existence of a modal control of system (5), (6) with incomplete feedback $v = Uy$.

On the basis of system (5), (6), we construct matrices $B \in M(n,m)$ and $C \in M(n,k)$ with entries $b_{ij} \equiv 0$, $i = 1,\ldots,p-1$, $j = 1,\ldots,m$, and $c_{ij} \equiv 0$, $i = p+1,\ldots,n$, $j = 1,\ldots,k$, where $M(n,k)$ is the space of $n \times k$ matrices. Let $J_0 = I \in M(n,n)$, and let $J_1 \in M(n,n)$ be the matrix whose entries on the first superdiagonal are equal to unity and whose remaining entries are zero. We set $J_q \doteq J_1^q$. The asterisk stands for transposition of matrices. (The transpose of a column vector is a row vector.)

**Theorem 1.** *Let the feedback $v = Uy$ bring system* (5), (6) *to the closed system* (7). *Then the coefficients $\gamma_i$ of system* (7) *satisfy the relations $\gamma_i = a_i - \operatorname{Sp} C^* J_{i-1} BU$, $i = 1,\ldots,n$.*

The proof is based on the following lemma.

**Lemma 1.** *Let $D_i = C^* J_i B$, $i \in \{0,\ldots,n-1\}$, and $D_i = \left\{d_{rj}^i\right\}$, $r = 1,\ldots,k$, $j = 1,\ldots,m$. Then $d_{rj}^i = \sum_{l=i+1}^n c_{l-i,r} b_{lj}$.*

**Proof.** For each $i \in \{0,\ldots,n-1\}$, we construct the matrix $J_i = \{\alpha_{st}^i\}_{s,t=1}^n$. Then $\alpha_{s,s+i}^i = 1$ for all $s = 1,\ldots,n-i$, and the remaining entries $\alpha_{st}^i$ are zero. Let $F_i = C^* J_i$. Then $F_i = \{f_{rl}^i\}$, $r = 1,\ldots,k$, $l = 1,\ldots,n$, where $f_{rl}^i = \sum_{s=1}^n c_{sr} \alpha_{sl}^i$. Since $\alpha_{l-i,l}^i = 1$ (if $l > i$) and the remaining entries $\alpha_{sl}^i$ are zero (and all $\alpha_{sl}^i$ are zero if $l \leq i$), it follows that $f_{rl}^i = c_{l-i,r} \operatorname{sgn} \max\{0, l-i\}$. Therefore,

$$d_{rj}^i = \sum_{l=1}^n f_{rl}^i b_{lj} = \sum_{l=i+1}^n c_{l-i,r} b_{lj}.$$

The proof of the lemma is complete.

**Proof of Theorem 1.** Let us compute the trace of the matrix $C^* J_{i-1} BU$. Let $D_{i-1} = C^* J_{i-1} B = \left\{d_{rj}^{i-1}\right\} \in M(k,m)$ and $U = \{u_{jr}\} \in M(m,k)$. Then, by Lemma 1,

$$\operatorname{Sp} C^* J_{i-1} BU = \operatorname{Sp} D_{i-1} U = \sum_{r=1}^k \sum_{j=1}^m d_{rj}^{i-1} u_{jr} = \sum_{r=1}^k \sum_{j=1}^m \sum_{l=i}^n c_{l+1-i,r} b_{lj} u_{jr}. \tag{8}$$

Let us substitute the control $v = Uy$ into system (5). We denote the right-hand side of the resulting relation by $\Delta$. Let us show that the coefficient of $z^{(n-i)}$ on the right-hand side coincides with (8), which implies the assertion of the theorem. Since $v_j = \sum_{r=1}^k u_{jr} y_r$ for all $j = 1,\ldots,m$ and $y_r = \sum_{s=1}^p c_{sr} z^{(s-1)}$ for $r = 1,\ldots,k$, we have

$$\Delta = \sum_{j=1}^m \sum_{l=p}^n b_{lj} v_j^{(n-l)} = \sum_{j=1}^m \sum_{l=p}^n b_{lj} \left( \sum_{r=1}^k u_{jr} y_r \right)^{(n-l)}$$
$$= \sum_{j=1}^m \sum_{l=p}^n \sum_{r=1}^k b_{lj} u_{jr} \left( \sum_{s=1}^p c_{sr} z^{(s-1)} \right)^{(n-l)} = \sum_{j=1}^m \sum_{l=p}^n \sum_{r=1}^k \sum_{s=1}^p b_{lj} c_{sr} u_{jr} z^{(n-l+s-1)}.$$

Let $i = l - s + 1$. Then $i$ ranges from $l - p + 1$ to $l$. Therefore,

$$\Delta = \sum_{r=1}^k \sum_{j=1}^m \sum_{l=p}^n \sum_{i=l-p+1}^l b_{lj} c_{l+1-i,r} u_{jr} z^{(n-i)}.$$

Note that if $i$ ranges from 1 to $l-p$, then $c_{l+1-i,r}=0$. Consequently,

$$\Delta = \sum_{r=1}^{k}\sum_{j=1}^{m}\sum_{l=p}^{n}\sum_{i=1}^{l} b_{lj}c_{l+1-i,r}u_{jr}z^{(n-i)}.$$

Now we note that if $l$ ranges from 1 to $p-1$, then $b_{lj}=0$, whence it follows that

$$\Delta = \sum_{r=1}^{k}\sum_{j=1}^{m}\sum_{l=1}^{n}\sum_{i=1}^{l} b_{lj}c_{l+1-i,r}u_{jr}z^{(n-i)}.$$

We change the summation order, replacing $\sum_{l=1}^{n}\sum_{i=1}^{l}$ by $\sum_{i=1}^{n}\sum_{l=i}^{n}$; then we obtain

$$\Delta = \sum_{r=1}^{k}\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{l=i}^{n} b_{lj}c_{l+1-i,r}u_{jr}z^{(n-i)} = \sum_{i=1}^{n} z^{(n-i)} \sum_{r=1}^{k}\sum_{j=1}^{m}\sum_{l=i}^{n} c_{l+1-i,r}b_{lj}u_{jr}.$$

The coefficient of $z^{(n-i)}$ coincides with (8). The proof of the theorem is complete.

Let us now find conditions guaranteeing that system (5), (6) has a modal control. We introduce the mapping vec : $M(n,m)\to \mathbf{R}^{nm}$ that unwraps every matrix $H=\{h_{ij}\}$, $i=1,\dots,n$, $j=1,\dots,m$, into a column vector by the rule

$$\operatorname{vec} H = \operatorname{col}(h_{11},h_{12},\dots,h_{1m},\dots,h_{n1},\dots,h_{nm}).$$

Note that $\operatorname{Sp}(A^*B)=(\operatorname{vec}A)^*\cdot(\operatorname{vec}B)$ for any matrices $A,B\in M(n,m)$. We construct the matrices

$$C^*J_0B, \quad \dots, \quad C^*J_{n-1}B \tag{9}$$

and the matrix $P=[\operatorname{vec}C^*J_0B,\dots,\operatorname{vec}C^*J_{n-1}B]\in M(mk,n)$. Let $a=\operatorname{col}(a_1,\dots,a_n)\in\mathbf{R}^n$.

**Theorem 2.** *System* (5), (6) *has a modal control if and only if the matrices* (9) *are linearly independent, and in this case, the feedback matrix $U$ bringing system* (5), (6) *to the form* (7) *with prescribed coefficients is given by* (11), *where* $w=\operatorname{vec}U^*$.

**Proof.** System (5), (6) has a modal control if and only if, for each $\gamma=\operatorname{col}(\gamma_1,\dots,\gamma_n)\in\mathbf{R}^n$, there exists a matrix $U\in M(m,k)$ such that

$$\gamma_i = a_i - \operatorname{Sp}C^*J_{i-1}BU = a_i - \operatorname{Sp}UC^*J_{i-1}B.$$

This is a system of $n$ equations with $mk$ unknowns $\{u_{jr}\}$. It can be rewritten in the vector form

$$a - P^*w = \gamma, \tag{10}$$

where $w=\operatorname{vec}U^*$. If the matrices (9) are linearly independent, then $\operatorname{rank}P=n$. In this case, $P^*P$ is a nondegenerate matrix, system (10) is solvable for any $\gamma$, and its solution has the form

$$w = P\left(P^*P\right)^{-1}(a-\gamma). \tag{11}$$

But if the matrices (9) are linearly dependent, then $\operatorname{rank}P<n$ and system (10) has no solution for the vector $\gamma=a-\beta$, where $\beta\notin\operatorname{Im}P^*$. The proof of the theorem is complete.

Note that in a system with an incomplete feedback, as well as in a system with a complete feedback, the possibility of bringing the system to a prescribed equation is independent of the coefficients $a_i$ of the equation and only depends on the coefficients $b_{lj}$ and $c_{sr}$ of the linear combinations of input and output signals.

**Example.** Consider system (5), (6) with $n = 6$, $m = 2$, $k = 3$, $z^{VI} = v_1 + v_2'$, $y_1 = z$, $y_2 = z''$, and $y_3 = z^{IV}$. For this system, we construct the matrices $B$ and $C$. One can readily see that the matrices (9) are linearly independent. Therefore, this system has the modal control $v = Uy$, where

$$U = \begin{pmatrix} -\gamma_6 & -\gamma_4 & -\gamma_2 \\ -\gamma_5 & -\gamma_3 & -\gamma_1 \end{pmatrix}.$$

This example shows that the condition $m + k - 1 \geq n$ can be weakened for system (5), (6). For the existence of a modal control, it is necessary that the dimensions $m$ and $k$ satisfy the condition $mk \geq n$. This readily follows from Theorem 2.

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